Let’s start with the simplest root of the imaginary unit.

Notice:

$$\sqrt{i} \equiv x + i\,y \implies i = \left(\sqrt{i}\right)^2 = \left(x^2 - y^2\right) + 2ixy \implies$$

$$\implies \begin{cases} x^2 = y^2 \\ \wedge \\ 2xy = 1 \implies y = \dfrac{1}{2x} \end{cases} \implies x^2 = \frac{1}{4x^2} \implies x^4 = \frac{1}{4}$$

$$\Downarrow$$

$$\boxed{y = \pm\frac{1}{\sqrt{2}}} \Longleftarrow \boxed{x = \pm\frac{1}{\sqrt{2}}}$$

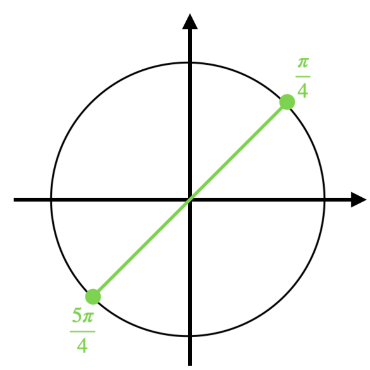
$$\therefore \sqrt{i} = \begin{cases} \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} = cos\left(\dfrac{\pi}{4}\right) + i\,sin\left(\dfrac{\pi}{4}\right) = e^{i\frac{\pi}{4}} \\ -\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}} = cos\left(\dfrac{5\pi}{4}\right) + i\,sin\left(\dfrac{5\pi}{4}\right) = e^{i\frac{5\pi}{4}} \end{cases}$$

Notice:

$$i = cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$$

$$\Downarrow$$

$$\sqrt{i} = \sqrt{-1 \cdot i^4} = i^{\frac{5}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{5}{2}} = e^{i\frac{5\pi}{4}}$$



We found 2 solutions, corresponding to the points in the unit circle with angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. Let's see what we get when trying to calculate the cubic root of the imaginary unit.

$\sqrt[3]{i} \equiv x + iy$	$x = ? \ \wedge \ y = ?$

$$\left(\sqrt[3]{i}\right)^3 = (x+iy)^3 \Longrightarrow i = \left(x^2 + 2ixy - y^2\right)(x+iy) =$$

$$= x^3 + ix^2y + 2ix^2y - 2xy^2 - xy^2 - iy^3 \qquad \Longrightarrow$$

$$\Longrightarrow \boxed{\left(x^3 - 3xy^2\right) + i\left(3x^2y - y^3\right) \equiv i}$$

$$\begin{cases} x^3 - 3xy^2 = 0 \\ \wedge \\ \boxed{3x^2y - y^3 = 1} \end{cases} \Longrightarrow x\left(x^2 - 3y^2\right) = 0 \Longrightarrow \begin{cases} \boxed{x = 0} \Longrightarrow \boxed{y = -1} \\ \vee \\ x^2 - 3y^2 = 0 \end{cases}$$

$$\Downarrow$$

$$\boxed{x^2 = 3y^2}$$

$$\Downarrow$$

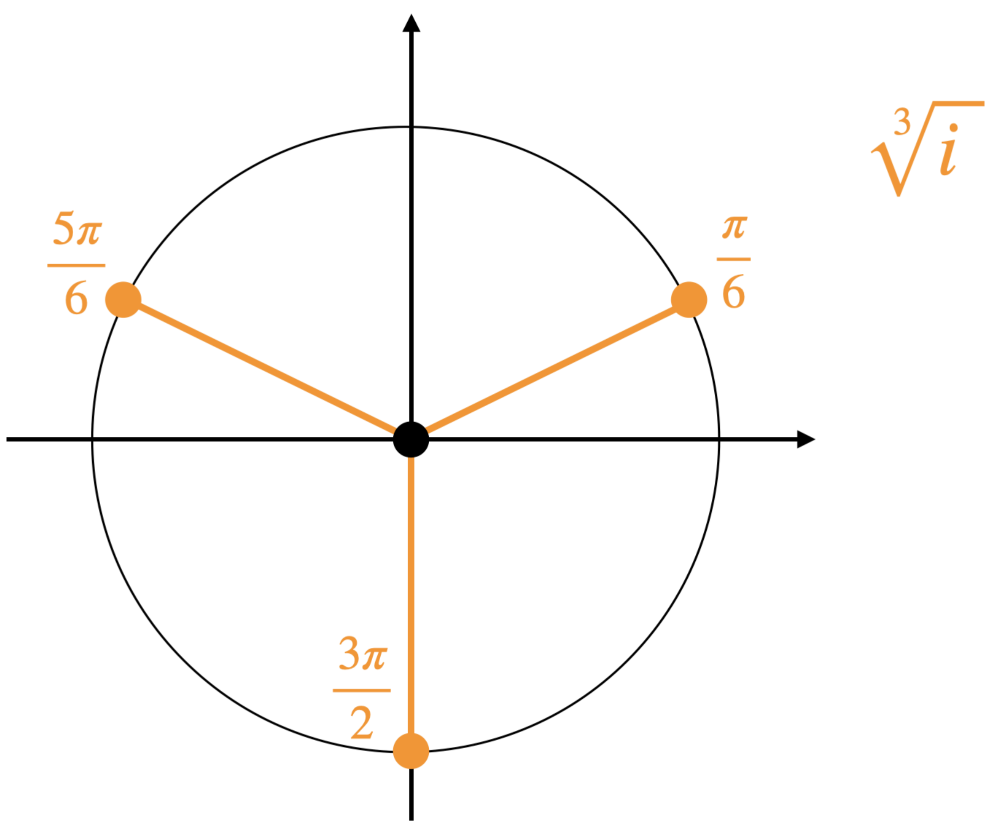
$$9y^3 - y^3 = 1 \Longrightarrow \boxed{y = \frac{1}{2}}$$

$$\Downarrow$$

$$\boxed{x = \pm\frac{\sqrt{3}}{2}}$$

$$\therefore \sqrt[3]{i} = \begin{cases} -i = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = e^{i\frac{3\pi}{2}} \\ \vee \\ +\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = e^{i\frac{\pi}{6}} \\ \vee \\ -\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = e^{i\frac{5\pi}{6}} \end{cases}$$

So, we found 3 solutions. Notice how each point in the unit circle correspond to the angles $\frac{3\pi}{2}$, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, which have a 120° (or $\frac{2\pi}{3}$) angle separating them. The consequence is that these solutions split the circle into 3 equal parts.



