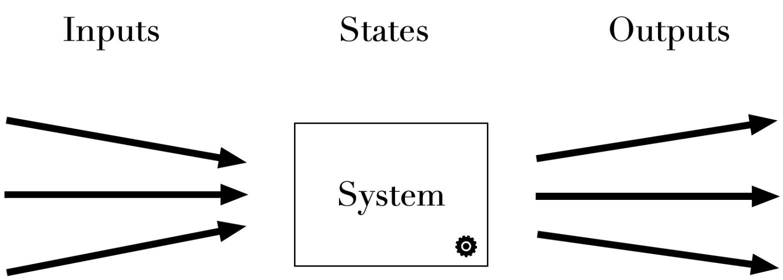
The Core of Dynamical Systems

By DiBeos

What is a system?

A system, in this context, is a mathematical object that has *inputs, outputs,* and *states*.



We will talk more about them throughout the document, but first of all, it is useful to think about "dynamical systems" as situations where the state (*position* and *position's rate of change*) evolves over time based on specific rules. The word "dynamic" is pretty intuitive. It refers to the change of state over time. These changes can be *continuous* or *discrete*.

Continuous

$$\frac{dx}{dt} = f(x, t)$$

Discrete

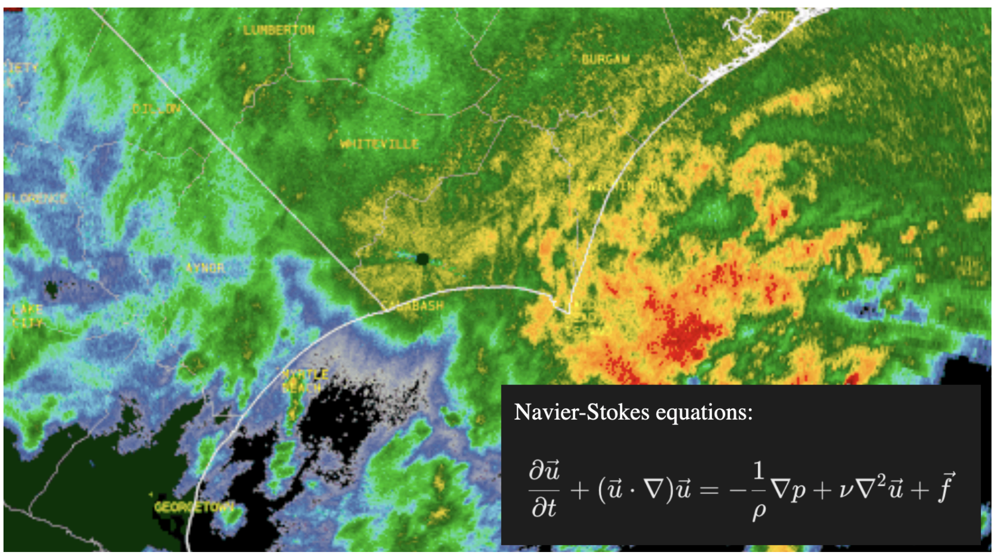
$$x_{n+1} = g\left(x_n\right)$$

The state of the system depends explicitly on time, and it is usually described by equations like:

$$\frac{dx}{dt} = f(x, t) \quad \text{(for the continuous case)}$$

$$x_{n+1} = g(x_n) \quad \text{(for the discrete case)}$$

Dynamical systems can also be categorized as being *simple*, and therefore *predictable*, or *complex*, and in this case *unpredictable*, like chaotic weather patterns, for example, even though being governed by deterministic rules.



Let us see our fist concrete example. Keep in mind though that we will start with a simple example and then will build up to more complex ones.

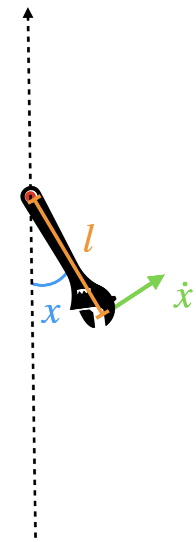
Equation of motion:

