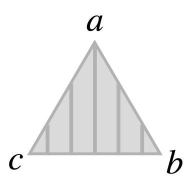
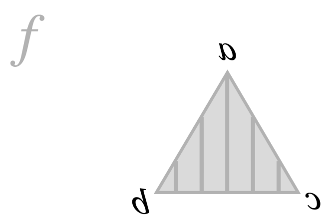
Subgroups

Say, we have this equation $frf = r^{-1}$. It tells us that to flip, then rotate, then flip again is the equivalent of rotating in reverse. But before seeing this play out on a Cayley Diagram, let's apply it to an object.

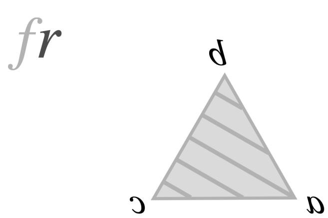
We start at e, since that's the initial state, and apply the reflection f.



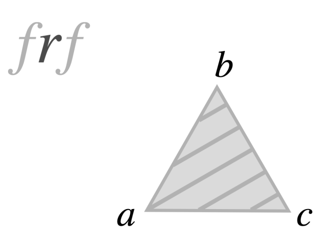
Then, we flip it, by applying f.



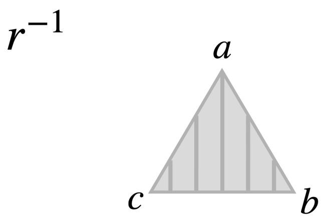
We apply r to the already flipped object, which in our case is rotating it by 120 degrees.



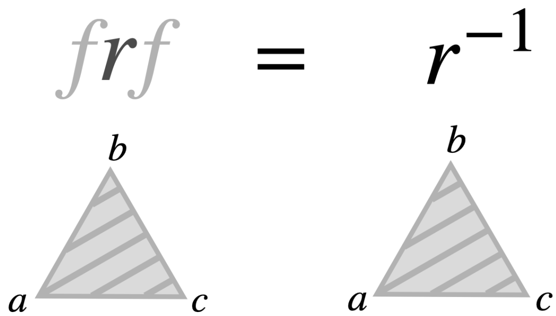
And finally, we flip it again in the state it finds itself, resulting in this.



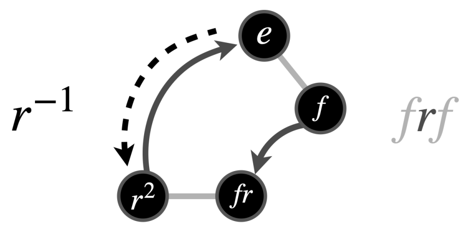
Now let's try to see when we rotate the initial state of the triangle 120 degrees in reverse.



We end up with the exact same set of circumstances for both.

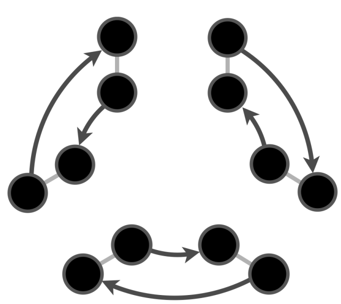


Let’s map it, starting at e, flip, so we end at f, then rotate, so we are at fr, and flip again, and we find ourselves at r^2. Now, go from e in reverse, and end at r^2 as well.

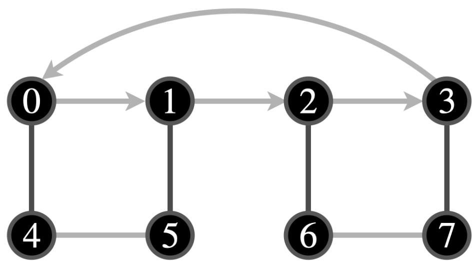


This specific pattern actually appears at every point in the whole diagram. The pattern can fit anywhere in S_3.

This tells us something interesting: Cayley Diagrams always have uniform symmetry. Every Cayley Diagram is regular, by definition. Diagrams which lack this regularity cannot be called Cayley Diagrams.

