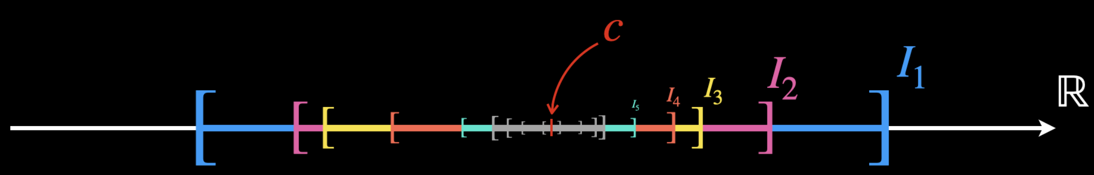
"*I see it, but I don't believe it!*" – Georg Cantor in a letter to Richard Dedekind, in 1877.

<u>Cantor's Theorem</u>: Let $\{I_n\}_{n\in\mathbb{N}}$ be a set with elements that form a sequence of <u>closed</u> and <u>bounded</u> intervals on $\mathbb{R}$, which are <u>decreasing by set inclusion</u> (nested)

$$\Downarrow$$

$$\boxed{\exists c \in I_n,\ \forall n \in \mathbb{N}}$$



Before proving it, let us carefully define every concept in the statement of this theorem. I_n, with n being a natural number, is a set with elements I_1, I_2, I_3, and so on. Each I_n (for a specific n) is an interval a_n, b_n, where a_n and b_n are real numbers, for all n natural, and a_n must be less than or equal to b_n.

$$\{I_n\}_{n\in\mathbb{N}} = \{I_1, I_2, I_3, \ldots\}$$

such that

$$s.t.\ \ I_n := [a_n, b_n]\ ,$$

$$\text{where}\ \ a_n, b_n \in \mathbb{R}\ ,\ \forall n \in \mathbb{N}$$

$$,\ \wedge\ a_n \leq b_n.$$

This last condition is necessary to ensure that I_n is indeed an interval in the real line, otherwise (if a_n is greater than b_n) it is not considered an interval.

$$\{I_n\}_{n\in\mathbb{N}} = \{I_1, I_2, I_3, \ldots\} \; s.t. \; I_n := [a_n , b_n] \,,$$
$$a_n , b_n \in \mathbb{R} \;,\; \forall n \in \mathbb{N} \,,\, \wedge \; \underline{a_n \leq b_n} \,.$$

$I_n = [a_n , b_n]$ *is an interval in* $\mathbb{R}$

$a_n > b_n \implies$ ~~$I_n = [a_n , b_n]$~~ $\rightsquigarrow$ ~~$[2 , 1]$~~ *(example)*

Beyond that, these intervals are <u>closed</u> and <u>bounded</u>.

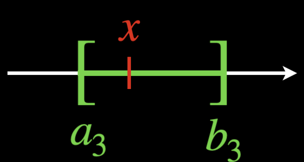
$$\{I_n\}_{n\in\mathbb{N}} = \{I_1, I_2, I_3, \ldots\} \; s.t. \; I_n := [a_n , b_n] \,,$$
$$a_n , b_n \in \mathbb{R} \;,\; \forall n \in \mathbb{N} \,,\, \wedge \; a_n \leq b_n \,.$$

$I_n := [a_n , b_n]$ → <u>closed</u>, <u>bounded</u>

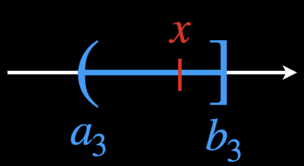
In this context, <u>closed</u> means that these intervals include the two endpoints.