gravitational acceleration

$$\ddot{x} + \frac{g}{l} sin(x) = 0 \quad \text{(Newton's notation)}$$

$$\frac{dx}{dt}(t) + \frac{g}{l} sin(x) = 0 \quad \text{(Leibniz's notation)}$$

pendulum's length



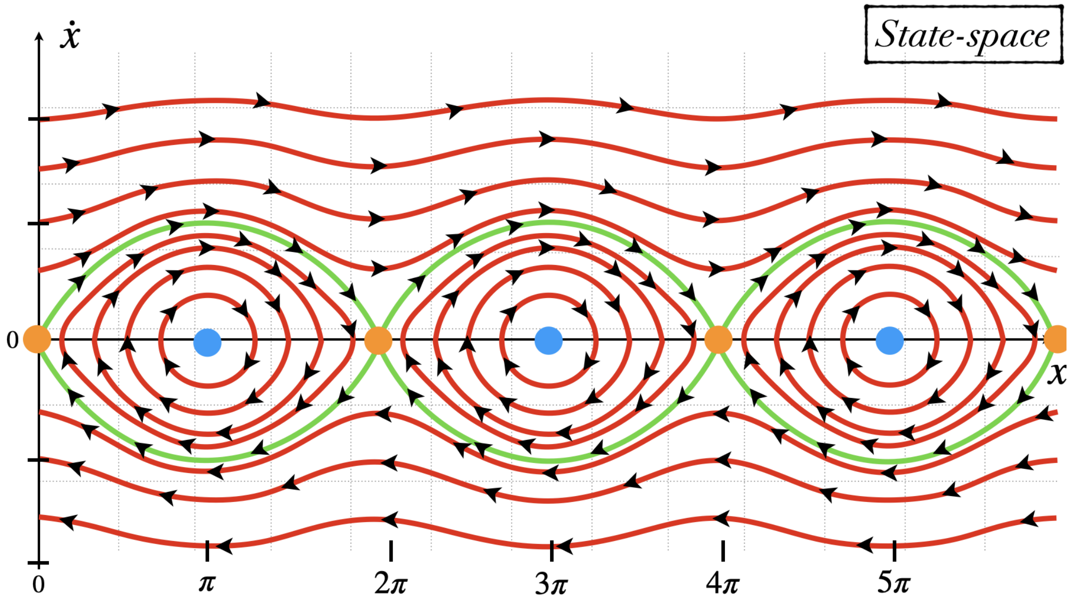
The *position* x represents the angular displacement of the pendulum from its vertical resting position. The *velocity* $\dot{x}$, which is the derivative of x, represents the rate of change of the position, and thus it describes how fast the pendulum is swinging.

Together, *position* and *velocity* fully describe the pendulum' state at any time. Using Newton's laws or energy principles, these variables can be used to predict how the system will evolve. This evolution is governed by the following *equation of motion*:

$$\boxed{\ddot{x} + \frac{g}{l} sin(x) = 0} \quad \text{(Newton's notation)}$$

$$\boxed{\frac{dx}{dt}(t) + \frac{g}{l} sin(x) = 0} \quad \text{(Leibniz's notation)}$$

This is not the most accurate representation in the context of Dynamical Systems (DS), though. In DS, we would rather plot a state-space, which describes the main quantities here, namely: the *position* and the *velocity*.



Some important definitions before moving on:

(I) *Rest (or equilibrium) points.*

These are points in the state-space where the system does not change over time, i.e. the derivatives of all state variables are zero. There exist 3 types:

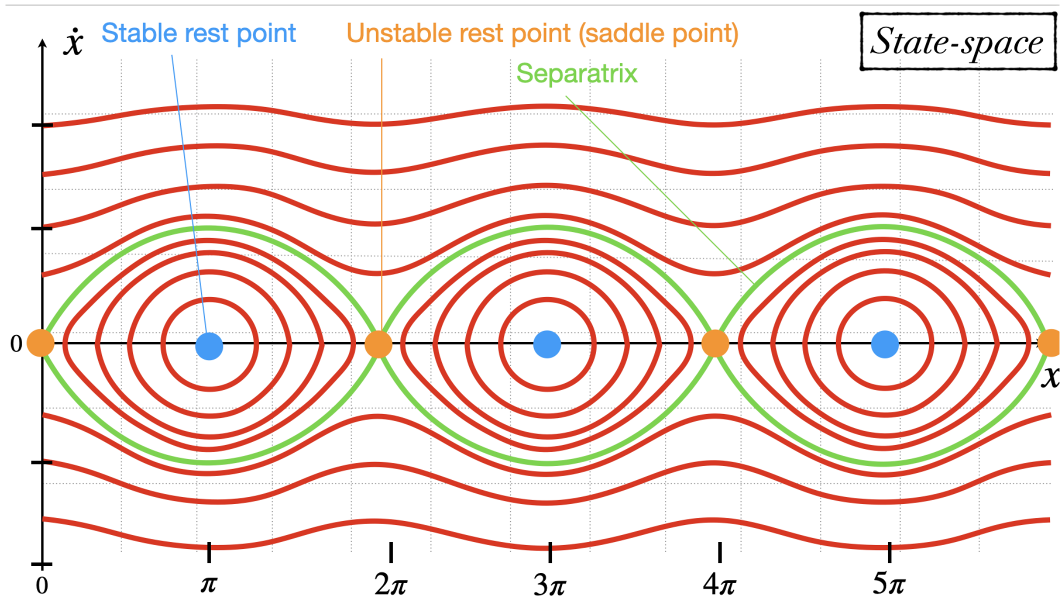
a) Stable (attracting) points – small perturbations return to the equilibrium point;

b) *Unstable (repelling) points* – small perturbations grow, moving the system away;

c) *Saddle points* – a mix of stable and unstable behavior depending on the direction.

(II) *Separatrix*.

This is a trajectory in the state-space that divides different types of motion. It usually separates *stable* and *unstable* regions, or *periodic* and *non-periodic* behaviors.



(III) *Stability*.

Closed orbits represent stable motion. Divergent orbits represent unstable motion.

(IV) *Attractors*.

