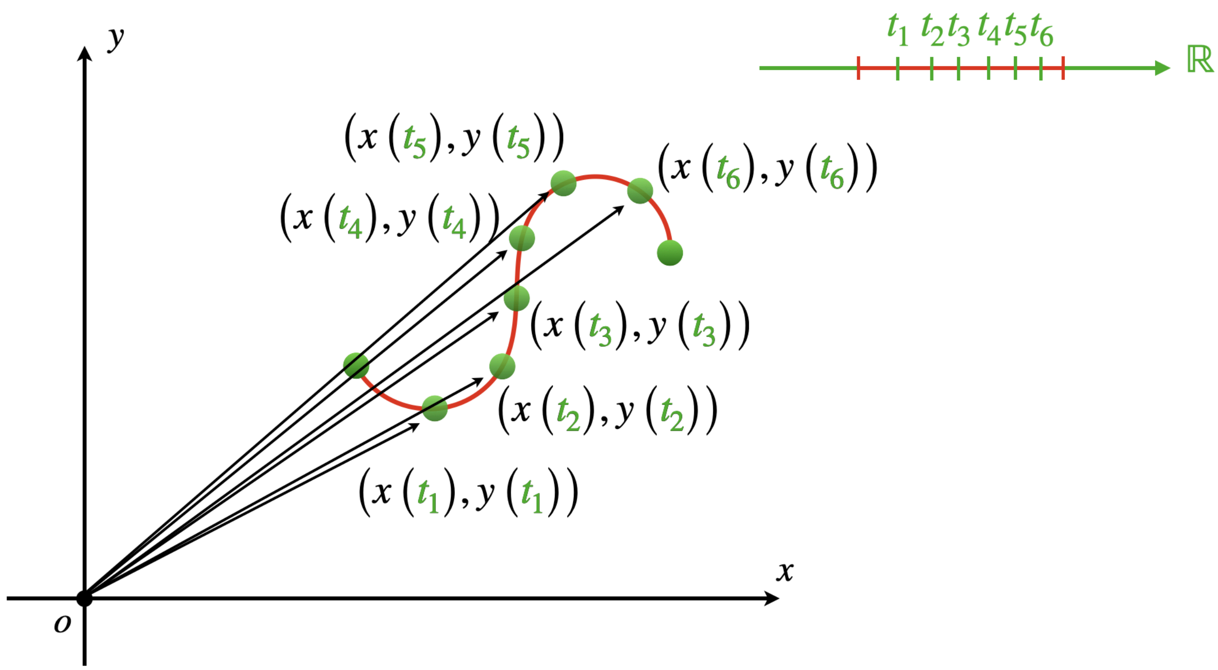
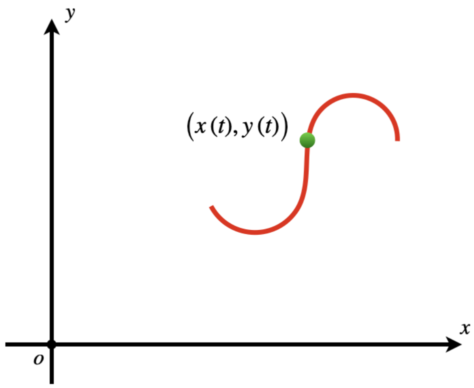
Imagine we have a point. And it moves around in the space where it lives. We can trace its path by simply painting where it has been. This will create a curve. Since in this case the curve lives on the plane of your screen, we can assign a coordinate system, with the origin wherever we want, and then parameterize the curve. This parameter t plays a very similar role to time (but not necessarily). It is useful though to think about it as the input, and the vector given by the coordinates of the point as the output.



So, for every input of time, there is an output of position. This will allow us to not only draw the curve formed by the path, but also to do Calculus on it. We could calculate limits, derivatives, integrals, and so on.

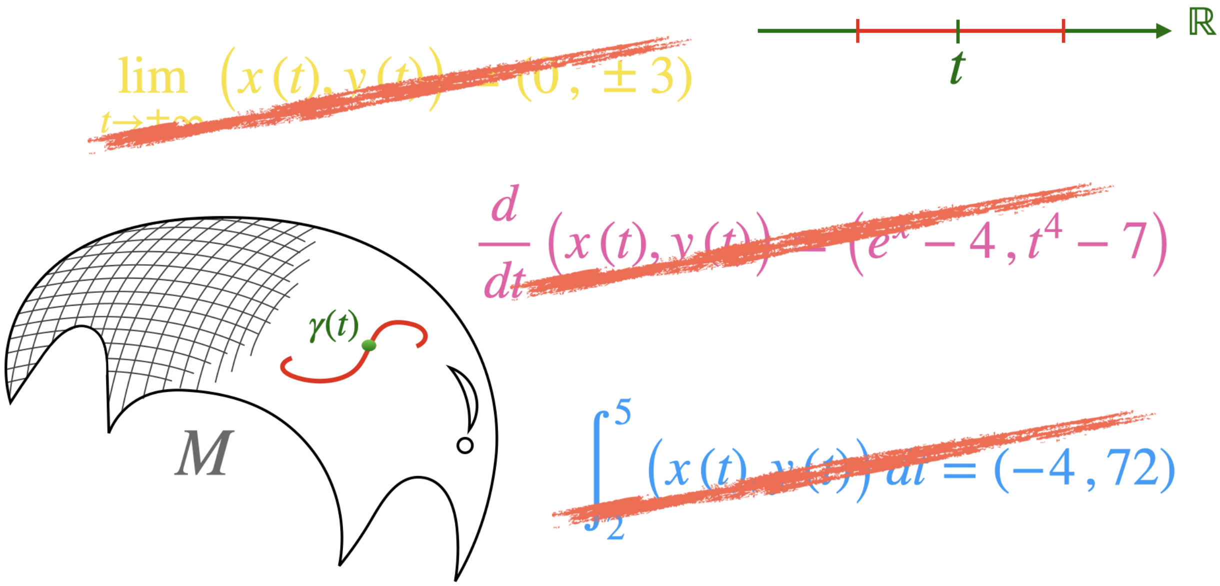
$$\lim_{t \to \pm\infty} \big(x(t), y(t)\big) = (0\,,\,\pm 3)$$