<u>closed</u> :

<u>*example*</u>: $x \in [a_3 , b_3] \iff a_3 \leq x \leq b_3$

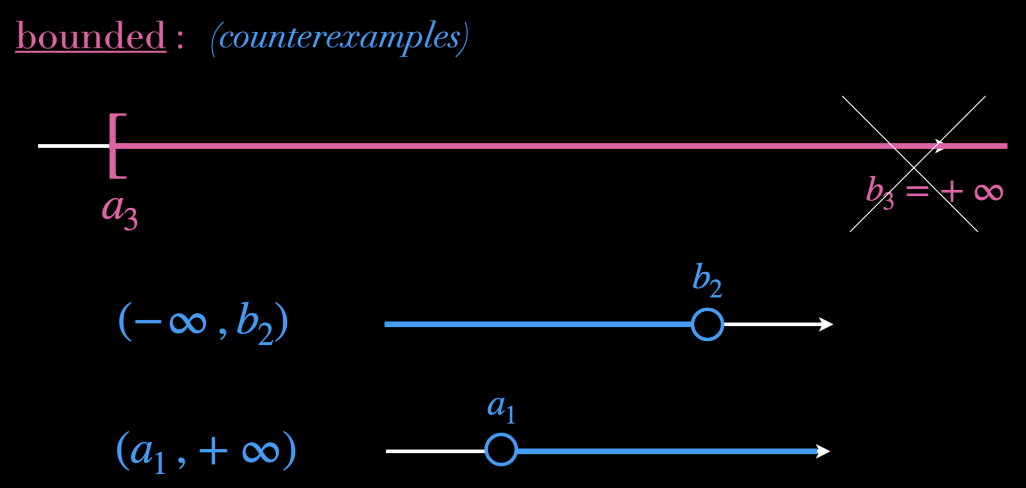


<u>*counterexample*</u>: $x \in (a_3 , b_3] \iff a_3 < x \leq b_3$



(BTW, consider becoming a member of the channel!) Thanks!

Bounded refers to the fact that these intervals do not extend infinitely in one or both directions. Counterexamples (so examples of **un**bounded intervals) would be the following:

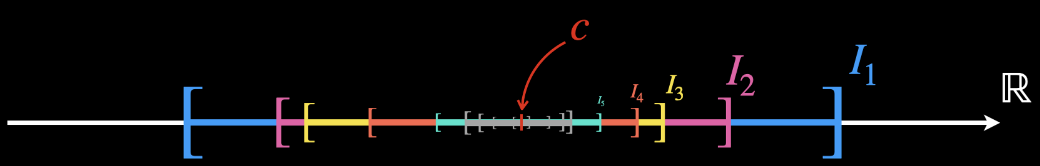


What is the implication of all of it? According to Cantor's theorem, this implies that it does not matter how many of these set inclusions there are, at the end of the day there will always exist at least one real, called c, belonging to all the intervals I_n.

Cantor's Theorem: Let $\{I_n\}_{n\in\mathbb{N}}$ be a set with elements that form a sequence of closed and bounded intervals on $\mathbb{R}$, which are decreasing by set inclusion (nested)

$$\Downarrow$$

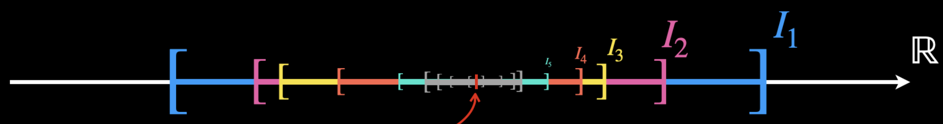
$$\boxed{\exists c \in I_n,\ \forall n \in \mathbb{N}}$$



In other words, the infinite intersection of all these intervals is non-empty. The fact that there exists such a real number, c, in this infinite intersection, is what we want to prove here.

decreasing by set inclusion (nested) :

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \dots$$



(implication)

$$\Longrightarrow \quad \exists c \in I_n, \; \forall n \in \mathbb{N} \quad \textit{i.e.} \quad \bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

$$I_1 \cap I_2 \cap I_3 \cap I_4 \cap \dots$$

$$\boxed{\exists c \in \bigcap_{n=1}^{\infty} I_n}$$

(that's what we want to prove!)