These are sets of points (or trajectories) in state-space toward which the system evolves over time, regardless of initial conditions. There are 3 types:

a) *Point attractor* – a stable rest point (e.g. damped pendulum at rest);

b) *Limit cycle* – a closed-loop attractor corresponding to sustained periodic motion (e.g. undamped oscillator);

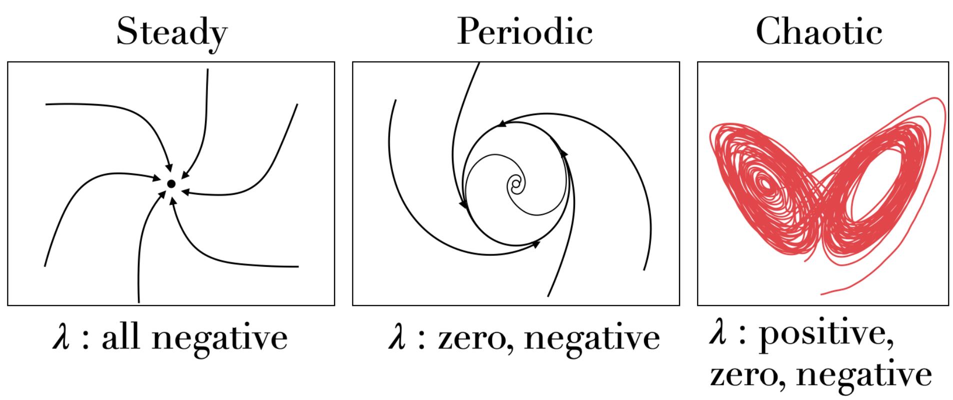
c) *Strange attractor* – a fractal structure associated with chaotic systems.

(V) *Chaos indicators*.

Slight changes in initial conditions result in vastly different trajectories. A strong indicator is the presence of strange attractors, where there are non-repeating yet bounded trajectories. Another indicator is the *Lyapunov exponent* which quantifies divergence of nearby trajectories.

$$\frac{du}{dt} = f(u) \quad \frac{du + \epsilon v}{dt} = f(u + \epsilon v) \quad v(t) \sim e^{\lambda t}$$

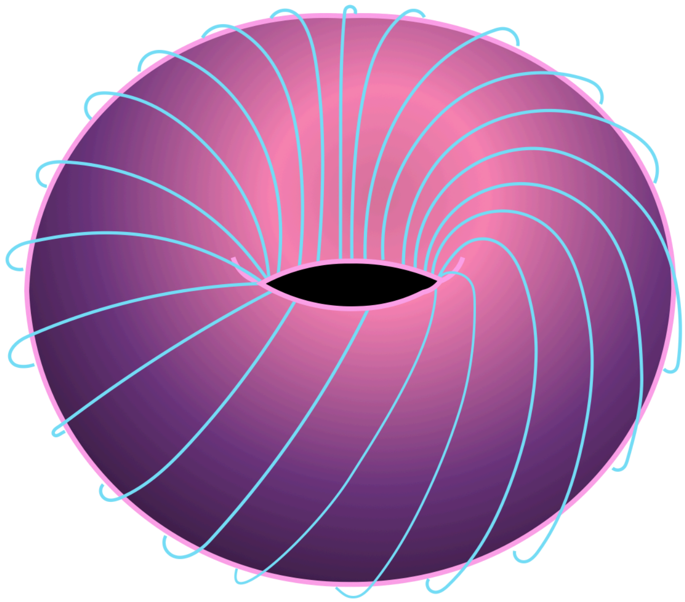
λ : Lyapunov exponent



(VI) *Flow*.

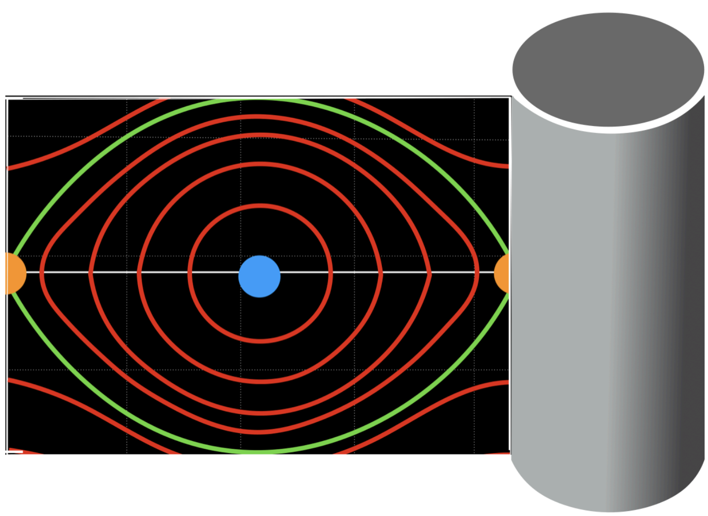
Flow is the smooth motion of trajectories in state-space driven by the system's equations. It represents the "current" of states. For example, for the pendulum, the flow creates spiraling trajectories in the presence of damping.

We have been talking about trajectories here, but it is time to be more specific now. An interesting type of trajectory (or *orbit* – they are the same) is the *quasi-periodic orbit*. These are trajectories that never exactly repeat, but can densely fill a toroidal surface in the space-state.

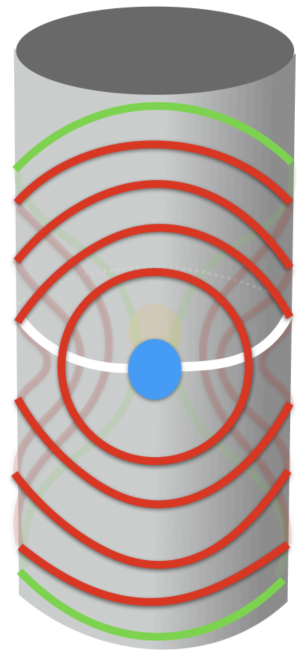


But, wait a second. How does a torus have anything to do with the state-space of a DS?

