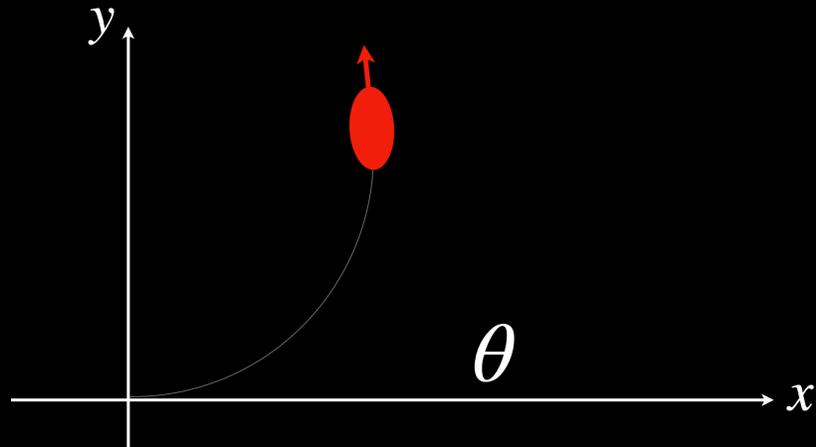
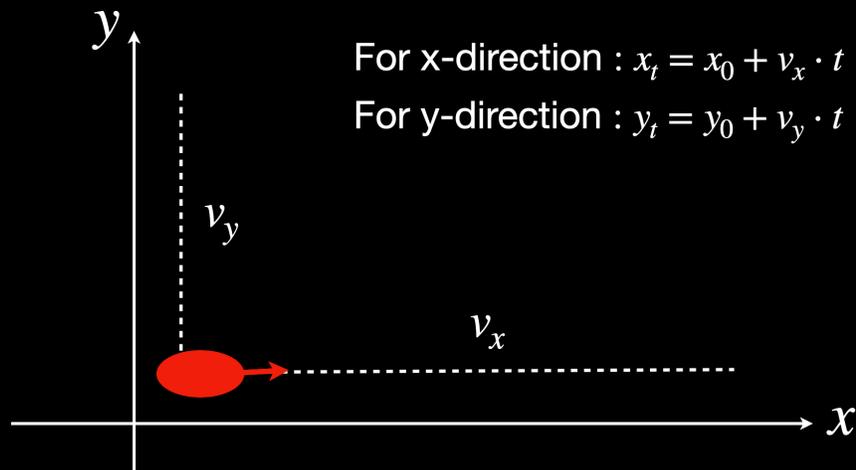


## Special Euclidean Group (2)

Say we have an object moving in a 2D Cartesian plane. Theta is an angle describing the object's orientation in the plane.



When the object moves with constant velocities  $v_x$  in the x-direction and  $v_y$  in the y-direction, its position changes linearly over time.



When the object combines translation (or moving forward) and rotation (or turning), its path becomes a curve rather than a straight line. Likewise, If you first move the object forward and then rotate it, it ends up in a different position than if you rotate it first and then move it forward. All of these are known as poses.

Now, there are actually more specifics to how the object behaves. Like say we want to define the composition of poses  $T_1$  and  $T_2$ .

$$T_1 T_2 = (x_1, y_1, \theta_1)(x_2, y_2, \theta_2) = (x_1 + \cos \theta_1 x_2 - \sin \theta_1 y_2, y_1 + \sin \theta_1 x_2 + \cos \theta_1 y_2, \theta_1 + \theta_2)$$

The matrix form you're looking at is a compact way to represent both rotation and translation in 2D space. It simplifies the process of combining these transformations through matrix multiplication.

$$T_1 T_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix}$$

R is a matrix within a matrix.

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

The t is the Translation Vector, a column vector

$$t = \begin{bmatrix} x \\ y \end{bmatrix}$$

It moves the object from one position to another by adding x and y to the existing coordinates.

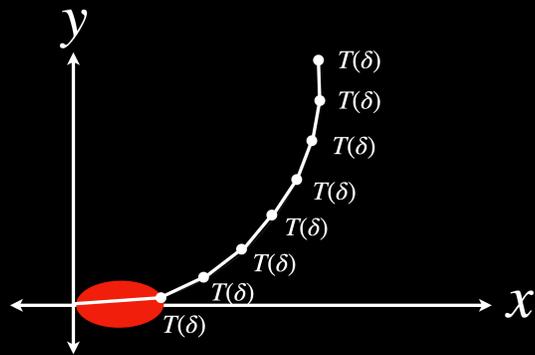
The 1's and 0's basically preserve properties so that the matrix can be translated back to the original equation.

When the matrices are multiplied, The result,

$$\begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix}$$

Shows the combined effect of the two rotations and the translational composition

The line is actually a combination of a bunch of tiny motions, infinitesimally small motions of the object to be exact, like a bunch of tiny rotations and translations.