First of all, we need to prove that all the elements of the set that we will define as A are less than or equal to all the elements of the set that we will define as B. Remember, I_n are intervals from a_n to b_n. We must show that a_n is less than or equal to b_k, for all natural numbers n and k.

$$A := \{a_n : \; n \in \mathbb{N}\} \qquad B := \{b_n : \; n \in \mathbb{N}\}$$

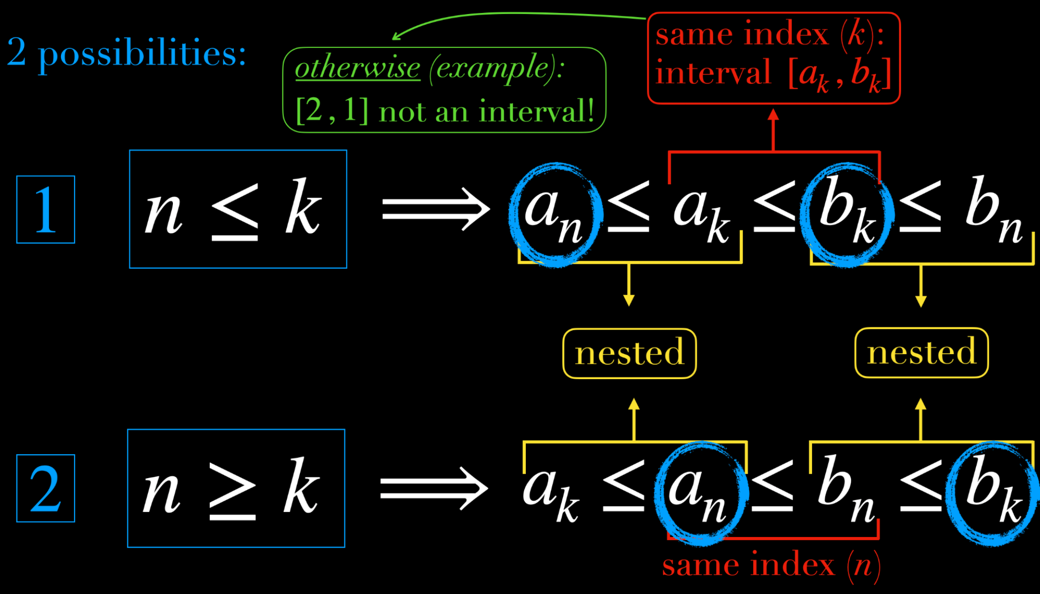
$$I_n = [a_n, b_n]$$

$$a_n \leq b_k \qquad \forall n, k \in \mathbb{N}$$

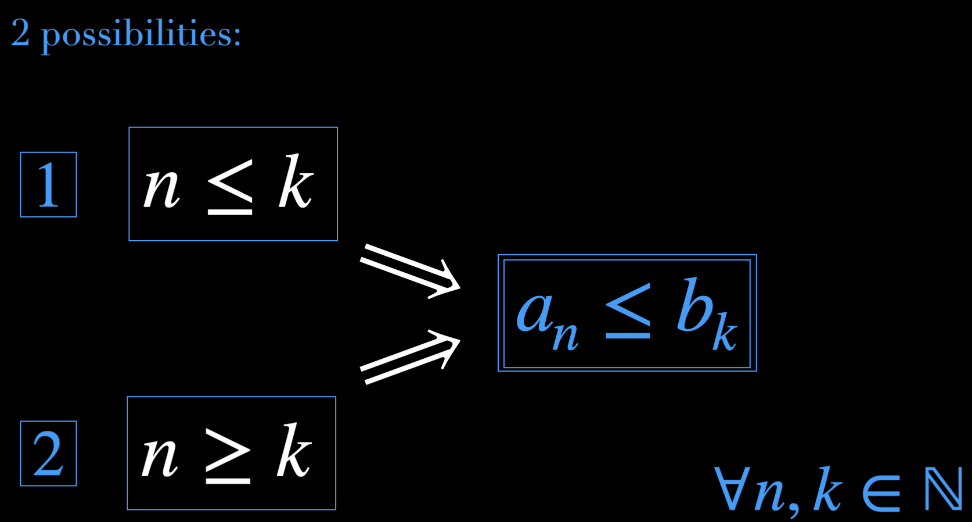
example 1: $n = 3; \; k = 7 \implies a_3 \leq b_7$

example 2: $n = 14; \; k = 2 \implies a_{14} \leq b_2$

In order to prove this fact, we notice that there are two possibilities here: either the index n is less than or equal to the index k, or n is greater than or equal to k. In the first case, a_n is less than or equal to a_k, and this is so because the intervals are nested. We also have that a_k is less than or equal to b_k, and this is so because they have the same index (k), and therefore they make up the same interval I_k. Otherwise, they would not form an interval! *[2,1]* is not an interval! Beyond that, we get that b_k is less than or equal to b_n, and this is so for the same reason as before, namely the intervals are nested. In the second case now, the index n is greater than or equal to the index k. We have that a_k is less than or equal to a_n (by set inclusion), and that a_n is less than or equal to b_n, because they have the same index (n), and therefore must form an interval. And finally, b_n is less than or equal to b_k, once again because the intervals are nested.