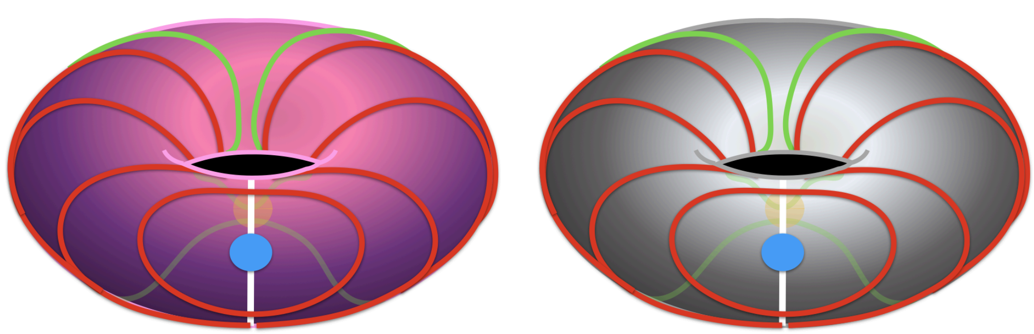
Well, this is quite interesting! Let's take a look at the pendulum' state-space again.



We notice that there is a pattern that repeats itself over and over again. Since the angular position x is periodic for intervals of length 2π, we can "wrap" the horizontal boundaries, $x = -\pi$ and $x = \pi$, to form a cylinder.



Additionally, if we consider the energy levels (called *iso-energy orbits*), the structure of the phase space suggests a repeating pattern for bounded trajectories. By compactifying angular velocity along energy levels, we can create a $2D$ toroidal representation.



Thus, the torus compactly encodes bounded oscillatory motion, but does not describe trajectories corresponding to unbounded (rotational) motion.

This is very useful, because, mathematically, it is simpler to study periodic orbits and flows on a torus since it is a well-defined, bounded, manifold:

$$T^2 = S^1 \times S^1$$

Here, S^1 is the unit circle, $\times$ is the Cartesian product, and T^2 the $2D$ torus that results from the operation.

The flow on the torus can be parameterized by *normalized energy levels*:

$$H(x, \dot{x}) = \frac{1}{2} m\, l^2\, \dot{x}^2 + mgl\,(1 - cos(x))$$

pull of gravity

length of the pendulum

mass

This is the *Hamiltonian* of the pendulum, which represents the total energy of the system. The first term $\left(\frac{1}{2} m\, l^2\, \dot{x}^2\right)$ is the energy related to the pendulum's motion (*kinetic energy*). On the torus, paths corresponding to constant energy ($H = constant$) form circular, or oval loops.

The motion on such a torus can often be described by angular coordinates (φ_1, φ_2), where time evolution is governed by linear equations like:

$$\varphi_1 = \omega_1 t + \varphi_1^0$$

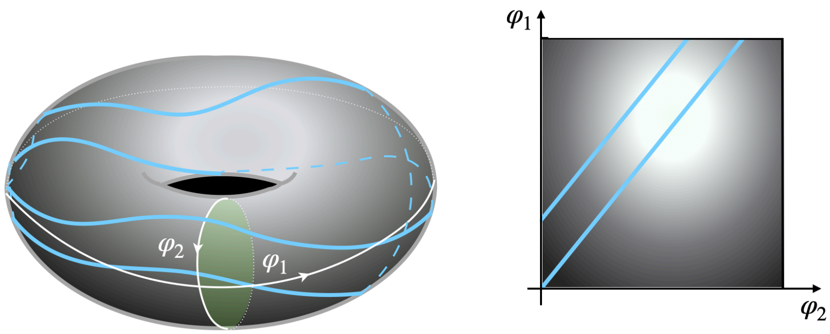
$$\varphi_2 = \omega_2 t + \varphi_2^0$$

φ_1 and φ_2 are the *angular positions* on the torus, just like latitude and longitude.

ω_1 and ω_2 are the *angular velocities* for time t.

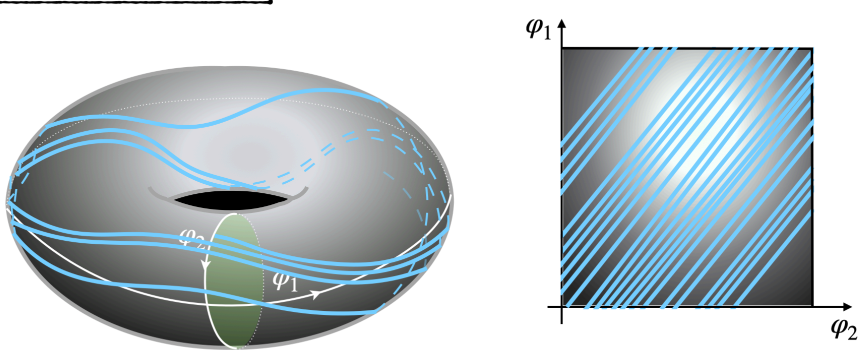
$\varphi_1^{\,0}$ and $\varphi_2^{\,0}$ are the *initial conditions*, so $\varphi_1^{\,0} = \varphi_1(0) \;\wedge\; \varphi_2^{\,0} = \varphi_2(0) \quad (t = 0)$.