Ok, things are getting interesting.

(BTW, consider becoming a member of the channel!) Thanks!

Let’s move on to the 4-th root now. I warn you though, this one is way more work. So, brace yourself.

$\sqrt[4]{i} \equiv x + i\,y$	$x = ? \;\; \wedge \;\; y = ?$

$$i = \left(\sqrt[4]{i}\right)^4 = \left(x + i\,y\right)^4 = (x^2 + 2ixy - y^2)^2$$

$$= x^4 + 2ix^3y - x^2y^2 + 2ix^3y - 4x^2y^2 - 2ixy^3$$

$$-x^2y^2 - 2ixy^3 + y^4 =$$

$$= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 =$$

$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) \equiv i \implies$$

$$\Longrightarrow \begin{cases} x^4 - 6x^2y^2 + y^4 = 0 \Longrightarrow t_x^2 - 6\,t_xt_y + t_y^2 = 0 \\ \wedge \\ 4x^3y - 4xy^3 = 1 \Longrightarrow \boxed{x^3y - xy^3 = \frac{1}{4}} \end{cases}$$

$$t_x^2 - 2\,t_xt_y + t_y^2 - 4\,t_xt_y = 0 \Longrightarrow (t_x - t_y)^2 = 4\,t_xt_y \Longrightarrow$$

$$\Longrightarrow t_x - t_y = \pm\,2\sqrt{t_xt_y} \Longrightarrow x^2 - y^2 = \pm\,2\,xy \Longrightarrow$$

$$t_x := x^2$$
$$t_y := y^2$$

$$\Longrightarrow \begin{cases} \boxed{x^2 - 2\,xy - y^2 = 0} \quad (\mathrm{I}) \\ \wedge \\ \boxed{x^2 + 2\,xy - y^2 = 0} \quad (\mathrm{II}) \end{cases}$$

(I):

$$x_{1,2} = \frac{2\,y_{1,2} \pm \sqrt{4\,y_{1,2}^2 + 4\,y_{1,2}^2}}{2} = \begin{cases} \boxed{y_1\left(1+\sqrt{2}\right)} \quad ① \\ \boxed{y_2\left(1-\sqrt{2}\right)} \quad ② \end{cases}$$

$$① \begin{cases} x_1 = y_1 \left(1+\sqrt{2}\right) \\ \wedge \\ \boxed{x_1^3 y_1 - x_1 y_1^3 = \frac{1}{4}} \end{cases}$$

$$\Downarrow$$

$$y_1^3\left(1+\sqrt{2}\right)^3 y_1 - y_1\left(1+\sqrt{2}\right) y_1^3 = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_1^4\left[\left(1+\sqrt{2}\right)^3 - \left(1+\sqrt{2}\right)\right] = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_1^4\left[\left(1+2\sqrt{2}+2\right)\left(1+\sqrt{2}\right) - 1 - \sqrt{2}\right] = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_1^4\left(3+3\sqrt{2}+2\sqrt{2}+4-1-\sqrt{2}\right) = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_1^4\left(6+4\sqrt{2}\right) = \frac{1}{4} \Longrightarrow$$

$$\boxed{y^{\pm}_{1_{(I)}} = \frac{\pm 1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}} \Longrightarrow \boxed{x^{\pm}_{1_{(I)}} = \frac{\pm\left(1+\sqrt{2}\right)}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}}$$

$$②\begin{cases} x_2 = y_2\left(1-\sqrt{2}\right) \\ \wedge \\ \boxed{x_2^3 y_2 - x_2 y_2^3 = \frac{1}{4}} \end{cases}$$

$$\Downarrow$$

$$y_2^3\left(1-\sqrt{2}\right)^3 y_2 - y_2\left(1-\sqrt{2}\right)y_2^3 = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_2^4\left[\left(1-\sqrt{2}\right)^3 - \left(1-\sqrt{2}\right)\right] = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_2^4\left[\left(1-2\sqrt{2}+2\right)\left(1-\sqrt{2}\right)-1+\sqrt{2}\right] = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_2^4\left(3-3\sqrt{2}-2\sqrt{2}+4-1+\sqrt{2}\right) = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_2^4\left(6-4\sqrt{2}\right) = \frac{1}{4} \Longrightarrow$$

$$\boxed{y^{\pm}_{2_{(I)}} = \frac{\pm 1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}} \Longrightarrow \boxed{x^{\pm}_{2_{(I)}} = \frac{\pm\left(1-\sqrt{2}\right)}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}}$$