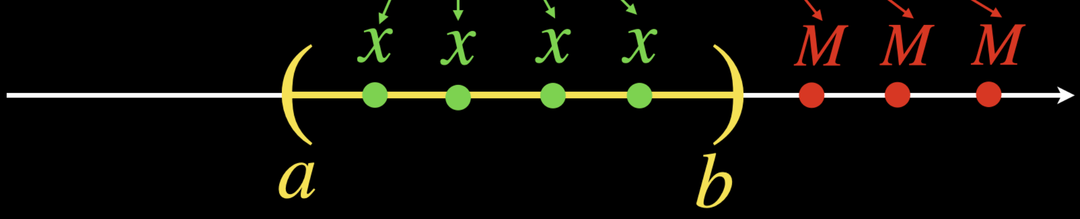


In any case, though, we conclude the same thing: a_n is less than or equal to b_k, no matter what combinations of natural numbers n and k we choose.



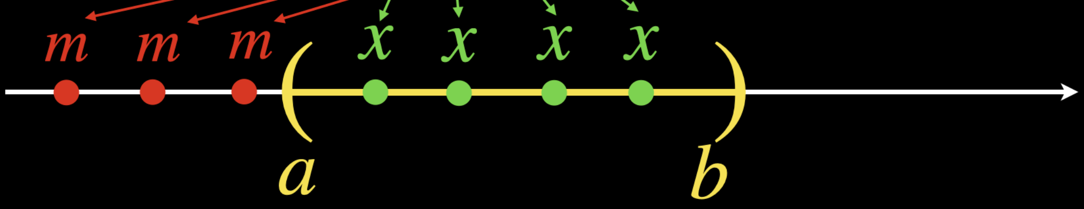
Before moving on, let us see some other important definitions:

(I) Upper bound: $\forall x \in I \,,\; x \leq M \implies$ M is an upper bound of I



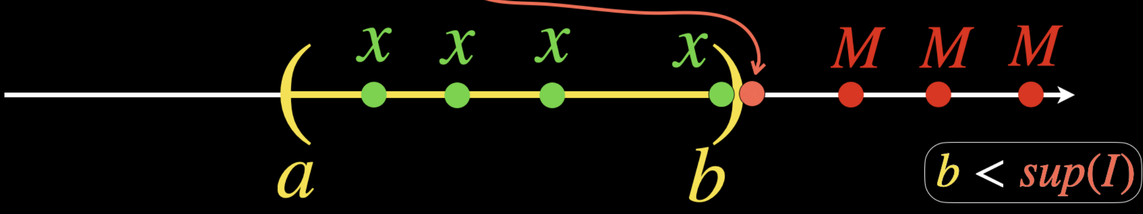
Notice: The interval I can have many upper bounds!

(II) Lower bound: $\forall x \in I \,,\; x \geq m \implies$ m is a lower bound of I



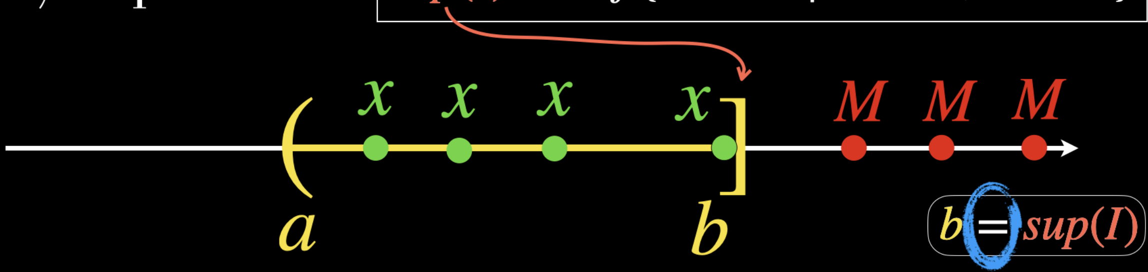
Notice: The interval I can have many lower bounds!