Periodic motion



$$\frac{\omega_2}{\omega_1} =: \nu \in \mathbb{Q}$$

Quasi-periodic motion



$$\frac{\omega_2}{\omega_1} =: \nu \in \mathbb{R}\backslash\mathbb{Q}$$

There are two possible cases:

(I) $\frac{\omega_2}{\omega_1} =: \nu \in \mathbb{Q} \Rightarrow$ *periodic* motion.

(II) $\frac{\omega_2}{\omega_1} =: \nu \in \mathbb{R}\backslash\mathbb{Q} \Rightarrow$ *quasi-periodic* motion (it fills the torus densely).

Proof:

$$\varphi_1(t) = \omega_1 t + \varphi_1^0$$

$$\varphi_2(t) = \omega_2 t + \varphi_2^0$$

Suppose that there is a certain moment $T \in \mathbb{R}$ in time at which the angular positions repeat, i.e. if a point travels along one of these trajectories on the torus, then at time T, the point gets back to its original position.

This mean that $\exists\, k_1, k_2 \in \mathbb{Z}$:

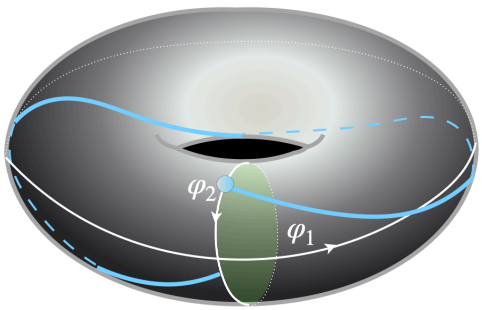
$$\varphi_1(T) = \varphi_1(0) + 2\pi k_1 \equiv \omega_1 T + \varphi_1^0 \Leftrightarrow \omega_1 T = 2\pi k_1 \Leftrightarrow \boxed{k_1 = \frac{\omega_1 T}{2\pi}} \in \mathbb{Z}$$

$$\varphi_2(T) = \varphi_2(0) + 2\pi k_2 \equiv \omega_2 T + \varphi_2^0 \Leftrightarrow \omega_2 T = 2\pi k_2 \Leftrightarrow \boxed{k_2 = \frac{\omega_2 T}{2\pi}} \in \mathbb{Z}$$

$$\Rightarrow \qquad \mathbb{Q} \ni \frac{k_2}{k_1} = \frac{\left(\frac{\omega_2 T}{2\pi}\right)}{\left(\frac{\omega_1 T}{2\pi}\right)} = \frac{\omega_2}{\omega_1} = \nu \qquad \Rightarrow \qquad \nu \in \mathbb{Q}$$

Therefore, if there are periodic trajectories, then ν must be rational. And, as a consequence, when ν is irrational, there is no periodic motion. ❐

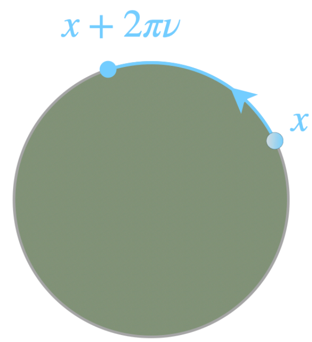
We will now show that, for $\nu \in \mathbb{R}\backslash\mathbb{Q}$, not only the motion is non-periodic, but actually that the set of all orbits is everywhere *dense* on the torus $\left(T^2\right)$.



$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$$

frequency

$$\frac{\omega_2}{\omega_1} =: \nu$$



$$\boxed{\varphi_1 = \omega_1 t + \varphi_1^0}$$
$$\boxed{\varphi_2 = \omega_2 t + \varphi_2^0}$$

$$\varphi_1(t) = \omega_1 t + \varphi_1^0 \quad \Rightarrow \quad \boxed{t = \frac{\varphi_1 - \varphi_1^0}{\omega_1}} \quad (*)$$

$$\varphi_2(t) = \omega_2 t + \varphi_2^0 \quad \text{and} \quad (*) \quad \Rightarrow \quad \varphi_2(t) = \omega_2\left(\frac{\varphi_1 - \varphi_1^0}{\omega_1}\right) + \varphi_2^0 \quad \Rightarrow$$

$$\Rightarrow \quad \varphi_2(t) = \frac{\omega_2}{\omega_1}\varphi_1 - \frac{\omega_2}{\omega_1}\varphi_1^0 + \varphi_2^0 \quad \Rightarrow \quad \frac{d\varphi_2}{d\varphi_1} = \nu \quad \Rightarrow \quad \varphi_2 = \nu\,\varphi_1 + \varphi_2^0$$

ν

we combined the 2 constants
and redefined the initial
position accordingly

$$\boxed{\varphi_2 = \nu\,\varphi_1 + \varphi_2^0}$$

$\rightarrow$ this equation means that for every increment 2π in φ_1 (a full circle), the coordinate φ_2 increases by $2\pi\nu$, forming a rotational shift:

$$\left(\varphi_1 \rightarrow \varphi_1 + 2\pi\right)$$

$$\Downarrow$$

$$\left(\varphi_2 = \nu\,\varphi_1 + \varphi_2^{\,0} \;\rightarrow\; \varphi_2 = \nu\left(\varphi_1 + 2\pi\right) + \varphi_2^{\,0} = \nu\,\varphi_1 + 2\pi\nu + \varphi_2^{\,0}\right)$$