(I): ✓ OK

(II): $\boxed{x^2 + 2xy - y^2 = 0}$

$$x_{1,2} = \frac{-2y_{1,2} \pm \sqrt{4y_{1,2}^2 + 4y_{1,2}^2}}{2} = \begin{cases} \boxed{y_1\left(-1+\sqrt{2}\right)} & ① \\ \boxed{y_2\left(-1-\sqrt{2}\right)} & ② \end{cases}$$

$$① \begin{cases} x_1 = y_1\left(-1+\sqrt{2}\right) \\ \wedge \\ \boxed{x_1^3 y_1 - x_1 y_1^3 = \frac{1}{4}} \end{cases}$$

$$\Downarrow$$

$$y_1^3\left(-1+\sqrt{2}\right)^3 y_1 - y_1\left(-1+\sqrt{2}\right) y_1^3 = \frac{1}{4} \Longrightarrow$$

$$\Longrightarrow y_1^4\left[\left(-1+\sqrt{2}\right)^3 - \left(-1+\sqrt{2}\right)\right] = \frac{1}{4} \Longrightarrow$$

$$\Rightarrow\ y_1^4\left[\left(1-2\sqrt{2}+2\right)\left(-1+\sqrt{2}\right)+1-\sqrt{2}\right]=\frac{1}{4}\ \Longrightarrow$$

$$\Longrightarrow\ y_1^4\left(-3+3\sqrt{2}+2\sqrt{2}-4+1-\sqrt{2}\right)=\frac{1}{4}\ \Longrightarrow$$

$$\Longrightarrow\ y_1^4\left(-6+4\sqrt{2}\right)=\frac{1}{4}\ \Longrightarrow$$

$$\boxed{y^{\pm}_{1_{(II)}}=\frac{\pm 1}{\sqrt[4]{8\left(-3+2\sqrt{2}\right)}}}\Longrightarrow\boxed{x^{\pm}_{1_{(II)}}=\frac{\pm\left(-1+\sqrt{2}\right)}{\sqrt[4]{8\left(-3+2\sqrt{2}\right)}}}$$

② $$\begin{cases} x_2=y_2\left(-1-\sqrt{2}\right) \\ \wedge \\ \boxed{x_2^3y_2-x_2y_2^3=\frac{1}{4}} \end{cases}$$

$$\Downarrow$$

$$y_2^3\left(-1-\sqrt{2}\right)^3y_2-y_2\left(-1-\sqrt{2}\right)y_2^3=\frac{1}{4}\ \Longrightarrow$$

$$\Longrightarrow\ y_2^4\left[\left(-1-\sqrt{2}\right)^3-\left(-1-\sqrt{2}\right)\right]=\frac{1}{4}\ \Longrightarrow$$

$$\Rightarrow\; y_2^4\left[\left(1+2\sqrt{2}+2\right)\left(-1-\sqrt{2}\right)+1+\sqrt{2}\right]=\frac{1}{4}\;\Longrightarrow$$

$$\Longrightarrow\; y_2^4\left(-3-3\sqrt{2}-2\sqrt{2}-4+1+\sqrt{2}\right)=\frac{1}{4}\;\Longrightarrow$$

$$\Longrightarrow\; y_2^4\left(-6-4\sqrt{2}\right)=\frac{1}{4}\;\Longrightarrow$$

$$\boxed{y^{\pm}_{2_{(II)}}=\frac{\pm 1}{\sqrt[4]{8\left(-3-2\sqrt{2}\right)}}}\;\Longrightarrow\;\boxed{x^{\pm}_{2_{(II)}}=\frac{\pm\left(-1-\sqrt{2}\right)}{\sqrt[4]{8\left(-3-2\sqrt{2}\right)}}}$$

$$\therefore\sqrt[4]{i}=\begin{cases}\pm\left(\dfrac{1+\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)\pm i\left(\dfrac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) & \boxed{1,(\mathrm{I})}\\[2ex] \pm\left(\dfrac{1-\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)\pm i\left(\dfrac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) & \boxed{2,(\mathrm{I})}\\[2ex] \pm\left(\dfrac{-1+\sqrt{2}}{\sqrt[4]{8\left(-3+2\sqrt{2}\right)}}\right)\pm i\left(\dfrac{1}{\sqrt[4]{8\left(-3+2\sqrt{2}\right)}}\right) & \boxed{1,(\mathrm{II})}\\[2ex] \pm\left(\dfrac{-1-\sqrt{2}}{\sqrt[4]{8\left(-3-2\sqrt{2}\right)}}\right)\pm i\left(\dfrac{1}{\sqrt[4]{8\left(-3-2\sqrt{2}\right)}}\right) & \boxed{2,(\mathrm{II})}\end{cases}$$

After this tremendous amount of work, we notice an “inconsistency” with the pattern observed so far, namely that the **4**-th root gave us **8** solutions. However…

There are actually just 4 solutions, not 8…

Let’s carefully analyze each of them in terms of their angles in the unit circle in order to find out which ones are duplicated.

```
Solution 1:
```

$$+\left(\frac{1+\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)+i\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=\underbrace{cos\theta_1}_{>0}+i\,\underbrace{sin\theta_1}_{>0} \implies$$

$$\implies \theta_1 = tan^{-1}\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}:\frac{1+\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=tan^{-1}\left(\frac{1}{\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}}\cdot\frac{\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}}{1+\sqrt{2}}\right)=tan^{-1}\left(\frac{1}{1+\sqrt{2}}\right)=\frac{\pi}{8}\ or\ \cancel{\frac{9\pi}{8}}$$

Solution 2:

$$-\left(\frac{1+\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) - i\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) = \underbrace{\cos\theta_2}_{<0} + i\,\underbrace{\sin\theta_2}_{<0} \implies$$

$$\implies \theta_1 = \tan^{-1}\left(\frac{-1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}} : \frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) =$$

$$= \tan^{-1}\left(\frac{-1}{\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}} \cdot \frac{\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}}{-1-\sqrt{2}}\right) = \tan^{-1}\left(\frac{1}{1+\sqrt{2}}\right) = \cancel{\frac{\pi}{8}} \text{ or } \frac{9\pi}{8}$$

