Getting to Gaussian Curvature

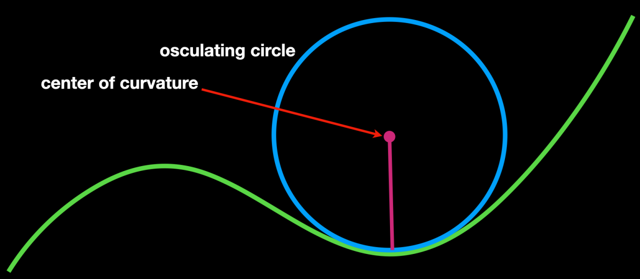
Aristotle said there are three ways to define a line. We can have a **straight loci**, lines that have zero curvature. The curvature \kappa of a straight line is always zero, because it does not curve at any point.

We can bend the line into a circle, and this was named as **circular loci.** These are paths that form a constant radius from a fixed point, or the center. The curvature of a circle is defined as the reciprocal of the radius r.

And, we can also have **mixed loci,** paths that are neither purely straight nor purely circular but may involve segments or combinations of both.
What is the radius of these 'mixed loci'?

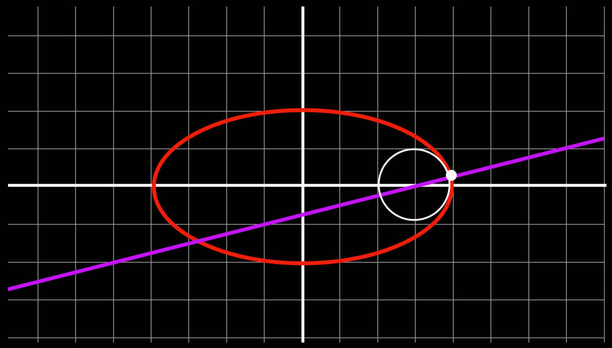
The radius of this curvature is the reciprocal of the curvature, $R = \frac{1}{\kappa}$

The circle defined by this radius is called the "osculating circle". This circle touches the curve at the point of interest and shares the same single point. The center of the osculating circle is called the *center of curvature*.

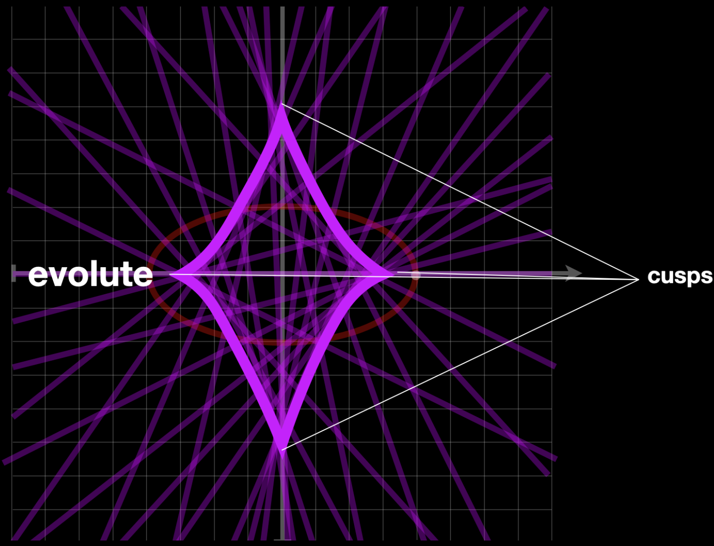


If we take an ellipse, we know that every one of these points on this curve has a certain degree of curvature, which is how sharply the curve bends at that point.

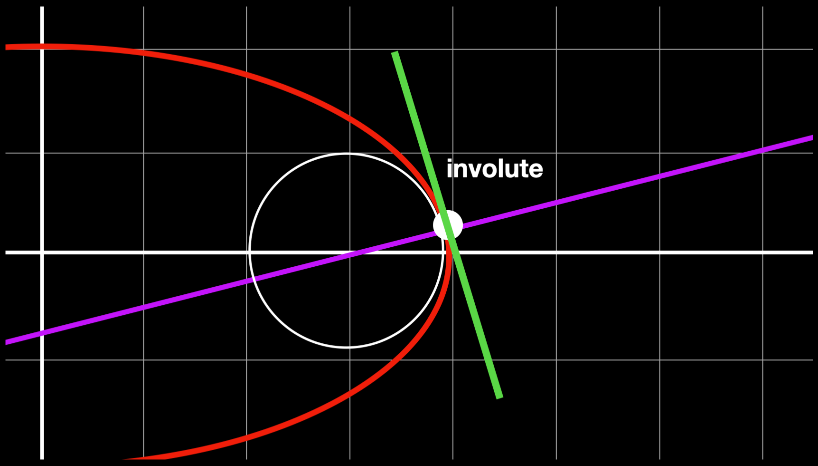
At every point on the curve, you can draw a small circle that just perfectly fits the curve at that point.