$$\frac{d}{dt}\big(x(t), y(t)\big) = \big(e^{x} - 4\,, t^{4} - 7\big)$$

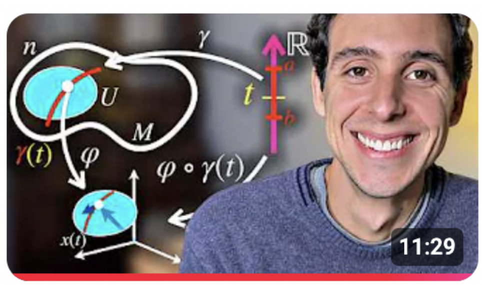


$$\int_{2}^{5} \big(x(t), y(t)\big)\, dt = (-4\,, 72)$$

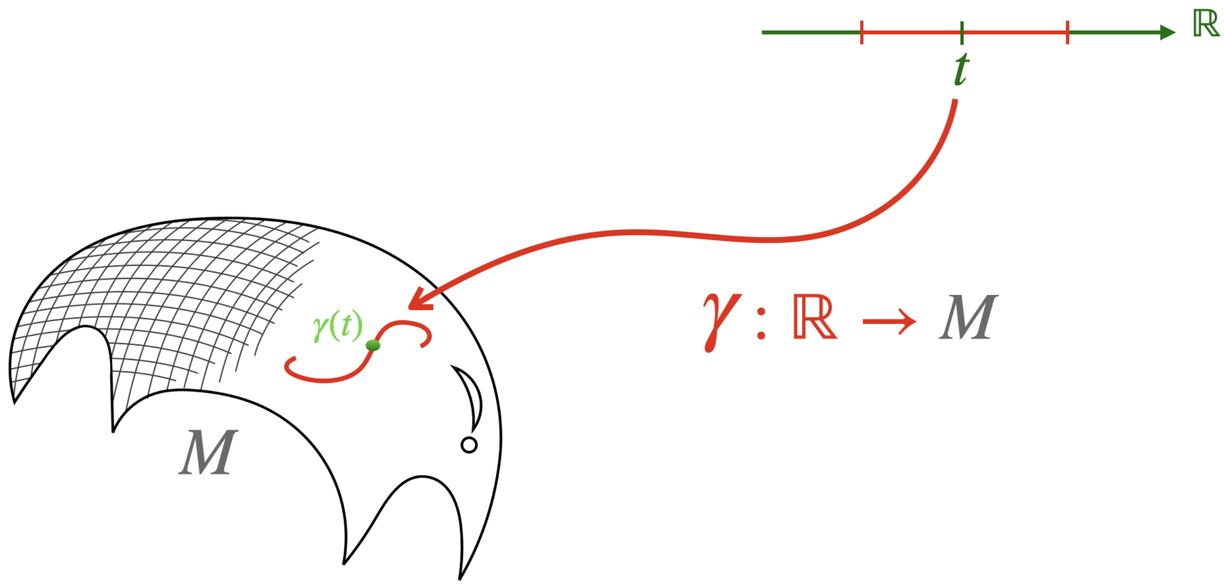
All of it is possible only and solely because this curve, which we will call γ, takes inputs in ℝ and returns outputs in $\mathbb{R}^2$. In other words, γ is a mapping from the Euclidean (flat) space ℝ (which is the real line) to the Euclidean (flat) space (or surface) $\mathbb{R}^2$. But what if we tell you that actually… actually… all of it is happening on a curved surface, and that we just thought it was flat because we were too close to perceive the curvature of the space around us. Well, if this is true then our situation really sucks because all of the Calculus stuff we just did (namely: limits, derivatives, integrals, etc) are just wrong.



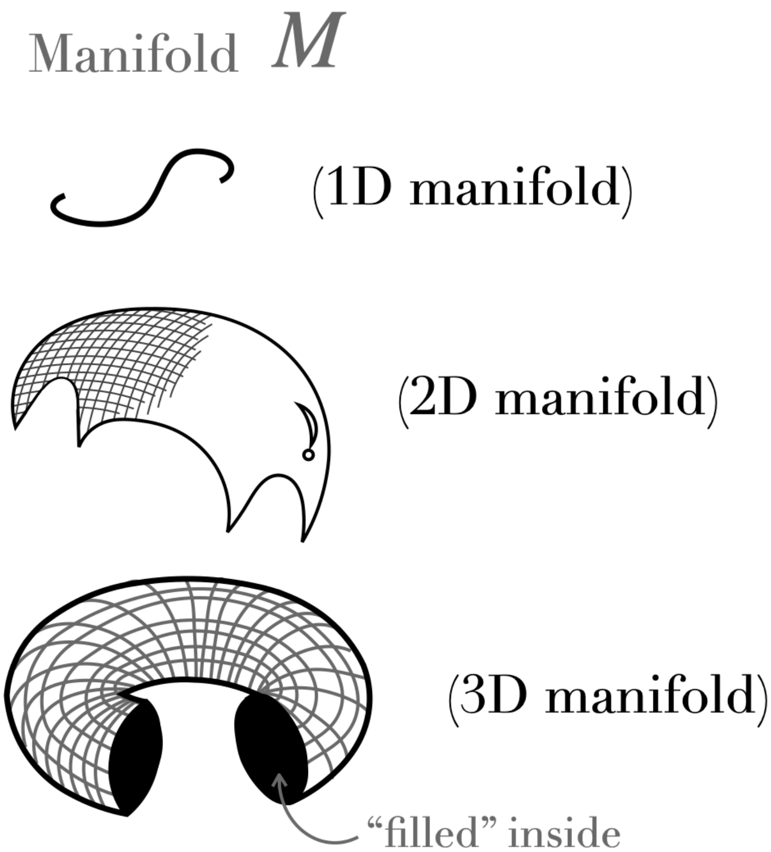
We cannot do Calculus anymore because now we found out that actually γ is a mapping from the Euclidean (flat) space $\mathbb{R}$ to the non-Euclidean (so, not flat) curved space M. We usually call this space M because it is a manifold. If you want to know in detail (and in a very intuitive way) what a manifold is, watch the related video on our channel.



How to do Calculus on an Abstract Manifold

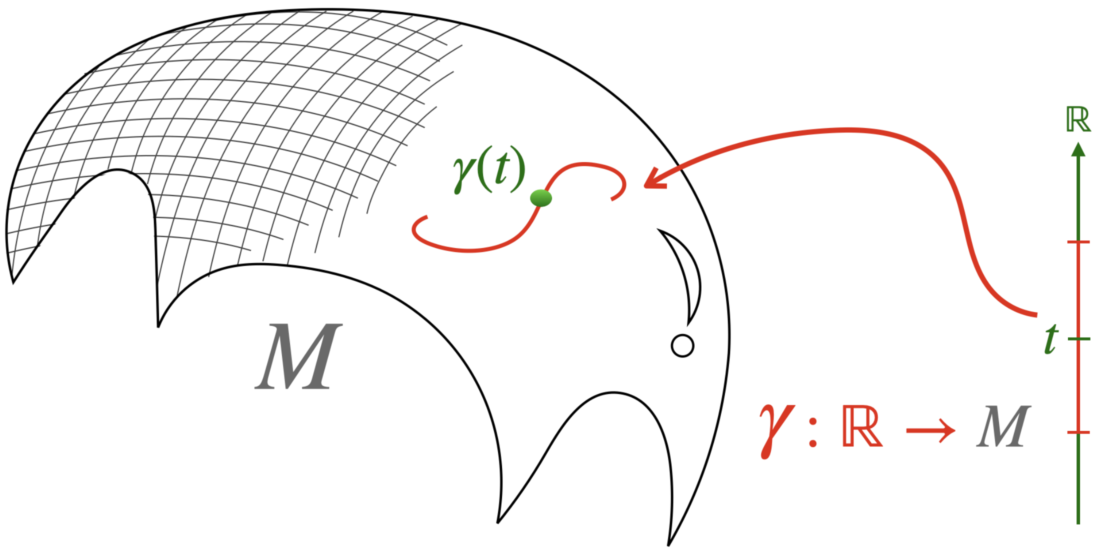


Just to say it quickly and not very rigorously, a manifold is a space (in 1, 2, 3 or even N dimensions) that can have, or not have, curvature.



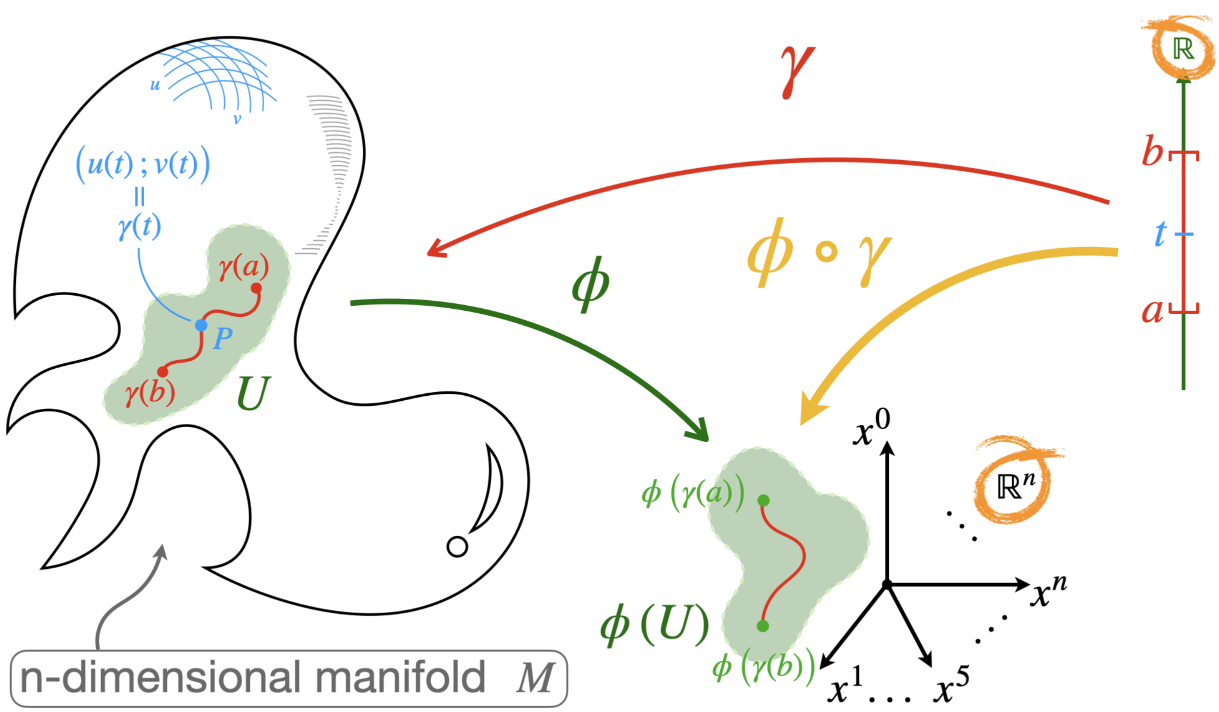
Most importantly, it is a space that looks like a Euclidean (flat) space locally. A very good example is the Earth, and the fact that we cannot perceive the curvature of the Earth just by standing on it. Locally, it looks flat. So we say that the Earth is locally a Euclidean space, or simply a manifold. Don’t ask a flat earther though. They don’t know the difference between a Euclidean space and a manifold.