Solution 3:

$$+\left(\frac{1-\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) + i\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) = \underbrace{\cos\theta_3}_{<0} + i\,\underbrace{\sin\theta_3}_{>0} \implies$$

$$\implies \theta_3 = \tan^{-1}\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} : \frac{1-\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) =$$

$$= \tan^{-1}\left(\frac{1}{\cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}} \cdot \frac{\cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}}{1-\sqrt{2}}\right) = \tan^{-1}\left(\frac{1}{1-\sqrt{2}}\right) = \frac{5\pi}{8} \text{ or } \cancel{\frac{13\pi}{8}}$$

Solution 4:

$$-\left(\frac{1-\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)-i\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)=\underbrace{cos\theta_4}_{>0}+i\,\underbrace{sin\theta_4}_{<0} \implies$$

$$\implies \theta_4 = tan^{-1}\left(\frac{-1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}:\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)=$$

$$=tan^{-1}\left(\frac{-1}{\cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}}\cdot\frac{\cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}}{-1+\sqrt{2}}\right)=tan^{-1}\left(\frac{1}{1-\sqrt{2}}\right)=\cancel{\frac{5\pi}{8}}\ or\ \frac{13\pi}{8}$$

We found the 4 solutions, corresponding to the angles $\frac{\pi}{8}$, $\frac{9\pi}{8}$, $\frac{5\pi}{8}$ and $\frac{13\pi}{8}$.
Hence, all the other solutions are duplicates of these 4. Let's prove that:

Solutions 5 and 6:

$$\pm\left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\underbrace{\left(-3+2\sqrt{2}\right)}_{<0}}}\right)\pm i\left(\frac{1}{\sqrt[4]{8\underbrace{\left(-3+2\sqrt{2}\right)}_{<0}}}\right)=\pm\left(\frac{-1+\sqrt{2}}{\sqrt[4]{8(-1)\left(3-2\sqrt{2}\right)}}\right)\pm i\left(\frac{1}{\sqrt[4]{8(-1)\left(3-2\sqrt{2}\right)}}\right)=$$

$$=\pm\frac{1}{\sqrt{i}}\left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)\pm\frac{i\cdot\sqrt{i}}{\sqrt{i}\cdot\sqrt{i}}\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)=$$

$$=\pm\left(\frac{1}{\pm\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}}\right)\left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)\pm\sqrt{i}\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)=$$

$$= \pm \left(\frac{1}{\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}} \right) \left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) \pm \sqrt{i} \left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) =$$

$$= \pm \left[\pm \left(\frac{\sqrt{2}}{1+i} \right) \right] \left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) \pm \left(\frac{\pm 1 \pm i}{\sqrt{2}} \right) \left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) =$$

$$= \left(\frac{\sqrt{2}}{1+i} \right) \left(\frac{-1+\sqrt{2}}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) + \left(\frac{1+i}{\sqrt{2}} \right) \left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}} \right) =$$

$$= \frac{-\sqrt{2}+2}{(1+i)\sqrt[4]{8\left(3-2\sqrt{2}\right)}} + \frac{1+i}{\sqrt{2}\ \sqrt[4]{8\left(3-2\sqrt{2}\right)}} =$$

$$= \frac{-\sqrt{2}+2}{(1+i)\sqrt[4]{8\left(3-2\sqrt{2}\right)}} + \frac{1+i}{\sqrt{2}\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} = \frac{-2+2\sqrt{2}\cancel{+1}\cancel{-1}+2i}{\sqrt{2}\,(1+i)\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} =$$

$$= \frac{-2+2\sqrt{2}+2i}{\sqrt{2}\,(1+i)\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} = \frac{\cdot\,(1-i)}{\cdot\,(1-i)} \quad \frac{\cancel{2}\left(\cancel{-1}+i+\sqrt{2}-i\sqrt{2}+i\cancel{+1}\right)}{\cancel{2}\sqrt{2}\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} =$$

$$= \frac{\sqrt{2}+i\left(2-\sqrt{2}\right)}{\sqrt{2}\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} = \frac{\cancel{\sqrt{2}}\left[1+i\left(\sqrt{2}-1\right)\right]}{\cancel{\sqrt{2}}\,\sqrt[4]{8\left(3-2\sqrt{2}\right)}} =$$

$$= \left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) + i\left(\frac{\sqrt{2}-1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right)$$

$$\left(\frac{1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) + i\left(\frac{\sqrt{2}-1}{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) = \underbrace{cos\theta_5}_{>0} + i\,\underbrace{sin\theta_5}_{>0} = cos\theta_6 + i\,sin\theta_6 \implies$$

$$\theta_6 = \theta_5 = tan^{-1}\left(\frac{\sqrt{2}-1}{\cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}} \cdot \cancel{\sqrt[4]{8\left(3-2\sqrt{2}\right)}}\right) = tan^{-1}\left(\sqrt{2}-1\right) = \frac{\pi}{8} \; or \; \cancel{\frac{9\pi}{8}}$$

$$\therefore \theta_1 = \theta_5 = \theta_6$$

Solution 7 and 8:

$$\pm\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(\underbrace{-3-2\sqrt{2}}_{<0}\right)}}\right)\pm i\left(\frac{1}{\sqrt[4]{8\left(\underbrace{-3-2\sqrt{2}}_{<0}\right)}}\right)=\pm\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8(-1)\left(3+2\sqrt{2}\right)}}\right)\pm i\left(\frac{1}{\sqrt[4]{8(-1)\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\pm\frac{1}{\sqrt{i}}\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)\pm\frac{i\cdot\sqrt{i}}{\sqrt{i}\cdot\sqrt{i}}\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\pm\left(\frac{1}{\pm\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}}\right)\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)\pm\sqrt{i}\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\pm\left(\frac{1}{\pm\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}}\right)\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)\pm\sqrt{i}\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\pm\left[\pm\left(\frac{\sqrt{2}}{1+i}\right)\right]\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)\pm\left(\frac{\pm1\pm i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\left(\frac{\sqrt{2}}{1+i}\right)\left(\frac{-1-\sqrt{2}}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)+\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)=$$