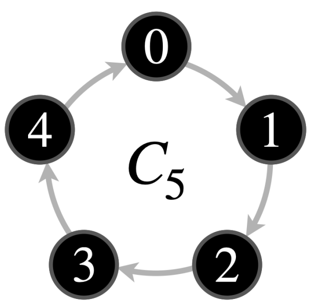


Take this diagram for example. It satisfies many of the necessary criteria, but lacks one: it is not regular. Equations about relationships among the generators change from one part of the diagram to another.

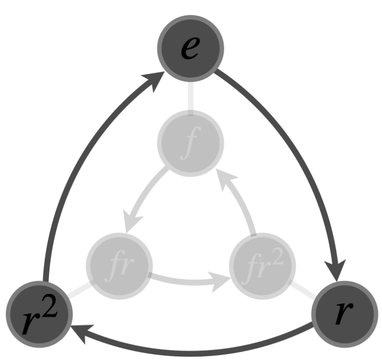


If you paid close attention to group theory in general, you'd notice that every group has at least one of another type of group inside of it, **cyclic groups**, or more appropriately, **orbits**.

Cyclic groups are the simplest types of groups that only have one action to them, and are commonly labeled as C_n, *n* standing for the number of elements or their order. like this C_5 with 5 elements, which uses the single action of rotation.



If we take a look at S_3 again, we notice a cyclic group inside of it, similar to the C_5 we just saw except with 3 elements. This orbit can be labeled as $\{e, r, r^2\}$. When one group is completely contained within another, it is called a **subgroup**.

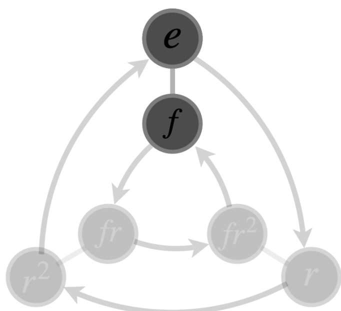


By the way, consider becoming a member of the channel. Thanks!

Hypothetically speaking, when a group *H* is a subgroup of *G*, the way to write it is *H*<*G*.

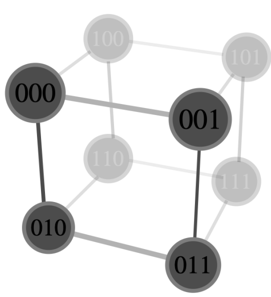
So the orbit of S_3 we just saw is the cyclic group C_3, and is written as $\{e, r, r^2\} < S_3$, or if we are not really concerned with formalities, it's just $C_3 < S_3$.

You can also find another group in S_3 , C_2. This group is also an orbit, and is labeled as *{e, f}*.



This may give you the impression that we pretty much need to look for cyclic groups or orbits inside of Cayley Diagrams, but not all subgroups are cyclic.

Take this group, $C_2 \times C_2 \times C_2$. There is a copy of the group V_4 inside of it, but it is not a cyclic group.



V_4

It would be a cyclic group if you could generate all the elements by repeatedly adding a single element from V_4. But you can't, so it's not a cyclic group.

Since $\{e, r, r^2\}$ is generated by a single action r, or rotation, it can be written more succinctly as <*r*>. In *{e, f }* is generated by a single action f, or flip, so it can be written as <*f*>.

But V_4 has order 2, which means that every element in the group can be obtained by the elements 001 and 010, so it is labeled as *<001, 010>*. We see that through the following calculations:

001+010 = 011

001 + 001 = 000

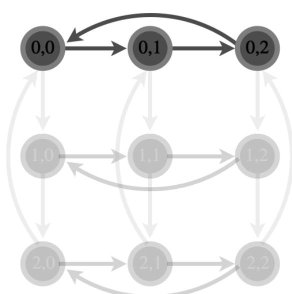
010 + 010 = 000

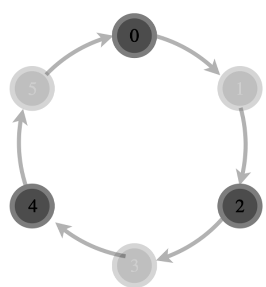
Thus, we were able to generate all 4 of the elements.

