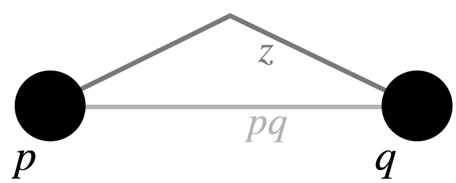
Geodesics

By DiBeos

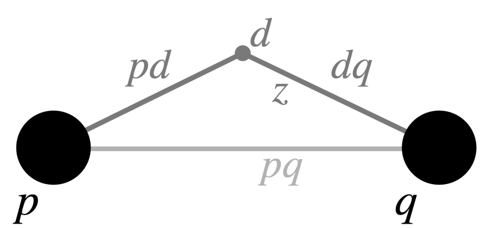
The point *p* finds itself in a flat space, called a 2 dimensional Euclidean surface. Now, say a point *q* also pops up on this surface, and we want to find the distance between the two. How can we *prove* that a straight line is the shortest path between the two points?

Assume there exists a path, other than the straight line *pq*.

Suppose this new path, *z*, is not a straight line. This means *z* must at least consist of two segments that together form an angle at some point d along *z*.



According to the triangle inequality, the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. Thus *pd* + *dq* > *pq*. This inequality shows that taking a detour through *d* results in a longer path than going straight from *p* to *q*.

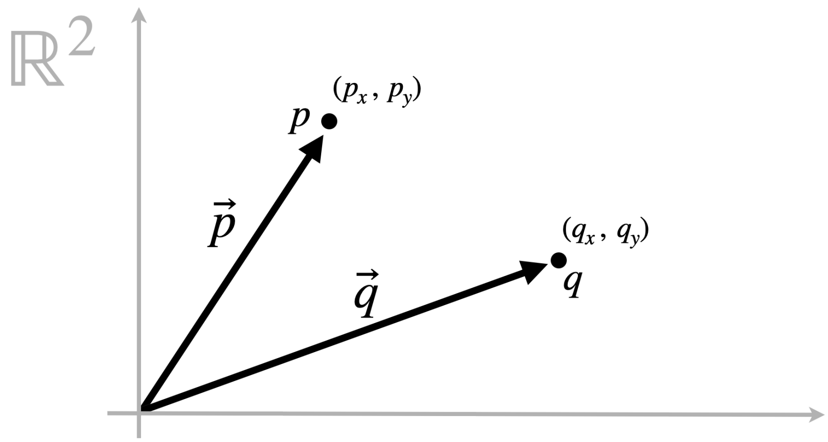


The argument can be generalized for any path z that deviates from the straight line pq.

To describe the relationship between the two points using more rigorous mathematical terminology we have to describe this flat space as a flat Euclidean space $\mathbb{R}^2$, which is a coordinate plane. Now we can place the points *p* and *q* on the coordinate plane. Each is described by a pair of coordinates (p_x, p_y) and (q_x, q_y).

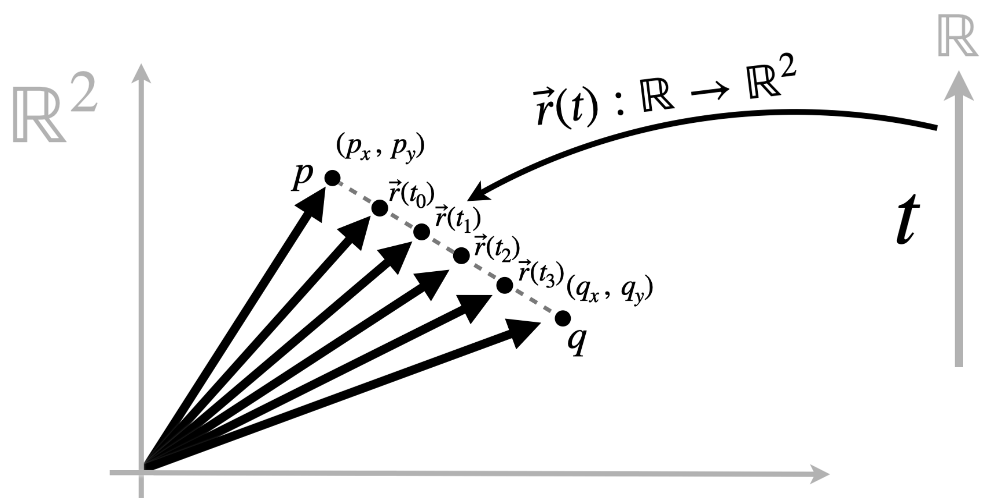
In $\mathbb{R}^2$ any point can be represented as a vector from the origin to that point. For instance, the point *p* with coordinates (p_x, p_y) is represented by the vector $\vec{p}$, while (q_x, q_y) becomes $\vec{q}$.