$$=\frac{-\sqrt{2}-2}{(1+i)\sqrt[4]{8\left(3+2\sqrt{2}\right)}}+\frac{1+i}{\sqrt{2}\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}}=$$

$$= \frac{-\sqrt{2}-2}{(1+i)\sqrt[4]{8\left(3+2\sqrt{2}\right)}} + \frac{1+i}{\sqrt{2}\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}} = \frac{-2-2\sqrt{2}+\cancel{1}-\cancel{1}+2i}{\sqrt{2}\,(1+i)\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}} =$$

$$= \frac{-2-2\sqrt{2}+2i}{\sqrt{2}\,(1+i)\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\frac{\cdot(1-i)}{\cdot(1-i)} = \frac{\cancel{2}\left(\cancel{-1}+i-\sqrt{2}+i\sqrt{2}+i+\cancel{1}\right)}{\cancel{2}\sqrt{2}\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}} =$$

$$= \frac{-\sqrt{2}+i\left(2+\sqrt{2}\right)}{\sqrt{2}\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}} = \frac{\cancel{\sqrt{2}}\left[-1+i\left(\sqrt{2}+1\right)\right]}{\cancel{\sqrt{2}}\,\sqrt[4]{8\left(3+2\sqrt{2}\right)}} =$$

$$= \left(\frac{-1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) + i\left(\frac{\sqrt{2}+1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right)$$

$$\left(\frac{-1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) + i\left(\frac{\sqrt{2}+1}{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) = \underbrace{\cos\theta_7}_{<0} + i\,\underbrace{\sin\theta_7}_{>0} = \cos\theta_8 + i\sin\theta_8 \implies$$

$$\theta_7 = \theta_8 = \tan^{-1}\left(\frac{-1-\sqrt{2}}{\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}}\cdot\cancel{\sqrt[4]{8\left(3+2\sqrt{2}\right)}}\right) = \tan^{-1}\left(-1-\sqrt{2}\right) = \frac{5\pi}{8} \; or \; \cancel{\frac{13\pi}{8}}$$

$$\therefore \theta_3 = \theta_7 = \theta_8$$

Great! Let’s finally gather everything we’ve found so far.

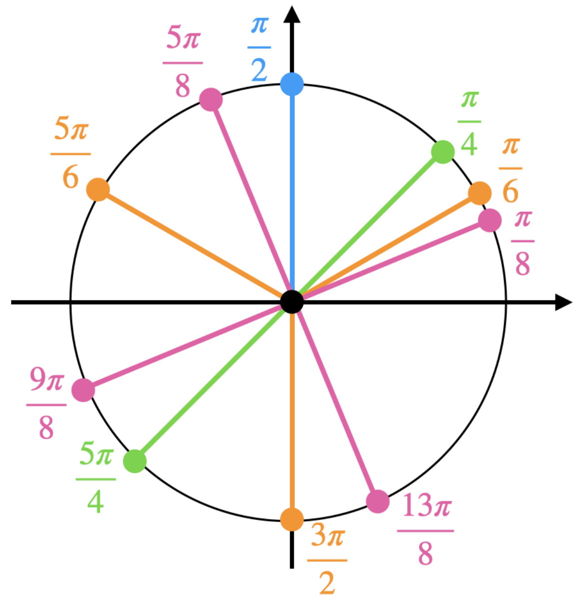
$i^{\frac{1}{1}}$ *has 1 solution.*

$i^{\frac{1}{2}}$ *has 2 solutions.*

$i^{\frac{1}{3}}$ *has 3 solutions.*

$i^{\frac{1}{4}}$ *has 4 solutions.*

$\vdots$



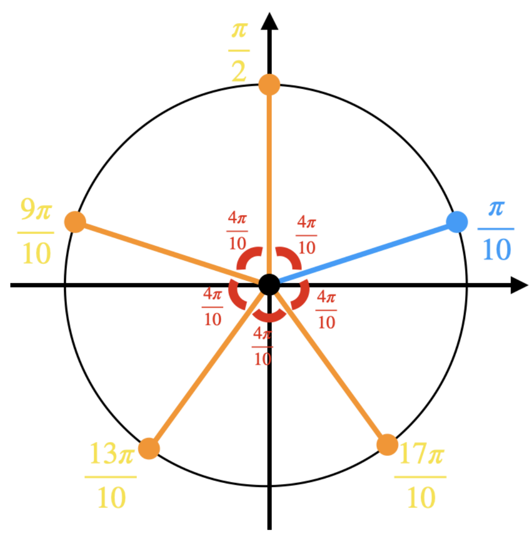
Fundamental Theorem of Algebra:
The number of distinct n-th roots of a complex number (like i) is always n.
So, for example, for $i^{1/4}$ *, there are exactly 4 distinct 4-th roots.*

What about $\sqrt[5]{i}$?

There should be 5 solutions, which is still manageable… Let's try!