That's described using this matrix

$$T(\delta) = \begin{bmatrix} \cos \omega\delta & -\sin \omega\delta & v_x\delta \\ \sin \omega\delta & \cos \omega\delta & v_y\delta \\ 0 & 0 & 1 \end{bmatrix}$$

This leads us to a simplified version of the matrix

$$T(\delta) \approx \begin{bmatrix} 1 & -\omega\delta & v_x\delta \\ \omega\delta & 1 & v_y\delta \\ 0 & 0 & 1 \end{bmatrix} = I + \delta\xi$$

$I$  is the identity matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

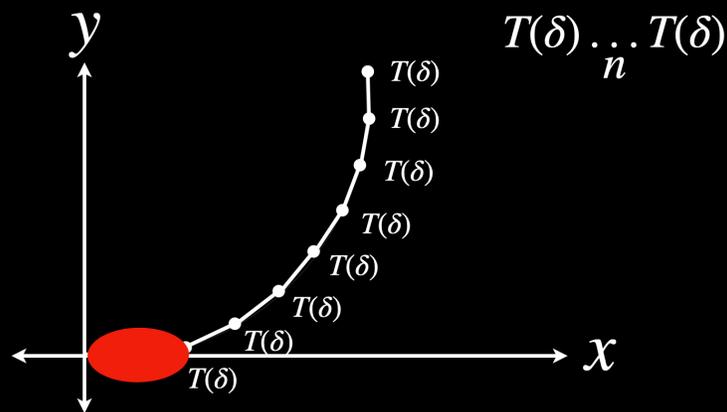
You see this squiggle right here? That's the 2D "twist" vector, or the "incremental transformation matrix".

$$T(\delta) \approx \begin{bmatrix} 1 & -\omega\delta & v_x\delta \\ \omega\delta & 1 & v_y\delta \\ 0 & 0 & 1 \end{bmatrix} = I + \delta\xi$$

This matrix combines Angular velocity, dictating how fast and in what direction (clockwise or counterclockwise) the object rotates. And Linear velocities dictating how the object moves along the x and y axes.

*(BTW, consider becoming a member of the channel!) Thanks!*

When it comes to the real world, t or time, can be quite large. So in order to manage it when that's the case, t needs to be split up into smaller time intervals, let's say n of them, each of length t over n.



The power n in the equation indicates that this small transformation is applied repeatedly, n times, to cover the entire duration t .

The most perfect solution possible is given when n is taken to infinity, since that's the most continuous and smooth transformation.

$$T(t) = \lim_{n \rightarrow \infty} \left( I + \frac{t}{n} \xi \right)^n$$

If you want to understand exponential growth through a mathematical lens, you break it down into infinitely many tiny steps that compound over time. For real numbers, this is expressed as this:

$$e^x = \lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n$$

The process of taking a twist matrix  $\{ \xi \}$  and converting it into a complete transformation matrix  $\{ e^{\xi t} \}$  is a mapping from the algebra of twists (or the infinitesimal generators of motion) to the group of rigid transformations (or the actual motions in space).

When we refer to the process of mapping 2D twist matrices  $\xi$  to 2D rigid transformations using the matrix exponential  $e^{\xi t}$  we are talking about what's known in mathematics and physics as the **exponential map**.

Rigid transformations are those that don't change the shape or size of an object; they only adjust its position or orientation. This means they include all the ways you can rotate and translate objects within a plane without distorting them.

We call the space of 2D rigid transformations, along with the composition operation, the special Euclidean group SE(2).

SE(2) is a Lie group for 3 reasons. First because it is a group, it satisfies all of 4 properties that a group must have:

Closure: If you compose two transformations (rotation and translation), the result is another transformation in SE(2).

Identity element: There is an identity transformation (no rotation, no translation) that leaves any object unchanged.

Inverse element: Every transformation has an inverse. For example, if you rotate an object by an angle  $\theta$  and translate it by  $(x, y)$ , you can reverse this with the inverse transformation.

Associativity: The order of applying transformations doesn't affect the final result, as long as they are grouped in the same way.

Second, because it is a topological group.

In a topological group, the two main group operations—composition (combining two transformations) and inversion (finding the reverse of a transformation)—must be continuous functions.

Third,  $SE(2)$  is a manifold.

Simply put, a manifold is a mathematical space that looks locally like Euclidean space (which is a flat, familiar space where we do calculus) but can have a more complex global structure.

Obviously there's a much more complete explanation for it, but that's the gist of it.

The key idea is that even if a manifold might have twists or curves on a large scale, if you zoom in close enough at any point, it behaves like flat space, allowing you to apply tools like calculus.

$SE(2)$  represents all possible rigid transformations (or combinations of rotations and translations) in a 2D space.

Globally,  $SE(2)$  is equivalent to the product manifold

$$S^1 \times \mathbb{R}^2$$

Locally, however, it looks like  $\mathbb{R}^3$

Plus,  $SE(2)$  is not just a manifold, but a smooth manifold.

Let's see an example of a transformation in  $SE(2)$ , which is the group of 2D rigid transformations (rotations and translations).

Rotate an object by 90 degrees counterclockwise

Translate the object 2 units along the  $x$ -axis and 3 units along the  $y$ -axis.

This transformation can be written in matrix form as:

$$T = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) & 2 \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's substitute the trigonometric values for

$$\cos\left(\frac{\pi}{2}\right) = 0 \quad \sin\left(\frac{\pi}{2}\right) = 1$$

So, the matrix becomes:

$$T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, let's apply this transformation to a point:

Let's say you have a point  $P = (1, 1)$  in 2D space. To apply the SE(2) transformation to this point, you can extend the point to homogeneous coordinates by writing P as (the third coordinate is 1 for homogeneous coordinates):

$$P_{\text{hom}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Now, multiply the transformation matrix  $T$  by the point  $P$ :

$$T \cdot P = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 1 + 2 \\ 1 \cdot 1 + 0 \cdot 1 + 3 \\ 0 \cdot 1 + 0 \cdot 1 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

After applying the transformation, the point  $P = (1, 1)$  is moved to the new position  $(1, 4)$ . So the transformation first rotates the point by 90 degrees and then translates it by 2 units in the x-direction and 3 units in the y-direction.

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