As you move along the curve, the center of the osculating circle will also move. If you could track the path of these centers as you progress along the entire curve, you would trace out a new shape. This new shape formed by the path of all these centers of curvature is called the evolute of the original curve. These pinchy ends are called cusps.



What if we take a curve, and instead of drawing a line in-line with the center of curvature and the touching point, we draw one that is perpendicular to it?
That is an involute.



Huygens' method was flawed: in order to find the radius of curvature, the evolute had to be provided, and as a result, the theory was useless for measuring arbitrary curves.

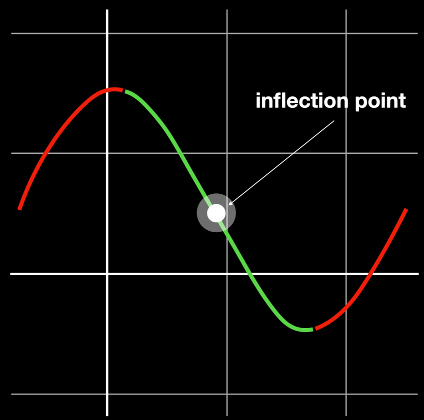
But, calculus introduces concepts like limits and infinitesimals, which are crucial for dealing with very small quantities. When studying curves, these concepts allow mathematicians to zoom in on an infinitesimally small segment of a curve, which is pretty much like a single point.

These 'single points' are actually called tangents (loosely speaking).

If a curve becomes flatter (less curved), the radius of curvature increases.

On the other hand, at a point where the curve bends sharply, the radius of curvature is smaller, indicating a tighter curve.

There was however a flaw in Newton's equations - they yielded "undefined" solutions at the points where the curve changes direction, known as points of inflection.



At an inflection point, the curve technically has zero curvature because it's the transition point between curving in one direction and curving in the opposite direction.

Newton's method for calculating curvature generally involves derivatives.

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

where f'(x) is the first derivative (slope) and f''(x) is the second derivative (rate of change of slope).

At an inflection point, the second derivative f''(x) is zero (since the slope stops changing from increasing to decreasing or vice versa). This results in the curvature calculation:

$$\kappa = \frac{0}{(1 + (f'(x))^2)^{3/2}} = 0$$

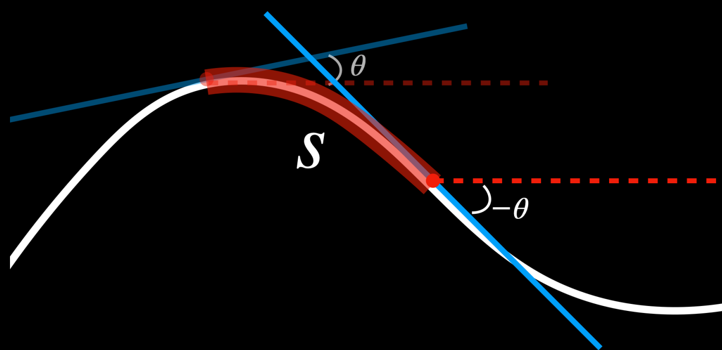
Newton's methods were still not that precise and universally applicable, until Euler came along.

$$\kappa = \frac{d\theta}{ds}$$

The κ as we established represents the curvature of the curve at a particular point. It tells us how sharply the curve is turning at that point. Higher values of κ mean the curve is bending more sharply, and lower values mean it is bending less sharply.
θ is the angle that the tangent line makes with a fixed direction, like the horizontal axis.

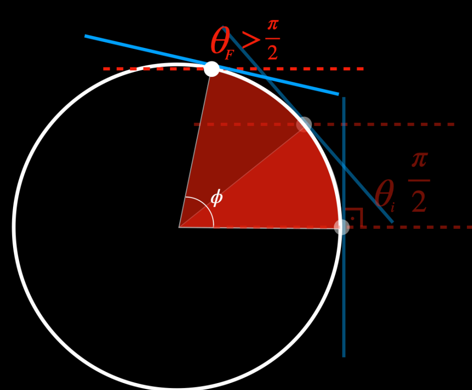
As you move along the curve, θ changes depending on how the direction of the curve changes.

s is the distance along the curve from a starting point to the point where you're measuring the curvature.