Before performing any calculation, though, we can visualize the solutions:

Split the circle into 5 equal pieces $\Longrightarrow$



$$i = e^{i\frac{\pi}{2}}$$

$$\Downarrow$$

$$i^{\frac{1}{5}} = e^{i\frac{\pi}{2}\cdot\frac{1}{5}} = e^{i\frac{\pi}{10}}$$

$\sqrt[5]{i} \equiv x + i\,y$	$x = ? \;\wedge\; y = ?$

$$i = (x + i\,y)^4\,(x + i\,y) =$$

$$= \left[\left(x^4 - 6x^2y^2 + y^4\right) + i\,\left(4x^3y - 4xy^3\right)\right] \cdot (x + i\,y) =$$

$$= \left(x^4 - 6x^2y^2 + y^4 + 4ix^3y - 4ixy^3\right) \cdot (x + i\,y) =$$

$$= x^5 + ix^4y - 6x^3y^2 - 6ix^2y^3 + xy^4 + iy^5 +$$

$$+4ix^4y - 4x^3y^2 - 4ix^2y^3 + 4xy^4 =$$

$$= \left(x^5 - 6x^3y^2 + xy^4 - 4x^3y^2 + 4xy^4\right) +$$

$$+i\,\left(x^4y - 6x^2y^3 + y^5 + 4x^4y - 4x^2y^3\right) =$$

$$= \left(x^5 - 10x^3y^2 + 5xy^4\right) + i\,\left(5x^4y - 10x^2y^3 + y^5\right) \equiv i$$

$$\Longrightarrow \begin{cases} 5x^4y - 10x^2y^3 + y^5 = 1 \\ \quad\wedge \\ x^5 - 10x^3y^2 + 5xy^4 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \boxed{\boxed{5x^4y - 10x^2y^3 + y^5 = 1}} \\ \wedge \\ x^5 - 10x^3y^2 + 5xy^4 = 0 \end{cases}$$

$$\Downarrow$$

$$\boxed{x = 0} \vee \boxed{\boxed{x^4 - 10x^2y^2 + 5y^4 = 0}}$$

$$\begin{tikzcd} & \boxed{x = 0} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\ i & \boxed{y = 1} \arrow[l, Rightarrow] \end{tikzcd}$$

$$\Longrightarrow \begin{cases} 5x^4y - 10x^2y^3 + y^5 = 1 \\ \wedge \\ x^4 - 10x^2y^2 + 5y^4 = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \boxed{5x^4y - 10x^2y^3 + y^5 = 1} \\ \wedge \\ \boxed{t_x^2 - 10\, t_x t_y + 5t_y^2 = 0} \end{cases}$$

$$t_x := x^2$$
$$t_y := y^2$$

$$\Longrightarrow \begin{cases} \boxed{5x^4y - 10x^2y^3 + y^5 = 1} \\ \quad\wedge \\ \boxed{t_x^2 - 10\,t_xt_y + 5t_y^2 = 0} \end{cases}$$

$$t_x^2 - 2\sqrt{5}\,t_xt_y + 5t_y^2 + 2\sqrt{5}\,t_xt_y - 10\sqrt{5}\,t_xt_y = 0 \quad\Longrightarrow$$

$$\Longrightarrow\; t_x^2 - 2\sqrt{5}\,t_xt_y + 5t_y^2 + 2\sqrt{5}\,t_xt_y - 10\sqrt{5}\,t_xt_y = 0 \quad\Longrightarrow$$

$$\Longrightarrow\; \left(t_x - \sqrt{5}\,t_y\right)^2 + \left(2\sqrt{5} - 10\right)t_xt_y = 0 \;\Longrightarrow$$

$$\Longrightarrow\; \left(t_x - \sqrt{5}\,t_y\right)^2 = \left(10 - 2\sqrt{5}\right)t_xt_y \;\Longrightarrow$$

$$\Longrightarrow\; t_x - \sqrt{5}\,t_y = \pm\sqrt{\left(10 - 2\sqrt{5}\right)t_xt_y} \;\Longrightarrow$$

$$\Longrightarrow\; x^2 - \sqrt{5}\,y^2 = \pm\,xy\sqrt{10 - 2\sqrt{5}} \;\Longrightarrow$$

$$\Longrightarrow \begin{cases} \boxed{x^2 - xy\sqrt{10 - 2\sqrt{5}} - y^2\sqrt{5} = 0} \quad \texttt{(I)} \\ \quad\wedge \\ \boxed{x^2 + xy\sqrt{10 - 2\sqrt{5}} - y^2\sqrt{5} = 0} \quad \texttt{(II)} \end{cases} \qquad \begin{aligned} t_x &:= x^2 \\ t_y &:= y^2 \end{aligned}$$

(I):

$$x_{1,2}=\frac{y\sqrt{10-2\sqrt{5}}\pm\sqrt{\left(10-2\sqrt{5}\right)y^2+4y^2\sqrt{5}}}{2}=$$

$$=\begin{array}{l}\nearrow\ =\dfrac{y_1\sqrt{10-2\sqrt{5}}+y_1\sqrt{10+2\sqrt{5}}}{2}=\\[2ex] \searrow\ =\dfrac{y_2\sqrt{10-2\sqrt{5}}-y_2\sqrt{10+2\sqrt{5}}}{2}=\end{array}$$

$$\begin{array}{l}\nearrow\ =\dfrac{y_1\sqrt{10-2\sqrt{5}}+y_1\sqrt{10+2\sqrt{5}}}{2}=\ \longrightarrow\\[2ex] \searrow\ =\dfrac{y_2\sqrt{10-2\sqrt{5}}-y_2\sqrt{10+2\sqrt{5}}}{2}=\ \longrightarrow\end{array}$$

①

$$\longrightarrow\ =\boxed{y_1\left(\frac{\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}}{2}\right)}$$

$$\longrightarrow\ =\boxed{y_2\left(\frac{\sqrt{10-2\sqrt{5}}-\sqrt{10+2\sqrt{5}}}{2}\right)}$$