Let us define a new angular variable on the unit circle S^1:

$$\boxed{\theta := \varphi_2}$$

$$\left(\varphi_1 \rightarrow \varphi_1 + 2\pi\right) \;\Rightarrow\; \left(\theta \rightarrow \theta + 2\pi\nu \quad mod\ 2\pi\right) \qquad \textit{(angular shift)}$$

The mapping $\theta \rightarrow \theta + 2\pi\nu$ defines a *discrete DS*. In the complex plane, since S^1 can be represented as $z \in \mathbb{C}$, with $|z| = 1$, the *angular shift* becomes:

$$\boxed{z = e^{i\theta} \rightarrow z \cdot e^{i2\pi\nu}}$$

Starting with an initial point $z = e^{i\theta} \in S^1$, its orbit under this DS is:

$$\boxed{O\left(e^{i\theta}\right) := \left\{e^{i\theta} \cdot e^{i2\pi\nu k}\right\}_{k\in\mathbb{Z}}}$$

Each point is obtained by successive rotations of the initial point $e^{i\theta}$ by $2\pi\nu$.
It is important to notice that proving that orbits of the unit circle rotation are dense, for $\nu \in \mathbb{R}\backslash\mathbb{Q}$, implies that the trajectories in the torus are also dense because the same logic applies to both dimensions, i.e. $1D$ projections on S^1 of the $2D$ torus dynamics:

Theorem: $\nu \in \mathbb{R}\backslash\mathbb{Q} \;\Rightarrow\; O\left(e^{i\theta}\right)$ **is *dense* on** S^1**.**

Proof:

By contradiction, assume that the orbits are *not dense* $\Rightarrow \exists\, U \in S^1$ (an open arc) :
$\nexists\, z \in O\left(e^{i\theta}\right)$, with $z \in U \quad \Leftrightarrow \quad \exists\, U \in S^1 : U \cap O\left(e^{i\theta}\right) = \emptyset$ (disjoint).

Let this arc U have length $\epsilon > 0$ (arbitrarily small).

Consider 2 point of an orbit that coincide on S^1:

$$\forall k, j \in \mathbb{Z}\ , \qquad e^{2\pi i\nu k} = e^{2\pi i\nu j} \qquad \Rightarrow$$

$$\Rightarrow \quad 2\pi\nu k \equiv 2\pi\nu j \quad mod\ 2\pi \quad \Rightarrow \quad 2\pi\nu k = 2\pi\nu j + 2\pi m \ , \quad m \in \mathbb{Z} \quad \Rightarrow$$

$$\Rightarrow \quad \nu(k - j) = m \ , \quad m \in \mathbb{Z} \quad \Rightarrow \quad \boxed{\nu = \frac{m}{k-j}}$$

But, if $k, j, m \in \mathbb{Z}\ (k \neq j)$, then $\nu \in \mathbb{Q}$. And this is a contradiction, since we assumed ν to be irrational. Hence, we must have that $k = j$, so that we cannot write $\nu = \frac{m}{k-j}$, and thus $\nu \in \mathbb{R}\backslash\mathbb{Q}$. In other words, no distinct point $e^{2\pi i\nu k}$ and $e^{2\pi i\nu j}$ $(k \neq j)$ can coincide with $\nu \in \mathbb{R}\backslash\mathbb{Q}$. This is the same as saying that all points in the orbit are distinct.

Now that the points in the orbit are distinct, we apply the *Pigeonhole Principle* – for a detailed explanation of it, as well as its proof, watch our corresponding YouTube video.

I Proved the Pigeonhole Principle (Or Did I?)

Consider $N + 1$ points in the orbit $\left\{ e^{2\pi i \nu k} \mid k = 0, 1, \dots, N \right\}$. Divide the circle into N equal arcs of length $\frac{2\pi}{N}$. By the *Pigeonhole Principle*, at least 2 of these points must fall into the same arc (because there are more points than arcs).

Let these 2 points correspond to indices $j \wedge l \ (j \neq l)$. The difference between their arguments $modulo$ 1 is:

$$\boxed{\beta := \nu\,(l - j) \quad mod\ 1} \qquad \Rightarrow$$

(The modulo 1 operation accounts for the periodicity of angles and ensures the differences between points are properly measured within a single rotation of S^1 *)*

$\Rightarrow \quad 0 < \beta < \frac{1}{N} \quad \Rightarrow$ the sequence of points $e^{2\pi i \beta p}$, $p \in \mathbb{Z}$, is dense on S^1 (as N can be made arbitrarily large).

Definition: The sequence of points generated by the orbit comes arbitrarily close to every point on S^1.