The origin serves as a common reference point for all vectors in the space. By starting every vector at the origin, we establish a uniform way to describe locations in the space.

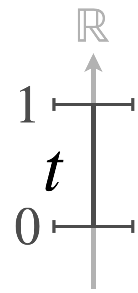


In order to find the line itself, we have to ***parametrize*** it. Which means that we need to describe every point along the line connecting $\vec{p}$ and $\vec{q}$ using a single variable that continuously varies over a defined interval. This single variable is usually *t*, which we can describe as time.

t exists in another 1 dimensional Euclidean space $\mathbb{R}$. We parametrize it to the Euclidean space through this mapping:



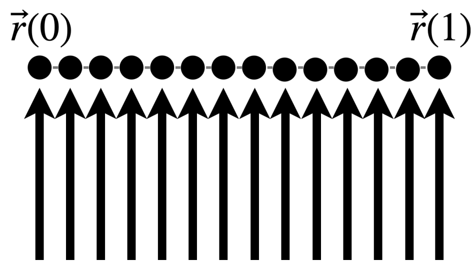
But, just putting it this way means we have described the infinite line, we didn't put an interval to it. Thus, we have to define an interval from the starting point p to the ending point q, which will be the equivalent of starting from 0 to 1.



By defining t in the interval *[0, 1]*, we ensure that when *t = 0*, the position vector is $\vec{p}$, and when *t = 1*, the position vector is $\vec{q}$.

For any value of *t* between 0 and 1, the formula shown gives us a point on the line segment directly between $\vec{p}$ and $\vec{q}$, effectively filling in the line segment.

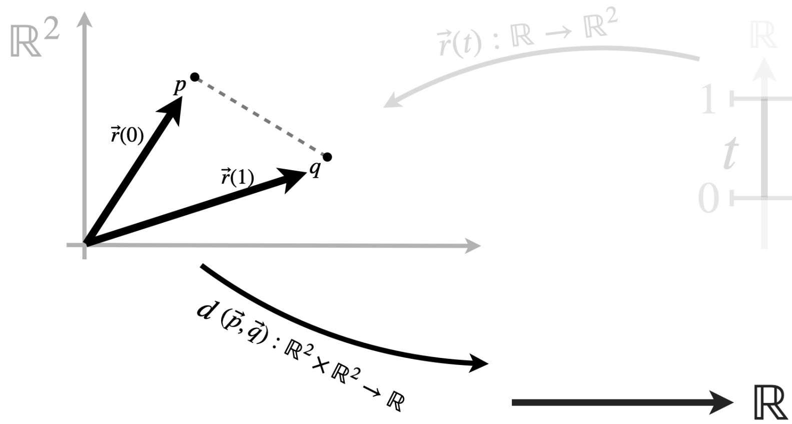
$$\vec{r}(t) = (1 - t)\vec{p} + t\vec{q}$$



This basically means that, if I replace *t* with 0, I will end up with the exact vector coordinates for $\vec{p}$ or (p_x, p_y) for example.