②

$$
① \begin{cases} x_1 = y_1 \left(\dfrac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2} \right) \\ \qquad \wedge \\ \boxed{5x_1^4 y_1 - 10x_1^2 y_1^3 + y_1^5 = 1} \end{cases}
$$

$$\Downarrow$$

$$
5y_1^4 \left(\frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2} \right)^4 y_1 - 10y_1^2 \left(\frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2} \right)^2 y_1^3 + y_1^5 = 1
$$

$$
5y_1^4 \left(\underbrace{\frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2}}_{:= A} \right)^4 y_1 - 10y_1^2 \left(\underbrace{\frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2}}_{:= A} \right)^2 y_1^3 + y_1^5 = 1
$$

$$\Downarrow$$

$$
5A^4 y_1^5 - 10A^2 y_1^5 + y_1^5 = 1
$$

$$\Downarrow$$

$$
y_1^5 \left(5A^4 - 10A^2 + 1 \right) = 1
$$

$$y_1^5\left(5A^4-10A^2+1\right)=1$$

$$\Downarrow$$

$$\boxed{y_{1_{(I)}}=\frac{1}{\sqrt[5]{5A^4-10A^2+1}}} \Longrightarrow \boxed{x_{1_{(I)}}=\frac{A}{\sqrt[5]{5A^4-10A^2+1}}}$$

$$A\equiv\frac{\sqrt{10-2\sqrt{5}}+\sqrt{10+2\sqrt{5}}}{2}$$

$$② \begin{cases} x_2=y_2\left(\underbrace{\frac{\sqrt{10-2\sqrt{5}}-\sqrt{10+2\sqrt{5}}}{2}}_{:=B}\right) \\ \wedge \\ \boxed{5x_2^4y_2-10x_2^2y_2^3+y_2^5=1} \end{cases}$$

$$\Downarrow$$

$$5y_2^4\,(B)^4\,y_2-10y_2^2\,(B)^2\,y_2^3+y_2^5=1$$

$$5B^4y_2^5-10B^2y_2^5+y_2^5=1$$

$$5B^4y_2^5 - 10B^2y_2^5 + y_2^5 = 1$$

$$y_2^5\left(5B^4 - 10B^2 + 1\right) = 1$$

$$\Downarrow$$

$$\boxed{y_{2_{(I)}} = \frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}} \quad \Longrightarrow \quad \boxed{x_{2_{(I)}} = \frac{B}{\sqrt[5]{5B^4 - 10B^2 + 1}}}$$

$$B \equiv \frac{\sqrt{10 - 2\sqrt{5}} - \sqrt{10 + 2\sqrt{5}}}{2}$$

(I): ✓ OK

(II): $\boxed{x^2 + xy\sqrt{10 - 2\sqrt{5}} - y^2\sqrt{5} = 0}$

$$x_{1,2} = \frac{-y_{1,2}\sqrt{10 - 2\sqrt{5}} \pm \sqrt{y_{1,2}^2\left(10 - 2\sqrt{5}\right) + 4\,y_{1,2}^2\sqrt{5}}}{2} =$$

$$x_{1,2} = \frac{-y_{1,2}\sqrt{10-2\sqrt{5}} \pm \sqrt{y_{1,2}^2\left(10-2\sqrt{5}\right)+4\,y_{1,2}^2\sqrt{5}}}{2} =$$

$$= \frac{-y_1\sqrt{10-2\sqrt{5}} + y_1\sqrt{10+2\sqrt{5}}}{2} =$$

$$=$$

$$= \frac{-y_2\sqrt{10-2\sqrt{5}} - y_2\sqrt{10+2\sqrt{5}}}{2} =$$

$$= \frac{-y_1\sqrt{10-2\sqrt{5}} + y_1\sqrt{10+2\sqrt{5}}}{2} = \boxed{y_1 \overbrace{\frac{\sqrt{10+2\sqrt{5}} - \sqrt{10-2\sqrt{5}}}{2}}^{A}} \tag{1}$$

$$= \frac{-y_2\sqrt{10-2\sqrt{5}} - y_2\sqrt{10+2\sqrt{5}}}{2} = \boxed{-y_2 \underbrace{\frac{\sqrt{10-2\sqrt{5}} - \sqrt{10+2\sqrt{5}}}{2}}_{B}} \tag{2}$$

① $\begin{cases} x_1 = -B\,y_1 \\ \quad\wedge \\ \boxed{5x_1^4y_1 - 10x_1^2y_1^3 + y_1^5 = 1} \end{cases}$

$$\Downarrow$$

$$5\,B^4\,y_1^4y_1 - 10\,B^2\,y_1^2y_1^3 + y_1^5 = 1$$

$$5\,B^4\,y_1^5 - 10\,B^2\,y_1^5 + y_1^5 = 1 \implies \boxed{y_{1_{(II)}} = \frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}} \quad \boxed{x_{1_{(II)}} = \frac{-B}{\sqrt[5]{5B^4 - 10B^2 + 1}}}$$

② $\begin{cases} x_2 = -A\,y_2 \\ \quad\wedge \\ \boxed{5x_2^4y_2 - 10x_2^2y_2^3 + y_2^5 = 1} \end{cases}$

$$\Downarrow$$

$$(\ldots)$$

$$\boxed{y_{2_{(II)}} = \frac{1}{\sqrt[5]{5A^4 - 10A^2 + 1}}} \qquad \boxed{x_{2_{(II)}} = \frac{-A}{\sqrt[5]{5A^4 - 10A^2 + 1}}}$$

$\therefore \sqrt[5]{i}$ has 5 solution: $\therefore \sqrt[5]{i} = \begin{cases} i \\ \left(\frac{A}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) & \boxed{1,(I)} \\ \left(\frac{B}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) & \boxed{2,(I)} \\ \left(\frac{-B}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) & \boxed{1,(II)} \\ \left(\frac{-A}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) & \boxed{2,(II)} \end{cases}$

$$\boxed{A \equiv \frac{\sqrt{10 - 2\sqrt{5}} + \sqrt{10 + 2\sqrt{5}}}{2}} \qquad \boxed{B \equiv \frac{\sqrt{10 - 2\sqrt{5}} - \sqrt{10 + 2\sqrt{5}}}{2}}$$