The Role of Differential Equations in DS

Differential equations (DE) are at the heart of DS. They are the actual bridge between the physical world and mathematical abstraction. A DE relates a function (like the position in terms of time) to its derivatives (like the velocity). Some example are:

(I) *Bouncing ball:*

height of the ball

$$\frac{dy}{dt} = v \quad \wedge \quad \frac{dv}{dt} = -g$$

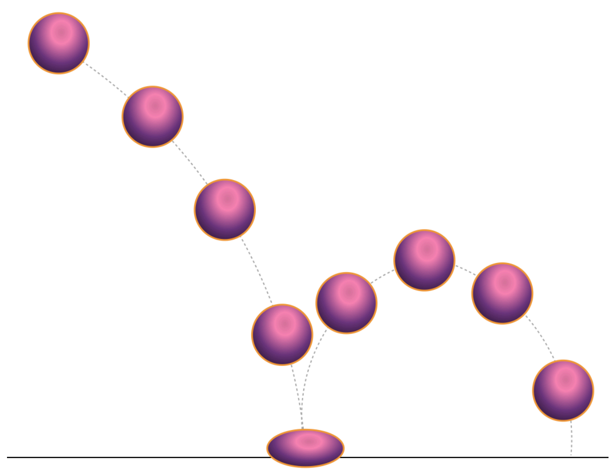
vertical velocity

gravity

$$\Downarrow$$

$$\frac{d^2y}{dt^2} = -g$$

It models the height and velocity of the ball, considering gravity and energy loss upon bouncing.



(II) *Population growth (Logistic model):*

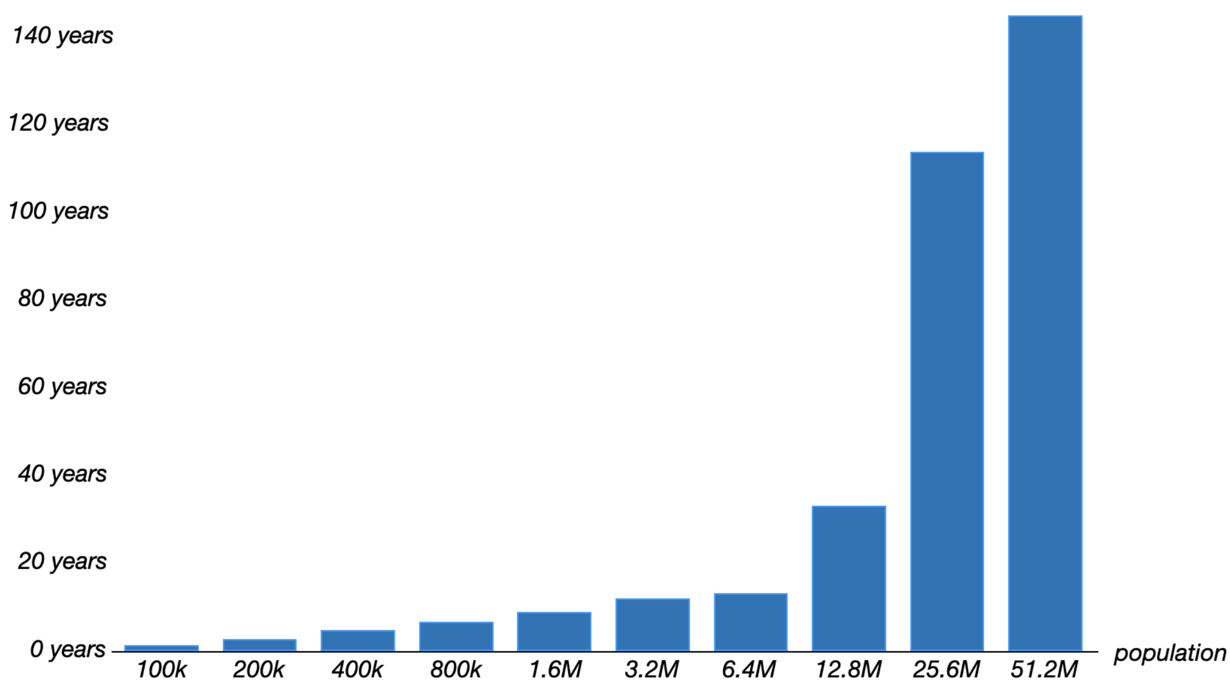
$N(t)$: population size

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$$

growth rate

carrying capacity (max. sustainable population)

It describes population growth with a limit imposed by resource availability (k)*. For small populations, growth is approximately exponential.*