(III) Supremum: $sup(I) := inf\,\{M \in \mathbb{R} \mid \forall x \in I,\, x \leq M\}$

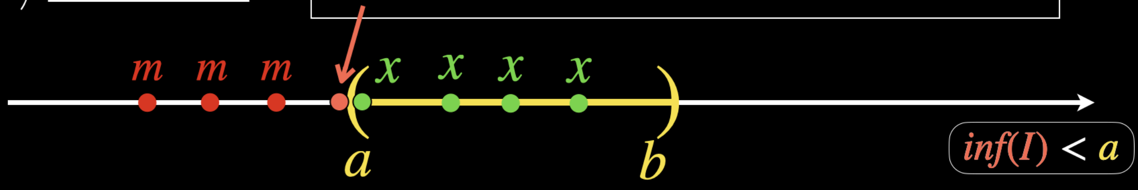


When the interval includes the endpoint b:

(III) Supremum: $sup(I) := inf\,\{M \in \mathbb{R} \mid \forall x \in I\,,\ x \leq M\}$

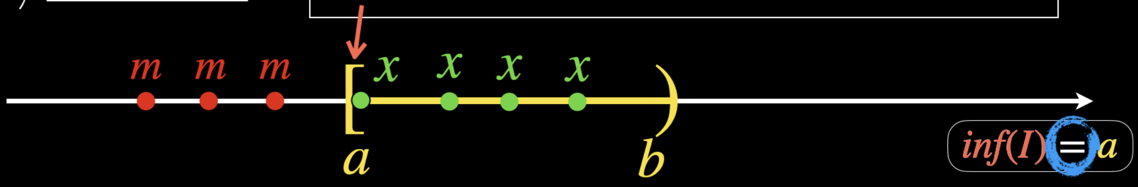


(IV) Infimum: $inf(I) := sup\,\{m \in \mathbb{R} \mid \forall x \in I\,,\ x \geq m\}$



When the interval includes the endpoint a:

(IV) Infimum: $inf(I) := sup\,\{m \in \mathbb{R} \mid \forall x \in I\,,\ x \geq m\}$



In conclusion:

(I) Upper bound: $\forall x \in I\ ,\ x \leq M \implies M$ is an upper bound of I

(II) Lower bound: $\forall x \in I\ ,\ x \geq m \implies m$ is a lower bound of I

(III) Supremum: $sup(I) := inf\,\{M \in \mathbb{R} \mid \forall x \in I\,,\ x \leq M\}$

(IV) Infimum: $inf(I) := sup\,\{m \in \mathbb{R} \mid \forall x \in I\,,\ x \geq m\}$

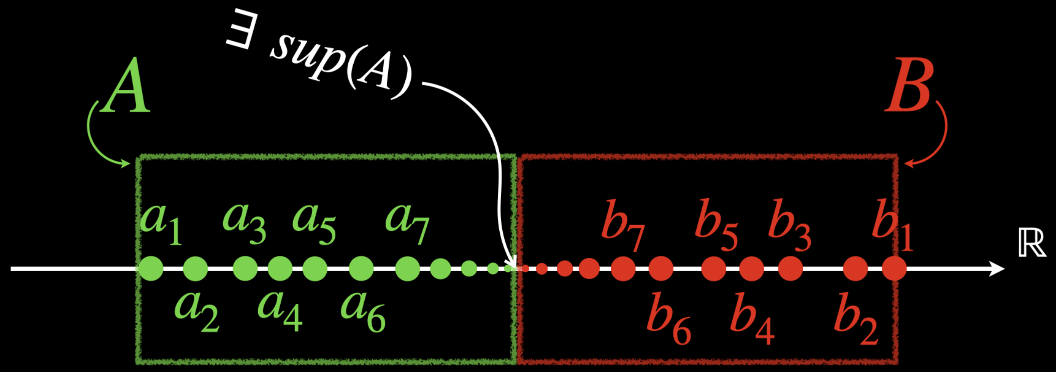
Notice: The interval I has only one supremum and only one infimum!