Let's try to find the visual representation (i.e., angles) of this result in the unit circle on the $\mathbb{C}$−plane:

$$\sqrt[5]{i} = \begin{cases} i = cos\theta_1 + i\,sin\theta_1 = e^{i\theta_1} \Longrightarrow \boxed{\theta_1 = \frac{\pi}{2}} \\ \left(\frac{A}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) = cos\theta_2 + i\,sin\theta_2 = e^{i\theta_2} & \Longrightarrow \\ \left(\frac{B}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) = cos\theta_3 + i\,sin\theta_3 = e^{i\theta_3} & \Longrightarrow \\ \left(\frac{-B}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5B^4 - 10B^2 + 1}}\right) = cos\theta_4 + i\,sin\theta_4 = e^{i\theta_4} & \Longrightarrow \\ \left(\frac{-A}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) + i\left(\frac{1}{\sqrt[5]{5A^4 - 10A^2 + 1}}\right) = cos\theta_5 + i\,sin\theta_5 = e^{i\theta_5} & \Longrightarrow \end{cases}$$

$$\implies \quad \theta_2 = tan^{-1}\left(\frac{1}{A}\right) = \boxed{\frac{\pi}{10}}$$

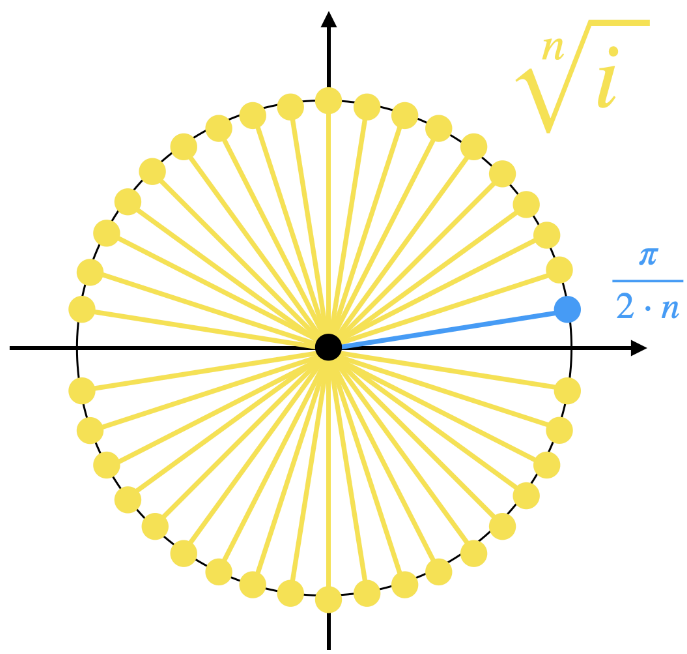
$$\implies \quad \theta_3 = tan^{-1}\left(\frac{1}{B}\right) = \boxed{\frac{17\pi}{10}}$$

$$\implies \quad \theta_4 = tan^{-1}\left(\frac{1}{-B}\right) = \boxed{\frac{13\pi}{10}}$$

$$\implies \quad \theta_5 = tan^{-1}\left(\frac{1}{-A}\right) = \boxed{\frac{9\pi}{10}}$$

$$\lim_{n\to\infty} i^{\frac{1}{n}} = 1$$

$\{i^{\frac{1}{n}} : n \in \mathbb{N}\}$ *is dense in* $\mathbb{S}^1$



It is important to emphasize that the fact that this set is dense in the unit circle does not mean that all points on the circle can be described by the n-th root of the imaginary unit (for n natural). All it means is that we found a one-to-one (injective) mapping from the natural numbers to the unit circle in the complex plane, but this mapping is not onto (surjective), otherwise we would be able to create a bijection from the natural numbers to complex numbers. This would be a contradiction, since complex numbers are innumerable and real numbers themselves are innumerable.

Some unanswered questions here would be:

1. What is the angle between these solutions in terms of n ?
2. We notice 2 distinct patterns depending on whether n is even or odd. Can you establish their main difference?
3. What happens if we pick n to be negative? What about irrational?

   These are just a few questions that come to my mind. But I do not know their answer. Anyway, I think it is a nice challenge for you guys to try to prove some of them. Let me know your thoughts in the comment section below.

Some important concepts and conclusion:

1. **Topological Explanation of Density**:

Density in the unit circle means that for any point on the circle and any arbitrarily small distance, there is at least one element from the set $\{i^{1/n}\}$ that lies within that distance of the point. However, being dense does ***not*** mean that the set covers all points on the circle—only that it can get arbitrarily close to any point.

2. **Role of the Real and Complex Number Set**:

The **real numbers** (and thus the complex numbers) are **uncountable**, as demonstrated by Cantor's diagonal argument, which shows that there are more real numbers than there are natural numbers. This highlights why the mapping from natural numbers to the complex numbers cannot be surjective—there are simply "too many" complex numbers for a mapping from natural numbers to cover them all.

3. **Visualizing Roots of Unity**:

Each n-th root of i lies on the unit circle, and as n increases, these roots start to "fill" the circle more and more densely. However, there are infinite points on the unit circle that cannot be precisely represented as an n-th root of i.

4. **Connection to Dirichlet's Approximation Theorem**:

This density is similar in spirit to results like **Dirichlet's Approximation Theorem** in number theory, which says that any real number can be approximated arbitrarily well by rational numbers with a small denominator. Here, the roots $i^{1/n}$ (which are specific points in the complex plane) can similarly approximate any point on the unit circle.

5. **Further Implications in Complex Dynamics**:

This can also be connected to **complex dynamics** or **iterated function systems**, where understanding the distribution and density of certain points in the complex plane (like roots of unity or fractional powers) plays a key role in studying the behavior of complex functions.

Comment: Please, if you find this document useful, let us know. Or if you found typos and things to improve, let us know as well. Your feedback is very important to us, since we are working hard every single day to deliver the best material possible. For contacting us: dibeos.contact@gmail.com