Anyway, now we have a very peculiar situation. The curve γ, that describes the path we are interested in, lives in the manifold M. And what is really tricky is to visually understand the fact that M does not live inside of any other space. Of course, it is really hard to represent it on screen, but all you need here is to convince yourself of the fact that M is a space of its own, such that it does not need another "larger" space, where to live in, in order to exist. In mathematical terms, we say that M is not embedded in a higher dimensional space.



M is not embedded in a higher dimensional space.

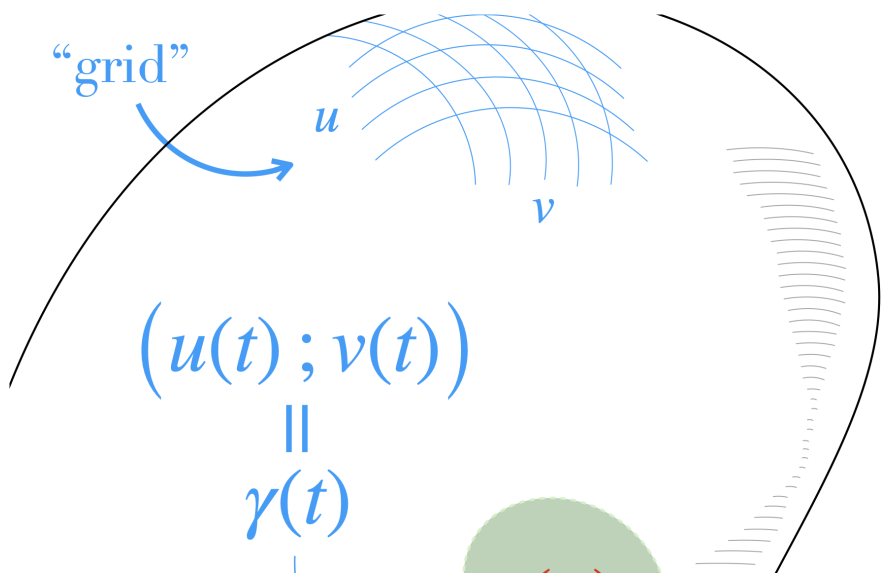
So, how do we fix the problem? I mean, we still want to be able to do Calculus on this manifold! And that's where *Differential Geometry* comes in. This subject gives us all the necessary tools to achieve our goal. So, now that we understand the situation from the intuitive point of view, and that we clearly stated the problem we are trying to solve, let's level up the discussion.



Picture an abstract manifold M, with dimension n, not embedded in a higher dimensional space, like $\mathbb{R}^n$. Then, we pick a point P in it, and a small neighborhood U around the point P. We can define a parameterized curve passing by P, called $\gamma(t)$ that goes from an interval $[a, b]$ in $\mathbb{R}$ to M. Its image in M is the interval $[\gamma(a)\ ,\ \gamma(b)]$. Let's also impose that the entire curve is inside the neighborhood U.

Next, we define a mapping $\varphi: M \supset U \to \mathbb{R}^n$, with image $\varphi(U) \subset \mathbb{R}^n$. n is the dimension of the manifold M as well as the dimension of the Euclidean space $\mathbb{R}^n$. Since we can create the composite mapping $\varphi \circ \gamma: \mathbb{R} \supset [a, b] \to \mathbb{R}^n$, we can also do Calculus here, because this is a mapping from a 1-dimensional Euclidean space to an n-dimensional Euclidean space. But what do we mean by "do Calculus" ?

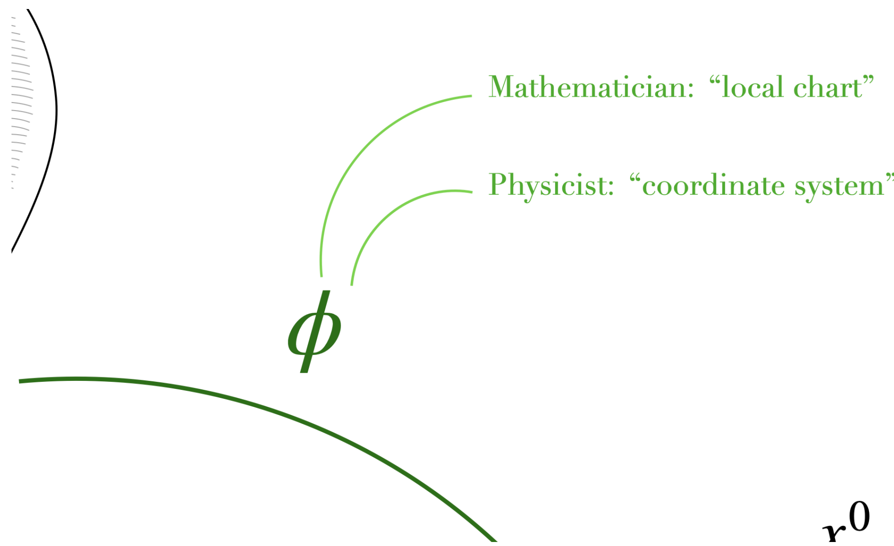
Well, we will see it shortly. First though, some important remarks before moving on:



1) The manifold M cannot be described using the traditional $(x, y, z,\ ...)$ Euclidean coordinates, because it is not necessarily embedded in $\mathbb{R}^n$. So, we

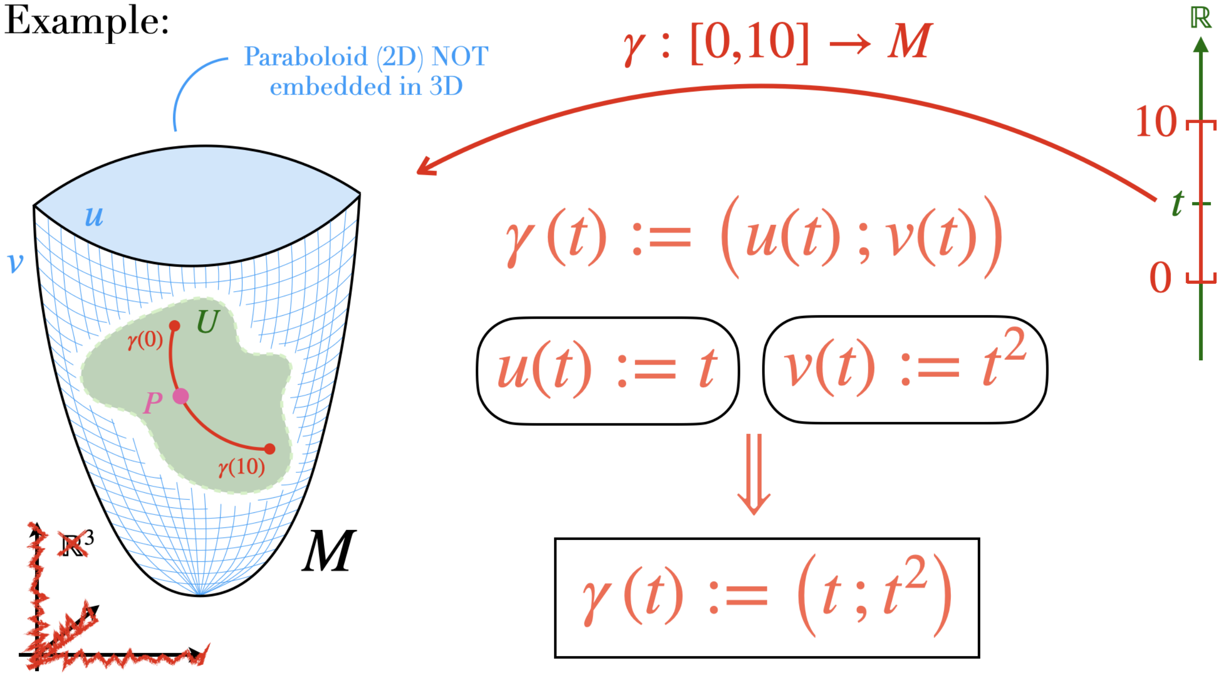
need to define instead intrinsic coordinates (u, v) , which will serve as a sort of "grid" used to measure distances in M. Actually, in order to measure distances, a metric must be defined. And, by the way, when applying Differential Geometry to General Relativity, this metric is usually referred to as g , and it describes *gravity* on the manifold.