Any number in-between 0 and 1 which we plug in instead of t will point to another vector coordinate and will eventually create the line if we were to plug in the infinite numbers between 0 and 1.

In order to know the exact length of the line, we need to assign it a scalar, given by another 1 dimensional Euclidean space $\mathbb{R}$. Done through the mapping *d*, or in full $d(\vec{p}, \vec{q}) : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$.



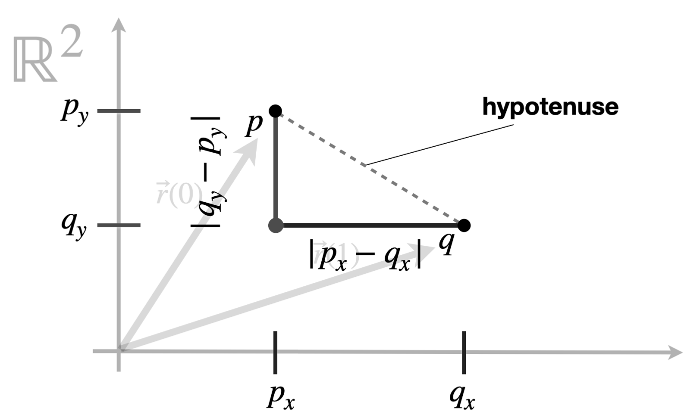
This scalar property is what allows distances to be compared directly, added, or used in various formulas without concern for orientation in space

More explicitly, it is calculated by using the Euclidean distance formula, which is kind of an elaboration of the Pythagorean Theorem:

$$d(\vec{p}, \vec{q}) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

To visualize what we're doing, draw a horizontal line from (p_x, q_x). This line segment represents the difference in the x-coordinates of the two points and measures $|p_x, q_x|$. This is one leg of the right triangle.

Draw a vertical line from (p_y, q_y). This line segment represents the difference in the y-coordinates of the two points and measures $|p_y, q_y|$. This is the other side of the right triangle.

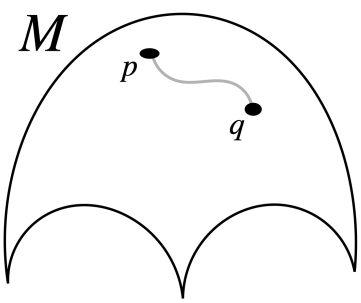


And that is how we find the hypotenuse, or the distance between *p* and *q*, found through this equation:

$$\textbf{hypotenuse} = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$$

The very same process can be repeated when we go to higher dimensions. Say we have a shape, formally called a manifold, that exists by itself. It's not embedded into any dimension, it is the dimensions, which in our case will be *n*.

We have our two points on the manifold, *p* and *q*. We trace a line that connects the two points. Any line to our liking can be traced between the two, as long as it connects the points.



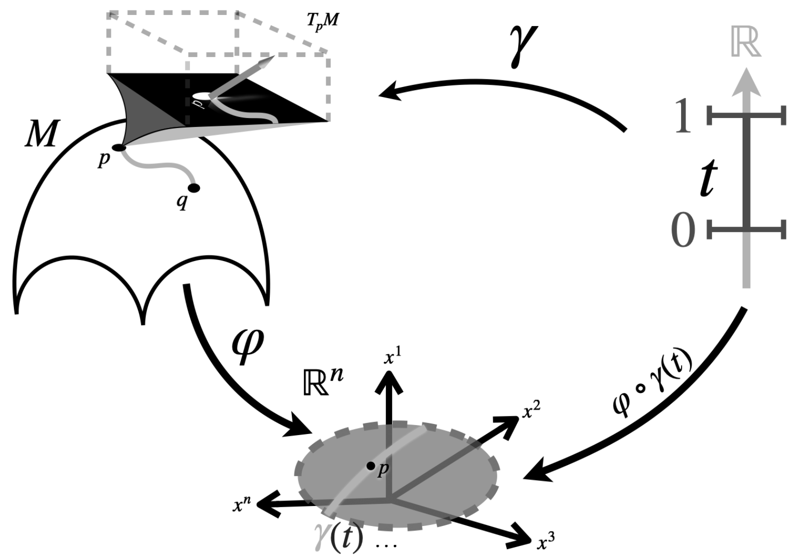
In curved manifolds, the notion of a "position vector" as used in Euclidean space does not directly apply because there's no fixed "origin". So instead, we describe paths using **curves**.