Now that we have these 4 definitions in place, let us continue the proof of Cantor's theorem: Since a_n is less than or equal to b_k , for all natural numbers *n* and *k*, then (by definition) all b_k's are upper bounds of the set *A*. And the minimum b_k (for some natural number *k*) is the supremum of *A*. So the supremum of *A* exists.

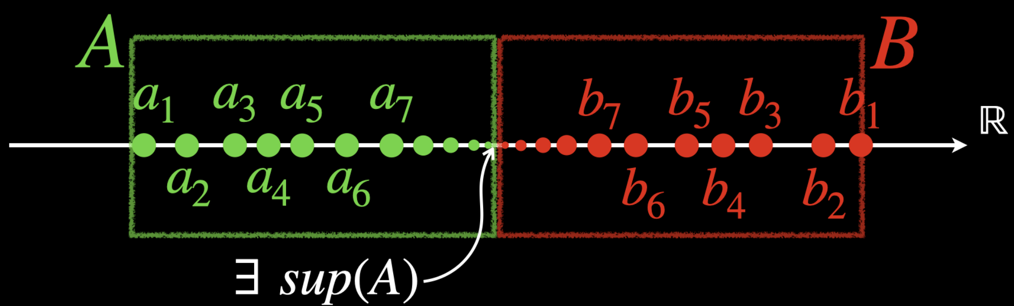
$$\boxed{a_n \le b_k} \quad \forall n, k \in \mathbb{N}$$

b_k are upper bounds of the set:

$$A = \{a_n : n \in \mathbb{N}\}$$



Thus, the supremum of *A* is less than or equal to b_k , for all natural numbers *k*, which implies that for all b_k's in the set *B*, b_k is less than or equal to the supremum of *A*. And therefore, the supremum of *A* is a lower bound of *B*.

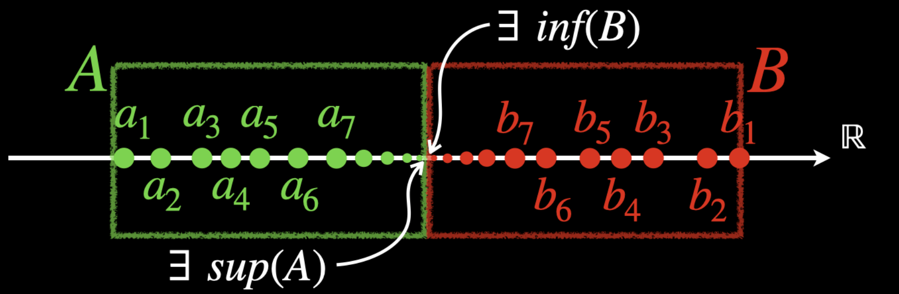


Thus, $\boxed{sup(A) \le b_k}$, $\forall k \in \mathbb{N} \implies \forall b_k \in B$, $b_k \ge sup(A) \implies$

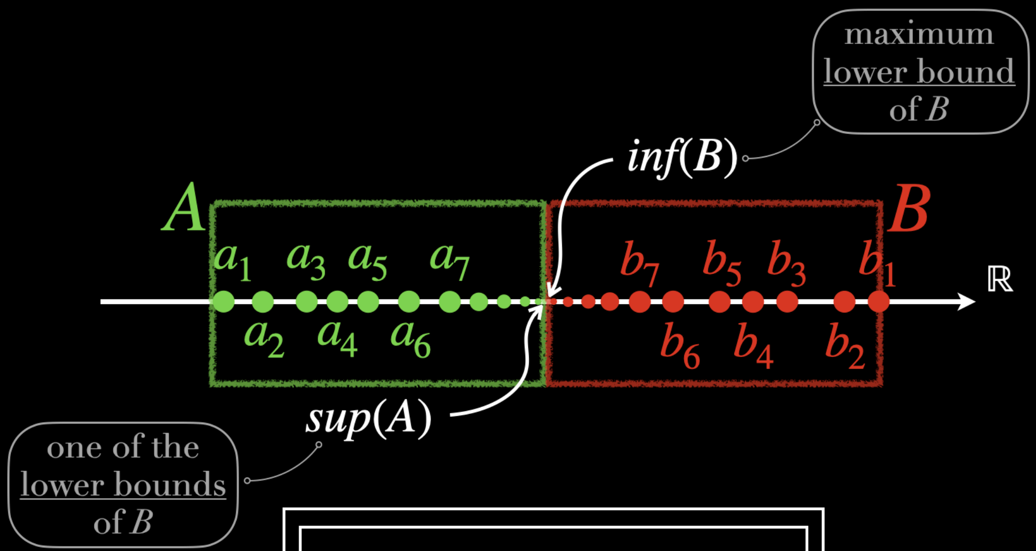
$\implies sup(A)$ is a lower bound of *B* *sup*(*A*) is a lower bound of *B*.