2) Another important remark is that the mapping $\varphi: M \supset U \to \mathbb{R}^n$ is called the local chart by mathematicians, and coordinate system by physicists.



Let's see a concrete example now:

Example:



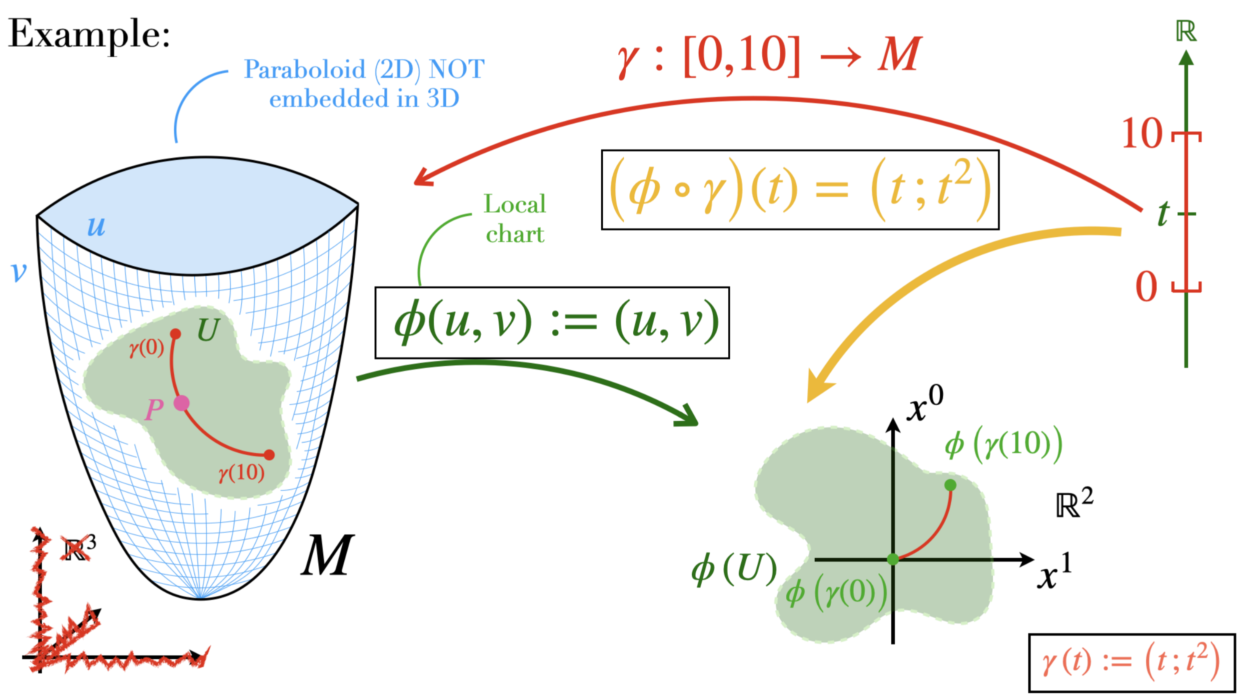
We have an abstract $2D$ manifold M , which is paraboloid-shaped, but it's <u>not</u> embedded in $\mathbb{R}^3$. Pick a point $P \in M$ and let U be a small neighborhood of P. This neighborhood U allows us to work locally on M. We cannot use the traditional (x, y, z) coordinates of Euclidean space. Instead, we will describe it using intrinsic coordinates (u, v).

Define $\gamma: \mathbb{R} \supset [0, 10] \rightarrow M$ as a parameterized curve on M. The domain $[0, 10] \subset \mathbb{R}$ serves as the interval for the parameter t , and the image of γ is the curve on M. Let the curve $\gamma(t) := (u(t)\ ;\ v(t))$ represent the path in terms of intrinsic coordinates (u, v) on M, where:

$$\boxed{u(t) := t} \qquad \wedge \qquad \boxed{v(t) := t^2}$$

The intrinsic coordinates act like a "grid" on M. In our case, we picked our "grid" to represent directions and distances on M, similar to polar, or geodesic, coordinates that vary smoothly as we move across the surface.

Let’s talk about the local chart φ now:



Define a mapping $\varphi\colon U \rightarrow \mathbb{R}^2$ which flattens the neighborhood U in M into $\mathbb{R}^2$:

$$\varphi(u, v) := (u, v)$$

This is a very simple mapping that takes the intrinsic coordinates (u, v) and directly maps them to $\mathbb{R}^2$.

Now, we compose φ *and* γ :

$$\varphi \circ \gamma\colon \mathbb{R} \supset [0, 10] \rightarrow \mathbb{R}^2$$

$$(\varphi \circ \gamma)(t) = \varphi(\,u(t)\,;\,v(t)\,) = \left(t, t^2\right)$$

, which is a parabola in $\mathbb{R}^2$.

This is a composite mapping from a 1-dimensional Euclidean space ($\mathbb{R}$) to a $2D$ Euclidean space $\left(\mathbb{R}^2\right)$, and therefore we can do Calculus with it:

$$(\phi \circ \gamma)(t) = (t\,;t^2) \qquad \phi(u,v) := (u,v) \qquad \gamma(t) := (t\,;t^2)$$

(1) Differentiation:

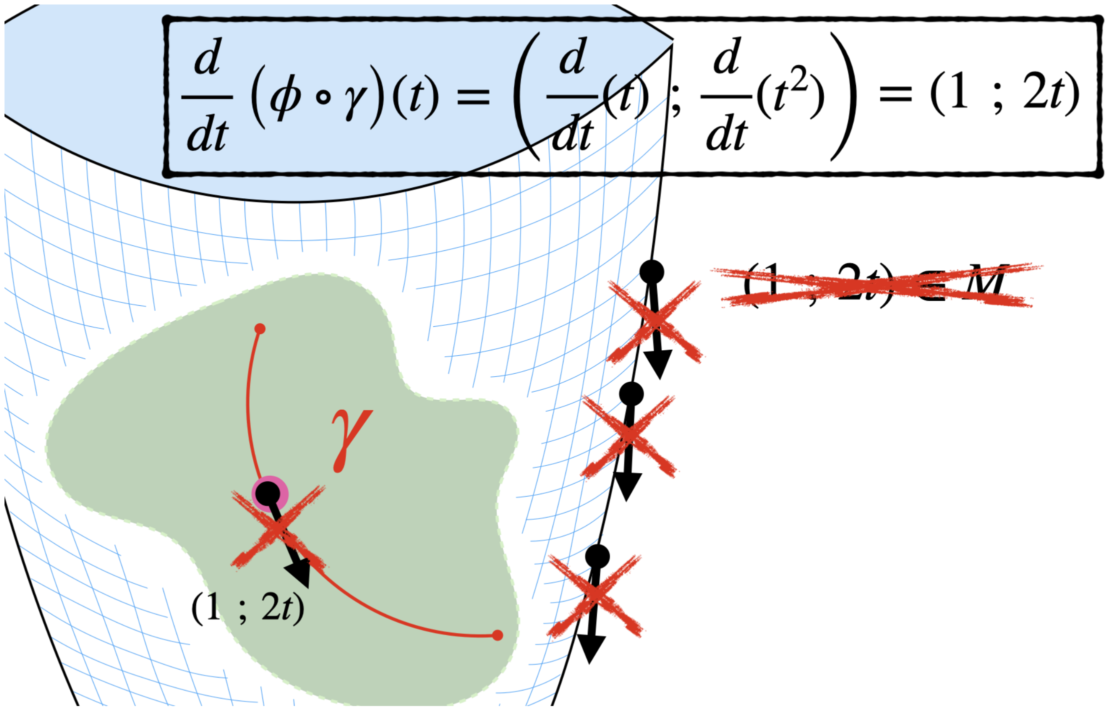
tangent vector from $t = 0$ to $t = 10$

$$\frac{d}{dt}(\phi \circ \gamma)(t) = \left(\frac{d}{dt}(t)\ ;\ \frac{d}{dt}(t^2)\right) = (1\ ;\ 2t)$$

(1) Differentiation: the derivative of φ composed with γ is the derivative of its coordinates, which results in the vector (1 ; 2t).

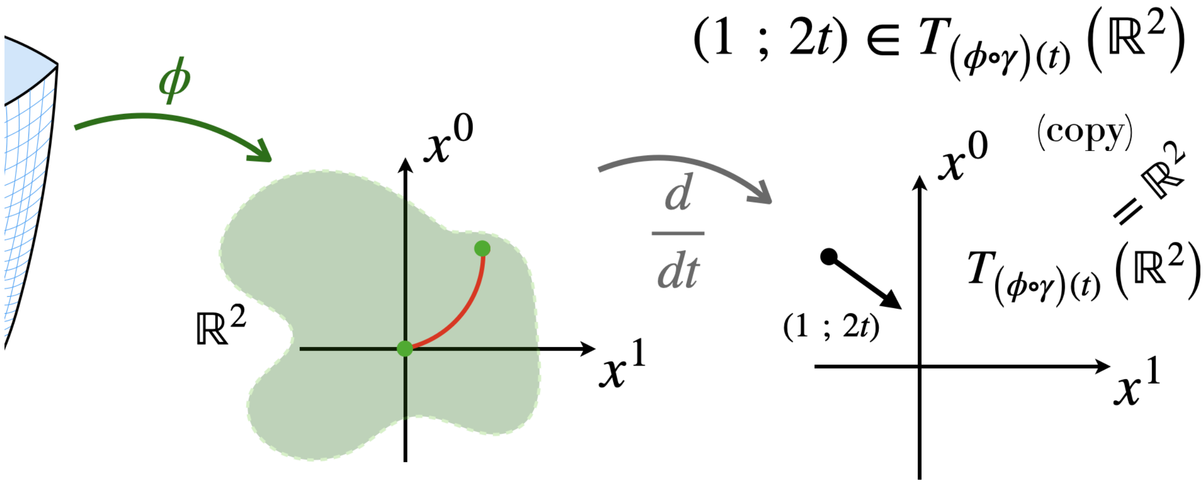
$$\frac{d}{dt}(\varphi \circ \gamma)(t) = \left(\frac{d}{dt}(t)\,;\frac{d}{dt}(t^2)\right) = (1, 2t)$$

, this is the tangent vector from $t = 0$ to $t = 10$. If t represents time, then this vector is the linear (or tangent) velocity at each point of γ . Now, an important question is: "where does the vector $(1, 2t)$ live?". One could say that this vector lives in the manifold M, but it doesn't make sense even from a visual point of view, because the manifold is curved, and a vector is "straight" (of course, assuming that we are using the simplistic view of vector as an arrow), so the tangent vector would stick out of M. But this representation is nonsense as well, because we said that the manifold is not embedded into another higher dimensional space, so it doesn't make sense to "draw" anything outside of the manifold M. There is no outside world, with respect to the manifold.



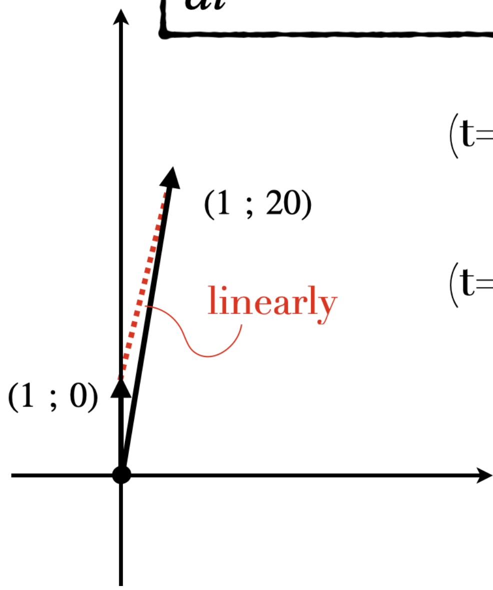
At this point, one could say that the tangent vector $(1, 2t)$ lives in the space $\mathbb{R}^2$ where φ lands. But this is also not completely true. The tangent vector actually lives in the tangent space $T_{(\varphi\circ\gamma)(t)}\left(\mathbb{R}^2\right)$ of the point $(\varphi \circ \gamma)(t)$, which is actually another (distinct) copy of $\mathbb{R}^2$.

$$\boxed{\frac{d}{dt}\left(\phi \circ \gamma\right)(t) = \left(\frac{d}{dt}(t) \ ; \ \frac{d}{dt}(t^2)\right) = (1 \ ; \ 2t)}$$



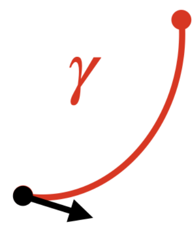
We notice that for $t = 0 : (1, 0)$ is the tangent vector (or “velocity vector” if you will) at the initial point $\gamma(0) \in M$. And for $t = 10 : (1, 20)$ is the tangent vector at the initial point $\gamma(10) \in M$. We can also see that this tangent vector increases linearly from the initial point to the final one along the curve γ.

$$\frac{d}{dt}(\phi \circ \gamma)(t) = \left(\frac{d}{dt}(t)\ ;\ \frac{d}{dt}(t^2)\right) = (1\ ;\ 2t)$$



(t=0) $$\frac{d}{dt}(\phi \circ \gamma)(0) = (1\ ;\ 0)$$

(t=10) $$\frac{d}{dt}(\phi \circ \gamma)(10) = (1\ ;\ 20)$$



(2) Magnitude of the Tangent Vector and Integration:

$$(\phi \circ \gamma)(t) = (t\,;t^2)$$ $$\phi(u,v) := (u,v)$$ $$\gamma(t) := (t\,;t^2)$$

(2) Magnitude of the Tangent Vector and Integration:

Euclidean length

$$\left|\frac{d}{dt}(\phi \circ \gamma)(t)\right| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

The magnitude (or Euclidean length) of the tangent vector is:

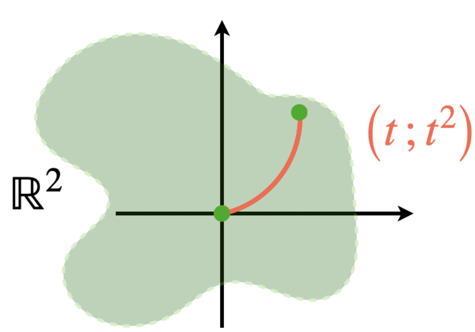
$$\left|\frac{d}{dt}(\varphi \circ \gamma)(t)\right| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

$$\left|\frac{d}{dt}(\phi \circ \gamma)(t)\right| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

$$\text{Arc length} = \int_0^{10} \sqrt{1 + 4t^2}\, dt$$

In order to find the arc length of the curve in $\mathbb{R}^2$, from $t = 0$ to $t = 10$, we integrate the magnitude of the tangent vector over this interval:

$$\text{Arc length} = \int_0^{10} \sqrt{1 + 4t^2}\, dt$$



This integral represents the length of the path (t, t^2) in $\mathbb{R}^2$ and can be evaluated as follows:

Let’s perform a change of variables, where

$$\boxed{t = \frac{1}{2} sinh(\theta)}$$

, and thus

$$\boxed{dt = \frac{1}{2} cosh(\theta) d\theta}$$

This lets us write the integral

$$\int_{0}^{10} \sqrt{1 + 4t^2}\, dt$$

this way

$$\int_{t=0}^{t=10} \sqrt{1 + 4\left(\frac{1}{2} sinh(\theta)\right)^2} \cdot \frac{1}{2} cosh(\theta)\, d\theta =$$

$$= \frac{1}{2} \int_{t=0}^{t=10} \sqrt{1 + sinh^2(\theta)} \cdot cosh(\theta)\, d\theta$$

At this point, we can use the following trigonometric inequality:

$$1 = cosh^2(\theta) - sinh^2(\theta)$$

So, we can write this integral as:

$$\frac{1}{2}\int_{t=0}^{t=10}\sqrt{cosh^2(\theta)} \cdot cosh(\theta)\, d\theta = \frac{1}{2}\int_{t=0}^{t=10} cosh^2(\theta)\, d\theta$$

The next trigonometric identity we will use is that:

$$cosh^2(\theta) = \frac{1+cosh(2\theta)}{2}$$

Using it in our current expression we get:

$$\frac{1}{2}\int_{t=0}^{t=10}\frac{d\theta}{2} + \frac{1}{2}\int_{t=0}^{t=10}\frac{cosh(2\theta)}{2}\, d\theta = \left(\frac{\theta}{4}\right)\Big|_{t=0}^{t=10} + \frac{1}{4}\int_{t=0}^{t=10} cosh(2\theta)\, d\theta$$

Going back to the definition $t = \frac{1}{2}\, sinh(\theta)$, we can write θ in terms of t :

$$\theta = sinh^{-1}(2t)$$

From it, we have this:

$$\left(\frac{sinh^{-1}(2t)}{4}\right)\Big|_{0}^{10} + \frac{1}{4}\cdot\left(\frac{1}{2}\, sinh(2\theta)\right)\Big|_{t=0}^{t=10}$$

In order to simplify the calculations, we will use the following trigonometric identity in the second term:

$$\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$$

We can evaluate the first term and perform this substitution in the second term:

$$\frac{\sinh^{-1}(20)}{4} - \frac{\sinh^{-1}(0)}{4} + \frac{1}{4}\cdot\left(\frac{2}{2}\sinh(\theta)\cosh(\theta)\right)\Big|_{t=0}^{t=10}$$

Next, we notice that $\sinh^{-1}(0) = 0$, we cancel out these 2’s in the second term, and we use once again the two facts that we have found, in other words, the expression of theta in terms of t, and vice versa:

$$\theta = \sinh^{-1}(2t) \quad \wedge \quad \cosh(\theta) = \sqrt{1 + 4t^2}$$

After, updating these results, we finally get:

$$\frac{\sinh^{-1}(20)}{4} + \frac{1}{4}\cdot\left(2t\sqrt{1 + 4t^2}\right)\Big|_{0}^{10} = \frac{\sinh^{-1}(20)}{4} + 5\sqrt{401} \simeq 101.05$$

And this is the approximate length of the path (t, t^2) in $\mathbb{R}^2$.

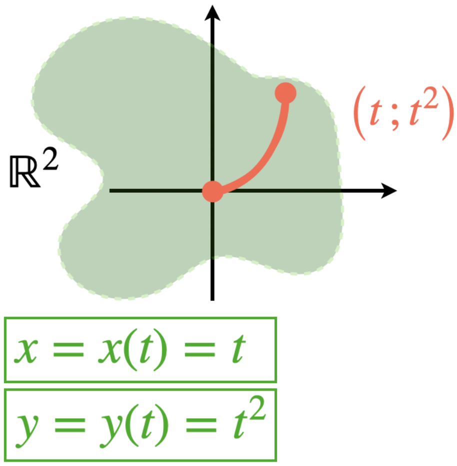
(3) Curvature:

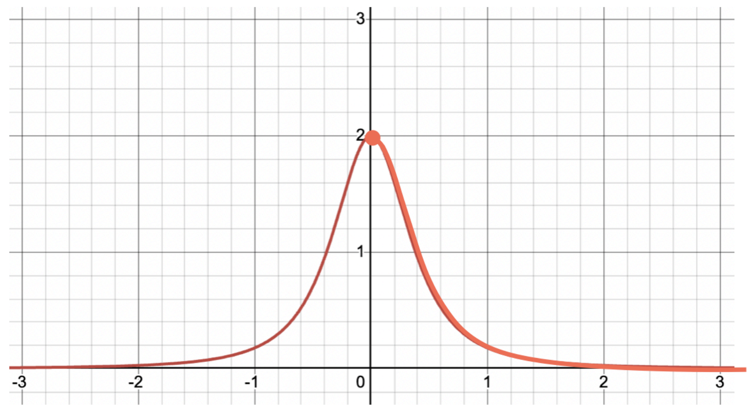
To find the curvature κ of the path (t, t^2) in $\mathbb{R}^2$, we use this formula:

$$\kappa = \frac{|x'y'' - y'x''|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

, where $x = t \quad \wedge \quad y = t^2$.

$$\kappa = \frac{\left|x'y'' - y'x''\right|}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} = \frac{\left|1 \cdot 2 - 2t \cdot 0\right|}{\left((1)^2 + (2t)^2\right)^{\frac{3}{2}}} = \frac{2}{\sqrt{\left(1 + 4t^2\right)^3}}$$

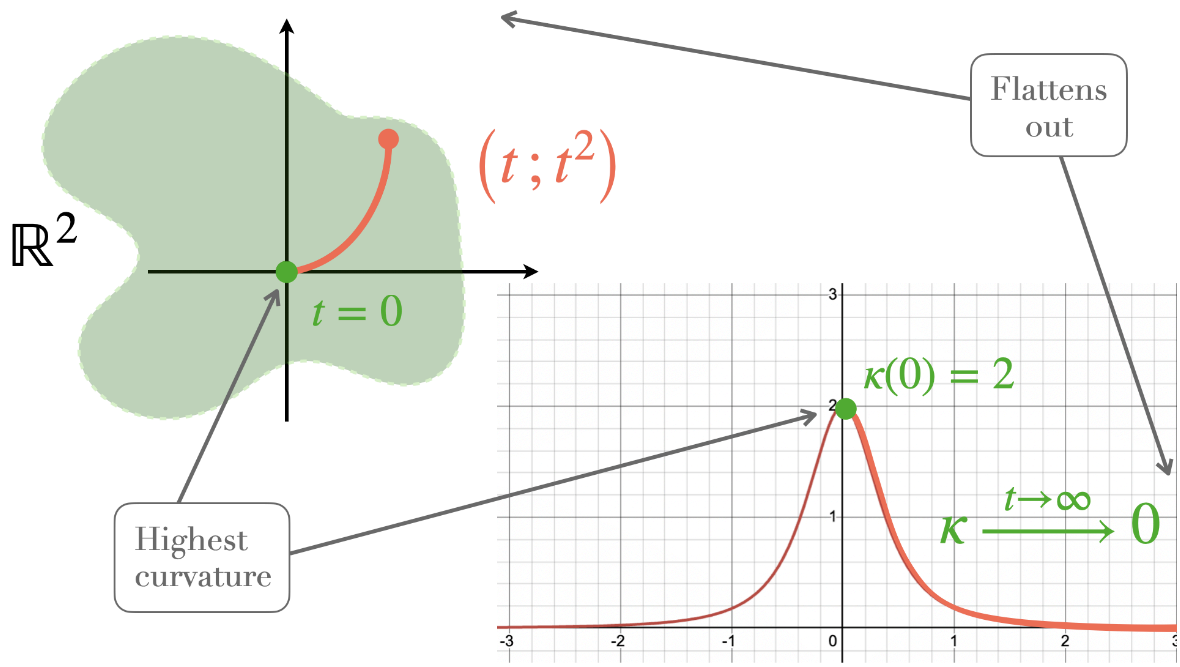




Let’s calculate it:

$$\kappa = \frac{|1\cdot2-2t\cdot0|}{\left((1)^2+(2t)^2\right)^{3/2}} = \frac{2}{\sqrt{\left(1+4t^2\right)^3}}$$

This curvature describes how sharply the curve $\left(t, t^2\right)$ bends at each point in $\mathbb{R}^2$:



⋆ $t \to 0$: $\kappa(0) = 2 \Rightarrow$ The curvature is highest at the start.