Every group has some kind of subgroup. The identity itself, {e}, is a copy of the group C_1, and it's formally called the **trivial subgroup**.

And technically, every group is a subgroup of itself $\{S_3 < S_3\}$ and this is formally called a **non-proper subgroup**.

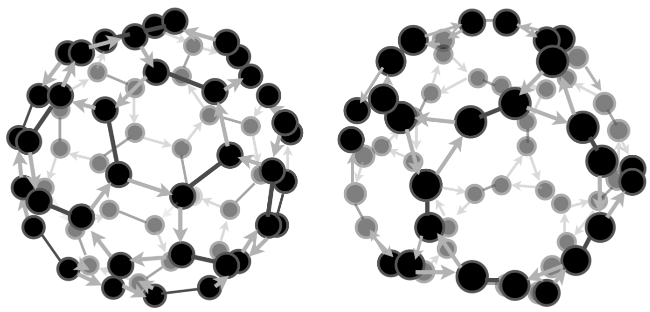
The examples we looked at so far had pretty easily identifiable subgroups, but try and spot them here:





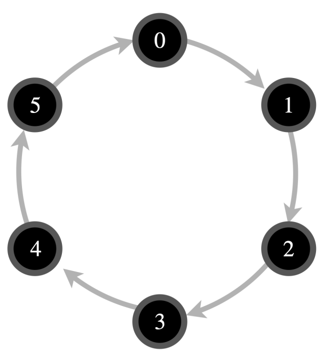
In this diagram of $C_3 \times C_3$ we identify a subgroup C_3. And in this diagram of C_6, we also find the group C_3.

Take these two representations of the group A_5. They're both A_5, but they emphasize different subgroups. One emphasizes a cyclic group of order 5 $\{C_5\}$, while the other better shows the cyclic group of order 3, $\{C_3\}$.

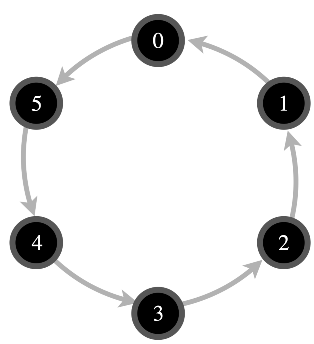


Take $C_6 = \{0, 1, 2, 3, 4, 5\}$. How would you generate its Cayley Diagram?

One is the most intuitive one, $C_6 =< 1 >$ So by repeatedly adding 1 we can reach all of the elements

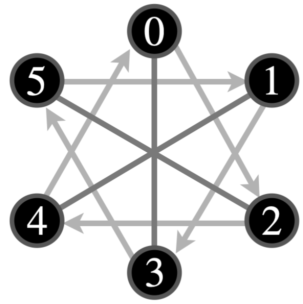


We can also have $C_6 =< 5 >$. Repeatedly adding 5, (which is just the equivalent of generating the group backwards by subtracting 1), covers all of the elements as well.

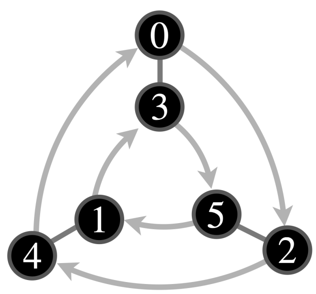


Here's a really unintuitive way of doing it: $C_6 =< 2, 3 >$. Neither 2 nor 3 can generate C_3 by themselves, but together they do.

$0 + 2 = 2$
$1 + 2 = 3$
$2 + 2 = 4$
$3 + 2 = 5$
$4 + 2 = 6 \equiv 0 \pmod{6}$
$5 + 2 = 7 \equiv 1 \pmod{6}$
$0 + 3 = 3$
$4 + 3 = 7 \equiv 1 \pmod{6}$
$5 + 3 = 8 \equiv 2 \pmod{6}$



We can actually also re-organize it into this. They are both diagrams representing $C_6 =< 2, 3 >$.



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