(III) *Galaxies motion (N-body problem):*

position vector

velocity vector

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i$$

gravitational constant

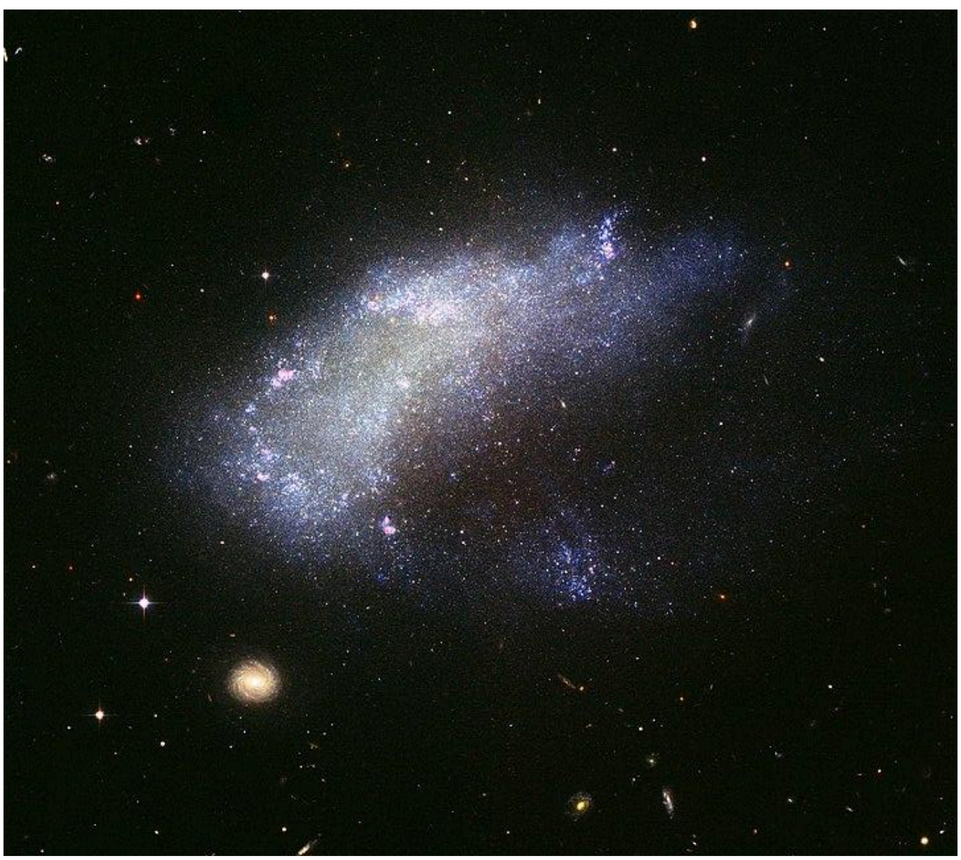
mass of the j-th galaxy

$$\frac{d\vec{v}_i}{dt} = -G \cdot \sum_{j \neq k} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|}$$

$$\frac{d\vec{v}_i}{dt} \equiv \frac{d^2\vec{r}_i}{dt^2}$$

$i \in \{1, \ldots, N\}$

It describes the gravitational interaction of N galaxies, leading to highly complex and chaotic motion.



(IV) *Atmosphere (Navier-Stokes Equations):*

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \overrightarrow{\nabla}\right) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + \vec{f}$$

pressure

external forces (gravity, Coriolis force, etc…)

velocity field of air

density of air

viscosity

This equation governs fluid motion in the atmosphere, including turbulence and weather patterns. Highly non-linear and chaotic.

(V) *Traffic flow (Lighthill-Whitham-Richards model): (conservation law)* traffic density

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\big(\rho v(\rho)\big) = 0$$

$\rho = \rho(x, t)$ velocity of cars

It describes traffic dynamics, including shock waves and congestion. The velocity function $v(\rho)$ *decreases as density increases.*

(VI) *Vortex formation (Fluid Dynamics):*

vorticity (rotation of fluid)

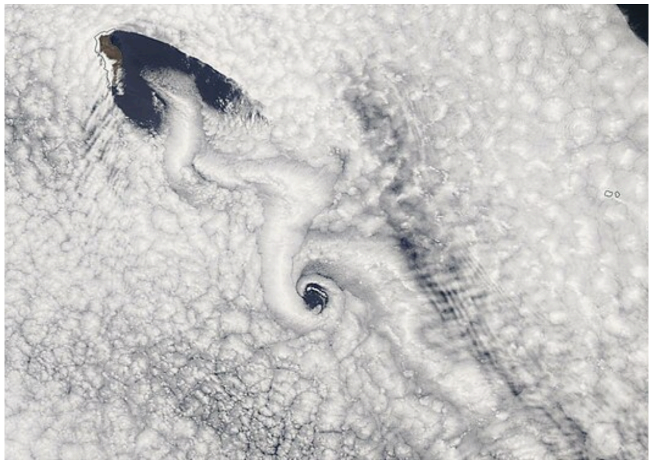
$$\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{\omega} = \left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{v} + \nu \nabla^2 \vec{\omega}$$

velocity field

kinematic viscosity

$\omega = \vec{\nabla} \times \vec{v}$

It captures the dynamics of vortices in fluids, including formation, dissipation, and interactions.



(VII) *Lorenz attractor:*

$$\frac{dx}{dt} = \sigma\left(y - x\right)$$

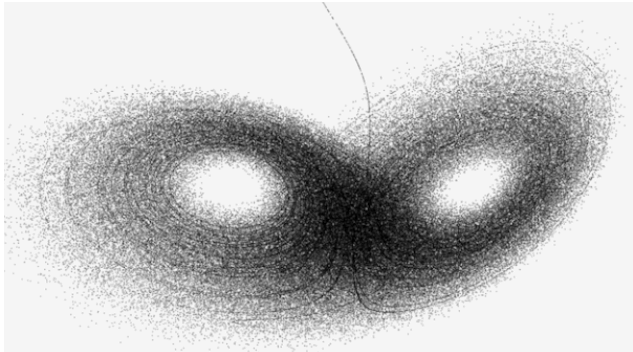
$x, y, z \rightarrow$ state variables (temperature differences, flow rates, etc)

$$\frac{dy}{dt} = x\left(\rho - z\right) - y$$

$$\frac{dz}{dt} = xy - \beta y$$

$\sigma, \rho, \beta \rightarrow$ system parameters

It models convection in fluids and exhibits chaotic behavior, with trajectories confined to a strange attractor.



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