⋆ $t \to \infty$: $\kappa \to 0 \Rightarrow$ The curve flattens out as it moves farther from the origin.

Now, in order to show the power of Differential Geometry to deal with a great variety of situations, let's change the local chart φ to a more interesting one:

$$\varphi:\ M \ni U \to \mathbb{R}^2$$

$$\varphi(u, v) = (u\,cos(v)\,,\ u\,sin(v))$$

γ, as well as the intrinsic coordinates (u, v) , stay the same:

$$\gamma: \mathbb{R} \ni [0, 10] \to M$$

$$\gamma(t) := \left(t\,,\ t^2\right) \equiv (u(t)\,,\ v(t))$$

As a consequence:

$$(\varphi \circ \gamma)(t) = \left(t\,cos\left(t^2\right),\ t\,sin\left(t^2\right)\right)$$

This is a curve in $\mathbb{R}^2$ with spiraling behavior.

Now, let's do Calculus with it:

(1) Differentiation:

$$\frac{d}{dt}(\varphi \circ \gamma)(t) = \frac{d}{dt}\left(t\,cos\left(t^2\right); t\,sin\left(t^2\right)\right) =$$

$$= \left(cos\left(t^2\right) - 2t^2 sin\left(t^2\right); sin\left(t^2\right) + 2t^2 cos\left(t^2\right)\right)$$

This is the tangent vector, from $t = 0$ to $t = 10$, that lives in the tangent space $T_{(\varphi\circ\gamma)(t)}\left(\mathbb{R}^2\right) \equiv \mathbb{R}^2$.

⋆ $t = 0$: $\frac{d}{dt}(\varphi \circ \gamma)(0) = (1, 0)$

⋆ $t = 10$: $\frac{d}{dt}(\varphi \circ \gamma)(10) =$

$$= (cos(100) - 200\,sin(100); sin(100) + 200\,cos(100))$$

$$\simeq (-197.14, -33.74)$$

(2) Magnitude of the Tangent Vector and Integration:

$$\left|\frac{d}{dt}(\varphi \circ \gamma)(t)\right| =$$

$$= \sqrt{\left[cos\left(t^2\right) - 2t^2 sin\left(t^2\right)\right]^2 + \left[sin\left(t^2\right) + 2t^2 cos\left(t^2\right)\right]^2} =$$

$$= [cos^2\left(t^2\right) - 4t^2 sin\left(t^2\right) cos\left(t^2\right) + 4t^4 sin^2\left(t^2\right) + sin^2\left(t^2\right) +$$

$$+ 4t^2 sin\left(t^2\right) cos\left(t^2\right) + 4t^4 cos^2\left(t^2\right)]^{1/2} = \sqrt{1 + 4t^4}$$

Then, the arc length is just the integral of this result. But unfortunately it is really hard to solve analytically, so we will just give its numerical approximation:

$$\text{Arc length} = \int_0^{10} \sqrt{1 + 4t^4}\, dt \simeq 667.52 \text{ (numerically)}$$

(3) Curvature:

$$\kappa = \frac{|x'y'' - y'x''|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

, where:

$$x(t) = t\, cos\left(t^2\right) \Rightarrow x'(t) = cos\left(t^2\right) - 2t^2\, sin\left(t^2\right) \Rightarrow$$

$$\Rightarrow x''(t) = - 2t\, sin\left(t^2\right) - 4t\, sin\left(t^2\right) - 4t^3\, cos\left(t^2\right) \Rightarrow$$

$$\Rightarrow \quad x''(t) = -\, 6t\,sin\left(t^2\right) - \, 4t^3\,cos\left(t^2\right)$$

$$y(t) \,=\, t\,sin\left(t^2\right) \quad \Rightarrow \quad y'(t) \,=\, sin\left(t^2\right) + \, 2t^2\,cos\left(t^2\right) \quad \Rightarrow$$

$$\Rightarrow \quad y''(t) \,=\, 2t\,cos\left(t^2\right) + \, 4t\,cos\left(t^2\right) - \, 2t^2 \cdot 2t\,sin\left(t^2\right) \quad \Rightarrow$$

$$\Rightarrow \quad y''(t) \,=\, 6t\,cos\left(t^2\right) - \, 4t^3\,sin\left(t^2\right)$$

$$x'y'' = \left[cos\left(t^2\right) - \, 2t^2\,sin\left(t^2\right)\right] \cdot \left[6t\,cos\left(t^2\right) - \, 4t^3 sin\left(t^2\right)\right] =$$

$$= \, 6t\,cos^2\left(t^2\right) - \, 4t^3 sin\left(t^2\right)\,cos\left(t^2\right) - \, 12t^3 sin\left(t^2\right)\,cos\left(t^2\right) +$$

$$+ \, 8t^5\,sin^2\left(t^2\right) \quad \Rightarrow$$

$$\Rightarrow \quad x'y'' \,=\, 6t\,cos^2\left(t^2\right) - \, 16t^3 sin\left(t^2\right)\,cos\left(t^2\right) + \, 8t^5\,sin^2\left(t^2\right)$$

$$y'x'' = \left[sin\left(t^2\right) + \, 2t^2\,cos\left(t^2\right)\right] \cdot \left[-\, 6t\,sin\left(t^2\right) - \, 4t^3 cos\left(t^2\right)\right] =$$

$$= -\, 6t\,sin^2\left(t^2\right) - \, 4t^3 sin\left(t^2\right)\,cos\left(t^2\right) - \, 12t^3 sin\left(t^2\right)\,cos\left(t^2\right)$$

$$- \, 8t^5\,cos^2\left(t^2\right) \quad \Rightarrow$$

$$\Rightarrow \quad y'x'' = -\, 6t\, sin^2\left(t^2\right) - \, 16t^3 sin\left(t^2\right) cos\left(t^2\right) - \, 8t^5\, cos^2\left(t^2\right)$$

$$x'y'' \, - \, y'x'' \, = \, 6t\, cos^2\left(t^2\right) - \, 16t^3\, sin\left(t^2\right) cos\left(t^2\right) + \, 8t^5\, sin\left(t^2\right) +$$

$$+ \, 6t\, sin^2\left(t^2\right) + \, 16t^3 sin\left(t^2\right) cos\left(t^2\right) + \, 8t^5 cos^2\left(t^2\right) =$$

$$= \left(6t \, + \, 8t^5\right) cos^2\left(t^2\right) + \left(6t \, + \, 8t^5\right) sin^2\left(t^2\right) =$$

$$= \, 6t \, + \, 8t^5 \qquad \Rightarrow$$

$$\Rightarrow \qquad x'y'' \, - \, y'x'' \, = \, 2t \cdot \left(3 \, + \, 4t^4\right)$$

$$(x')^2 \, + \, (y')^2 = \left[cos\left(t^2\right) - \, 2t^2 sin\left(t^2\right)\right]^2 + \left[sin\left(t^2\right) + \, 2t^2 cos\left(t^2\right)\right]^2 =$$