However, vectors can't directly poke out of the points because there's nothing to poke out into. Remember, all of these points are on a manifold not embedded in a space. So in order to draw a vector poking out of a point, say *p*, we have to use a tangent space. We access the tangent space by performing derivatives.

An elaborate explanation of how this is done can be found here:
What are Tangent Spaces in Differential Geometry?

Once we access the tangent space, we draw a vector on it that will be tangent to the curve, or touch the curve at just one point, which in our example is *p*. *p* is dependent on time *t*. It works the same way as we described previously, mapped through gamma to our curve. But since we don't perform calculus directly on the manifold, we actually map it to a Euclidean space in order to do so.

(Consider becoming a member of the channel!) Thanks!



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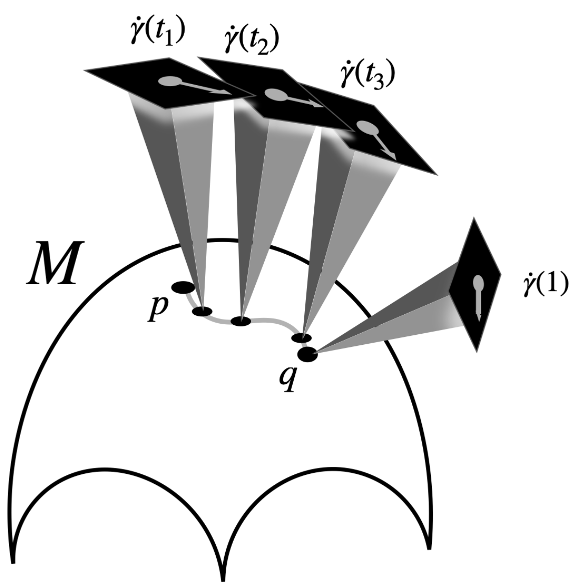
How to do Calculus on an Abstract Manifold

The entire curve is made of these tangent vectors. Each of which is found by performing derivatives for each point of our choosing. We do this infinite times up to $\dot{\gamma}(t_0)$. The numbers that result from the derivatives of each tangent vector specify the vector length, describing how much the curve changes at that point.



Once we find all the individual vectors along the curve, we want to sum their lengths to find the length of the curve. This is done through the distance formula:

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\, dt}$$

length of gamma (the line)

the length squared of this tangent vector at each point *t*

tangent vector to the curve at $\gamma(t)$

The *g* is the **Riemann metric**. The tangent space does not come with a measuring stick that gives tangent vectors a concept of length and angle. A Riemannian metric puts a measuring stick on every tangent space, and it does so through the inner product.

We found a random curve between the point *p* and *q*, and its length. But what we actually want is the minimal length – the shortest curve between the points. The distance *d*(p, q) is the infimum of the lengths of all possible smooth curves γ that connect *p* and *q*.

So instead of calculating all of these possible paths by hand, we find the infimum of all such curves. In other words, we encode all the infinite possibilities and quote on quote spit out the distance.

$$d(a, b) = \inf\{L(\gamma) \mid \gamma : [0,1] \to M, \gamma(0) = a, \gamma(1) = b\}$$

d by definition, is the shortest path possible. In more proper terminology, *d* is the minimal path. This specific curve γ has the minimum length between *p* and *q* is one of the ***geodesics*** of the manifold. Geodesics is the straightest path possible, or the minimal distance between two points. It is the closest thing to a straight line on a curved surface.

The universe in general tends to favor the shortest path for everything, the minimal distance in a curved space. This field is known as geodesics.

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