The first derivative that you probably learned is the one of the function $f(x) = x^n$, which is:

$$f'(x) = n \cdot x^{n-1}$$

, as long as $n \in \mathbb{N}$ and $n \neq 0$.

But how can you prove that? Well, you can calculate a lot of things in Calculus only knowing this rule, but learning how to derive it, from the derivative's own definition, can be really useful in order to exercise your muscles of "proving stuff in math". And it is not as easy as it seems, at all.

By definition,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

, for $h \in \mathbb{R}$.

Now, considering our function $f(x) = x^n$, we can write its derivative using the definition of a limit:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

In order to simplify this expression we need to use the Binomial Theorem. Let's introduce it by noticing the following:

$$(x + h)^1 = (x + h) \ ;$$
$$(x + h)^2 = x^2 + 2hx + h^2 \ ;$$
$$(x + h)^3 = x^3 + 3hx^2 + 3h^2x + h^3 \ ; \ (...)$$
$$(x + h)^n = \ ???$$

Well, let's find a formula to express $(x + h)^n$.

$$(x+h)^n = ???$$

▲

$$\boxed{(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k} \text{, where } \overbrace{\binom{n}{k}}^{\text{Binomial Coefficient}} := \frac{n!}{(n-k)!\,k!}$$

$$\boxed{\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}}$$

Let's show that the 'tetrahedron equation', and therefore the binomial theorem, is indeed valid. We will do it by induction.

$\vdash$ ▲ :

▲ $\boxed{(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k}$ $\quad \binom{n}{k} := \frac{n!}{(n-k)!\,k!}$

Base case: $(n = 1)$

$$(x+h)^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} h^k = \binom{1}{0} x^{1-0} h^0 + \binom{1}{1} x^{1-1} h^1$$

$$= \overset{=1}{\frac{1!}{(1-0)!0!}} x^1 \overset{=1}{h^0} + \overset{=1}{\frac{1!}{(1-1)!1!}} \overset{=1}{x^0} h^1$$

$$= x + h$$

The base case holds. Now we proceed to the inductive step. So, we assume that the 'tetrahedron equation' is true for n and we want to prove it for $n + 1$:

$\vdash \blacktriangle :$

$\blacktriangle$ $\boxed{(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k}$

$\binom{n}{k} := \frac{n!}{(n-k)!\,k!}$

Inductive step: (assume that $\blacktriangle$ is true, and $\vdash$ for $n+1$)

$$(x+h)^{n+1} = (x+h)\cdot(x+h)^n = (x+h)\cdot\sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^k$$

$$= x\sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^k + h\sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^k$$

$$= x\sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^k + h\sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^k =$$

$$= \sum_{k=0}^{n}\binom{n}{k} x^{n+1-k} h^k + \sum_{k=0}^{n}\binom{n}{k} x^{n-k} h^{k+1} =$$

for the second sum only

Change of variables:

$j = k+1$

$k = j-1$

$$= \sum_{k=0}^{n}\binom{n}{k} x^{n+1-k} h^k + \sum_{j=1}^{n}\binom{n}{j-1} x^{n-j+1} h^j =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{j=1}^{n} \binom{n}{j-1} x^{n-j+1} h^j =$$

$$= \binom{n}{0} x^{n+1-0} \underbrace{h^0}_{1=} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{j=1}^{n} \binom{n}{j-1} x^{n-j+1} h^j =$$

$$= \frac{\overset{1}{\cancel{n!}}}{(n \cancel{-} 0)!\,\underbrace{0!}_{=1}} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{j=1}^{n} \binom{n}{j-1} x^{n-j+1} h^j =$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{k\,\cancel{j}=1}^{n} \binom{n}{\underset{k}{\cancel{j}}-1} x^{n-\overset{k}{\cancel{j}}+1} h^{\overset{k}{\cancel{j}}} =$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{k\,\cancel{j}=1}^{n} \binom{n}{\underset{k}{\cancel{j}}-1} x^{n-\overset{k}{\cancel{j}}+1} h^{\overset{k}{\cancel{j}}} =$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} h^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n-k+1} h^k =$$

$$= x^{n+1} + \sum_{k=1}^{n} \left\{ \left[\binom{n}{k} + \binom{n}{k-1} \right] \cdot x^{(n+1)-k} h^k \right\}$$

$$= x^{n+1} + \sum_{k=1}^{n} \left\{ \left[\binom{n}{k} + \binom{n}{k-1} \right] \cdot x^{(n+1)-k} h^k \right\}$$

$$\frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!} =$$

$$= \frac{n!}{(n-k)!\,k\cdot(k-1)!} + \frac{n!}{(n-k+1)\cdot(n-k)!\,(k-1)!} =$$

$$= \frac{(n-k+1)\cdot n! + n!\cdot k}{(n-k+1)\cdot(n-k)!\cdot k\cdot(k-1)!} = \frac{(n+1)\cdot n!}{\big((n+1)-k\big)!\cdot k!} =$$

$$= \frac{(n+1)!}{\big((n+1)-k\big)!\cdot k!} = \binom{n+1}{k}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} \cdot x^{(n+1)-k} h^k =$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} \cdot x^{(n+1)-k} h^k$$

$$\therefore \boxed{(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k}$$

$\blacksquare$

Perfect! *(BTW, consider becoming a member of the channel!) Thanks!*

Now, going back to where we stopped, we can confidently substitute the 'tetrahedron equation' (that we just proved) into the 'icosahedron equation'.

Binomial Coefficient

$$\boxed{(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k}\ \text{, where } \binom{n}{k} := \frac{n!}{(n-k)!\,k!}$$

$$\boxed{\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}}$$

This gives us that:

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k - \cancel{x^n}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k - \cancel{x^n}}{h} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[\binom{n}{1} x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \ldots + \binom{n}{n} x^{n-n} h^n \right] =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[\frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}\, 1!} x^{n-1} h^1 + \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)!}\, 2!} x^{n-2} h^2 + \right.$$

$$\left. + \ldots + \frac{\cancel{n!}}{\underbrace{(n-n)!}_{1}\, \cancel{n!}} \overbrace{x^0}^{1 =} h^n \right] =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[n \cdot x^{n-1} h + \frac{n \cdot (n-1)}{2} x^{n-2} h^2 + \ldots + h^n \right] =$$

$$= \lim_{h \to 0} \left[n \cdot x^{n-1} + \underbrace{\frac{n \cdot (n-1)}{2} x^{n-2} h}_{0} + \underbrace{\ldots}_{0} + \underbrace{h^{n-1}}_{0} \right] = \; n \cdot x^{n-1}$$

$$\frac{d}{dx} \left(x^n \right) = n \cdot x^{n-1}$$

$\blacksquare$

And that's exactly the result we expected! We can conclude that, indeed, the derivative of x^n is $n \cdot x^{n-1}$.

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