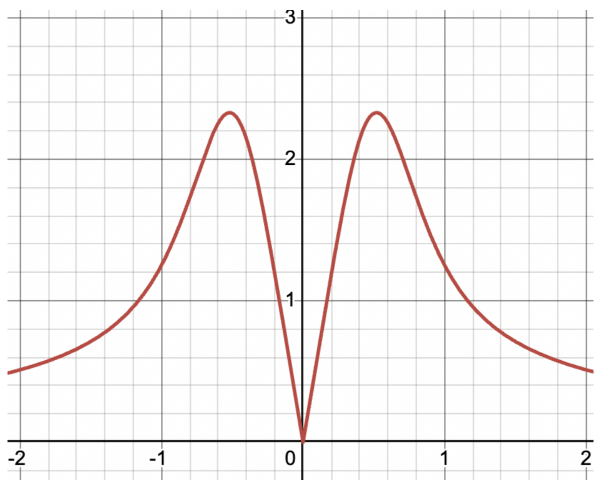
$$= \, cos^2\left(t^2\right) - \, 4t^2 sin\left(t^2\right) cos\left(t^2\right) + \, 4t^4 sin^2\left(t^2\right) + \, sin^2\left(t^2\right) +$$

$$+ \, 4t^2 sin\left(t^2\right) cos\left(t^2\right) + \, 4t^4 cos^2\left(t^2\right) =$$

$$= \left(1 \, + \, 4t^4\right) cos^2\left(t^2\right) + \left(1 \, + \, 4t^4\right) sin^2\left(t^2\right) \qquad \Rightarrow$$

$$\Rightarrow \qquad (x')^2 \, + \, (y')^2 \, = \, 1 \, + \, 4t^4$$

$$\therefore \quad \kappa = \frac{\left|2t \cdot \left(3+4t^4\right)\right|}{\sqrt{\left(1+4t^4\right)^3}}$$



$$\star \quad t = 0: \quad \kappa(0) = \frac{\left|2 \cdot 0 \cdot \left(3+4 \cdot 0^4\right)\right|}{\sqrt{\left(1+4 \cdot 0^4\right)^3}} = 0$$

$$\star \quad t = 10: \quad \kappa(10) = \frac{|2 \cdot 10 \cdot (3+4 \cdot 10000)|}{\sqrt{(1+4 \cdot 10000)^3}} = \frac{|20 \cdot (3+40000)|}{\sqrt{(40001)^3}}$$

$$= \frac{60+800000}{40001\sqrt{40001}} = \frac{800060}{40001\sqrt{40001}} \simeq \frac{800060}{800015000}$$

$$\simeq 0.001$$

Since the curvature along the curve is ≈ 0 from , $t = 0$ to $t = 10$, then this curve is a good approximation of a true geodesic. Do you see why? Let us know if you wish to know more about it: **dibeos.contact@gmail.com**

$$\star \quad t \to \infty: \quad \lim_{t \to \infty} \frac{\left|2t \cdot \left(3+4t^4\right)\right|}{\sqrt{\left(1+4t^4\right)^3}} = \lim_{t \to \infty} \frac{2t \cdot \left(3+4t^4\right)}{\sqrt{\left(1+4t^4\right)^3}} =$$

$$= \lim_{t \to \infty} \frac{2 \cdot \left(\frac{3}{t^4}+4\right)}{\sqrt{\frac{1}{t^{10}}\left(1+4t^4\right)^3}} = 2 \lim_{t \to \infty} \frac{\left(\frac{3}{t^4}+4\right)}{\sqrt{\left(\frac{1}{t^{10/3}}+4\frac{t^4}{t^{10/3}}\right)^3}} =$$

$$= 0$$

Hence, when $t \to \infty$, this curve behaves just as a geodesic. The geodesics are the equivalent to "straight lines" in curved manifolds.

$$\kappa = \frac{\left|2t \cdot \left(3+4t^4\right)\right|}{\sqrt{\left(1+4t^4\right)^3}}$$

So, again, this is the curvature of the path in $\mathbb{R}^2$, i.e. it lets us calculate how sharply the curve bends at each point. It is important to notice that this expression is also called the "geodesic curvature" in the case in which the manifold M has what is called a "Riemannian structure", i.e. a way to measure distances and angles intrinsically (like a metric).

We would like to finish with a quick discussion of what this problem could represent when applied to a real world scenario in mathematical physics. More precisely, in General Relativity.

A paraboloid, just like the manifold that we studied here, can serve as an intuitive, local, approximation of the curvature around a planet, for example. Of course, there are some obvious limitations to that, so let's talk about it.

In General Relativity, the presence of mass (like a planet) causes *spacetime (a 4-dimensional manifold)* to curve, which we describe mathematically using the metric tensor, that we did not discuss here, but let us know if you would like us to post a video dedicated to it.

The exact shape of spacetime curvature around a planet is actually more complex than a simple paraboloid. This type of curvature has a "well" shape, where the curvature is stronger closer to the planet and flattens out farther away. The geodesics on the paraboloid (so, the paths with zero curvature) approximate the behavior of free-falling particles in a gravitational field. A geodesic on the paraboloid would roughly correspond to the path of a particle moving along a straight line in curved spacetime, which shows how gravity affects motion.
Now, let's talk about some of the limitations of this model:

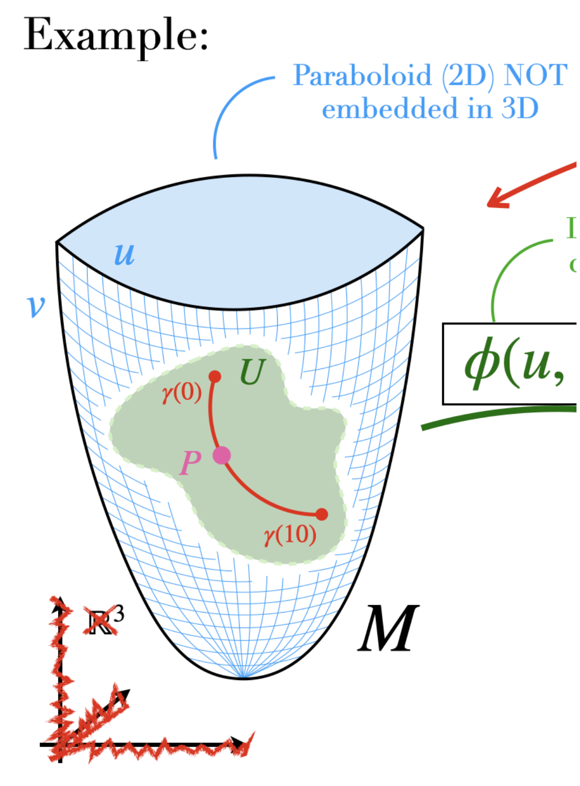
(1) First of all, the actual curvature of spacetime around a planet is $4D$, and it affects both space and time. Our paraboloid instead is a $2D$ manifold, where all the dimensions are supposed to represent just space.

(2) The exact mathematical description of spacetime curvature around a spherical mass like a planet is given by the Schwarzschild metric, which has a more complex curvature profile than a paraboloid. The Schwarschild solution describes how spacetime "warps" in all directions around the mass, not just in one, like, "bowl" shape.

(3) Around a planet, the gravitational field is radially symmetric and diminishes with the distance according to an inverse-square law. The paraboloid here, though, doesn't have any of these properties.

Intrinsic coordinates:

We now proceed by describing the intrinsic coordinates (u, v) in M avoiding referencing (x, y, z) , since M is not embedded in $\mathbb{R}^3$.



⋆ u represents the intrinsic *distance* from a reference point (which would be called vertex of the paraboloid if it were embedded in $\mathbb{R}^3$).

$$u := d\left(p_0 , p\right) \qquad ; \quad u \in [\,0, \infty\,)$$

$$d \,:\, \left\{p_0\right\} \times M \,\rightarrow\, [\,0\,, \infty\,)$$

$$\left(p_0 , p\right) \mapsto d\left(p_0 , p\right)$$

⋆ v represents the intrinsic *angle* between a reference direction (defined by a fixed point p_{ref} on M) and the direction from p_0 to any point p .

$$v := \theta\left(p_0 , p , p_{ref}\right) \qquad ; \quad v \in [\,0\,, 2\pi\,)$$

$$\theta \,:\, \left\{p_0\right\} \times M \times \left\{p_{ref}\right\} \,\rightarrow\, [\,0\,, 2\pi\,)$$

$$\left(p_0 , p , p_{ref}\right) \mapsto \theta\left(p_0 , p , p_{ref}\right)$$

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dibeos.contact@gmail.com