Consider a circle of radius r. If you start at the rightmost point and move counter-clockwise, θ is initially 90 degrees.

As you move along the circle, the arc length from the starting point increases directly with the angle you've moved through, measured in radians. If you've moved through an angle ϕ in radians, the arc length s is $s = r\phi$.



As you move along, the tangent angle \theta increases directly with ϕ. Since the circle's circumference is proportional to r times the angle in radians, $\theta = \phi$ and $s = r\phi$, so $\frac{ds}{d\phi} = r$ Therefore,

$$\frac{d\theta}{d\phi} = 1 \text{ and } \frac{d\theta}{ds} = \frac{d\theta}{d\phi} \cdot \frac{d\phi}{ds} = 1 \cdot \frac{1}{r} = \frac{1}{r}$$

This was picked up by Gauss, who advanced our understanding of curvature in his Theorema Egregium.

Say we have a flat, flexible sheet. Now, start bending that sheet into different shapes, like a cylinder, twist it around, or shape it like a U.

This bending doesn’t involve stretching the material; you're simply repositioning it in space. The way we observe the way a 2 dimensional sheet behaves in a 3 dimensional space is called extrinsic curvature.

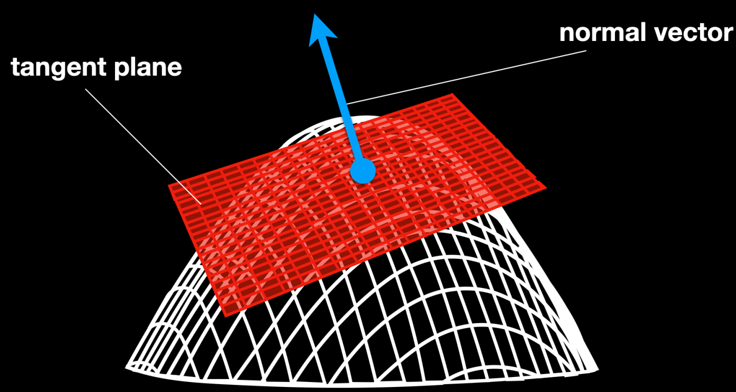
But, what if we did want to study the shape itself, not the way it is embedded into the space around it, and study the behavior of curvatures in themselves?

Well that’s intrinsic curvature, the better understanding of which came after extrinsic curvature.

We focus on Gaussian curvature.

Surfaces can have “hills” and “valleys”, and the analysis of how they bend introduces the idea of a Tangent Plane.

Remember the tangent line? Well this one is similar, except instead of being a line, it’s a plane that touches the surface at just one point. Perpendicular to this tangent plane is a line called the normal vector, which sticks straight out of the surface.



If we slice the surface along a plane that contains both the normal vector and one of these directions, the intersection forms a curve on the surface. This curve shows how the surface bends in that particular direction and is known as the Normal Section Curvature.

However, among all these, two directions yield special curvatures:

1. The direction where the curvature is maximized.
2. The direction where the curvature is minimized.

These are called the Principal Curvatures, usually denoted as κ_1 (maximum) and κ_2 (minimum)

The Gaussian Curvature, denoted as K, is defined as the product of these two principal curvatures $K = \kappa_1 \cdot \kappa_2$

If K > 0 :
- Both principal curvatures have the same sign (both positive or both negative).
- The surface curves in the same direction along both principal directions.
- This resembles the shape of a peak (like the top of a hill) or a valley (like the bottom of a bowl).

If K < 0:
-Since there are infinitely many directions in the tangent plane, there are infinitely many normal section curvatures at that point.
- The principal curvatures have opposite signs.
- The surface curves in opposite directions along the principal directions. - This creates a saddle point.

If K = 0
- At least one of the principal curvatures is zero.
- The surface is flat in at least one direction.
- This resembles a cylinder.

Challenge:

Consider a curve with the function $f(x) = \frac{1}{2}x^2$. At what point on this curve does the curvature κ reach its maximum value?

To help you along:
The formula for the curvature of a curve defined by $y = f(x)$ is:

$$\kappa = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

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