Remark:

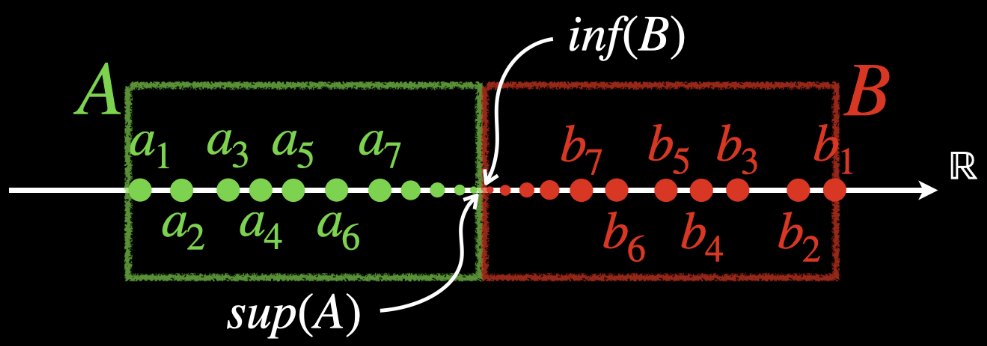
B has other lower bounds *as well, other than sup(A), but if we pick the maximum one out of all of these lower bounds, we get its* infimum:



So the infimum of *B* exists. Since the infimum of *B* is the maximum out of all the lower bounds of *B*, and since the supremum of *A* is one of the lower bounds of *B*, then the infimum of *B* is greater than or equal to the supremum of *A*. Using again the definitions of upper and lower bounds, we notice that b_n is greater than or equal to the infimum of *B* (since the infimum of *B* is a lower bound of *B*), and that the supremum of *A* is greater than or equal to a_n (since the supremum of *A* is an upper bound of *A*).



$$inf(B) \geq sup(A)$$



$$b_n \geq \mathit{inf}(B) \geq \mathit{sup}(A) \geq a_n$$

inf(B) is a <u>lower bound</u> of *B*

sup(A) is an <u>upper bound</u> of A

$$b_n \geq \mathit{inf}(B) \geq \mathit{sup}(A) \geq a_n \implies$$

$$\implies \bigcap_{n=1}^{\infty} I_n \supseteq [\mathit{sup}(A), \mathit{inf}(B)] \neq \varnothing \implies$$

in the "worst case scenario": $\mathit{inf}(B) = \mathit{sup}(A)$

$$\implies \bigcap_{n=1}^{\infty} I_n \neq \varnothing$$

$$\therefore \exists c \in I_n, \ \forall n \in \mathbb{N}$$

Q.E.D.

Cantor's theorem is very important in all fields of mathematics because it fundamentally reshaped our understanding of the infinity and the structures of the real number line. This theorem proves that the set of real numbers is uncountable and strictly larger in cardinality than the set of natural numbers, and thus it shows that not all infinities are equal. They have different sizes. This discovery opened many doors. Some of them were the developments of set theory, topology and modern analysis.

Comment: Please, if you find this document useful, let us know. Or if you found typos and things to improve, let us know as well. Your feedback is very important to us, since we are working hard every single day to deliver the best material possible. For contacting us: dibeos.contact@gmail.com