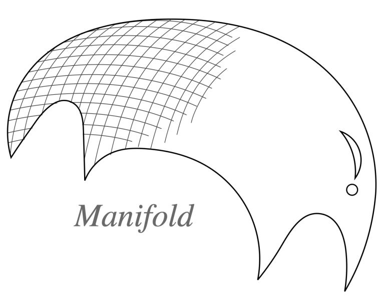
Schwarzschild solution

By DiBeos

$$\boxed{G_{\mu\nu} \equiv \overbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}}^{\text{curvature of the manifold (spacetime)}} = \overbrace{8\pi T_{\mu\nu}}^{\text{distribution of matter}}}$$

This is definitely one of the most important equations you will ever study in your life. The left-hand-side represents the *curvature of the manifold* and the right-hand-side the *distribution of matter* in it. It is called *Einstein's equation* (or *Einstein field equations – EFE*), and this manifold is often described as the spacetime around us.

Just to give you an intuition behind what a manifold is, think of it as a shape or surface that can be stretched and bent but looks flat if you zoom in close enough. For example, the surface of the Earth is a curved 2D manifold, but it feels flat when you're standing on it. In this context, spacetime is a 4D manifold that describes the "stage" where everything happens.



Do not let this equation intimidate you. We will focus on its mathematics here (and more specifically on one of its solutions), however since this equation is often used to model applied problems in nature, let us very briefly describe its physical interpretation.

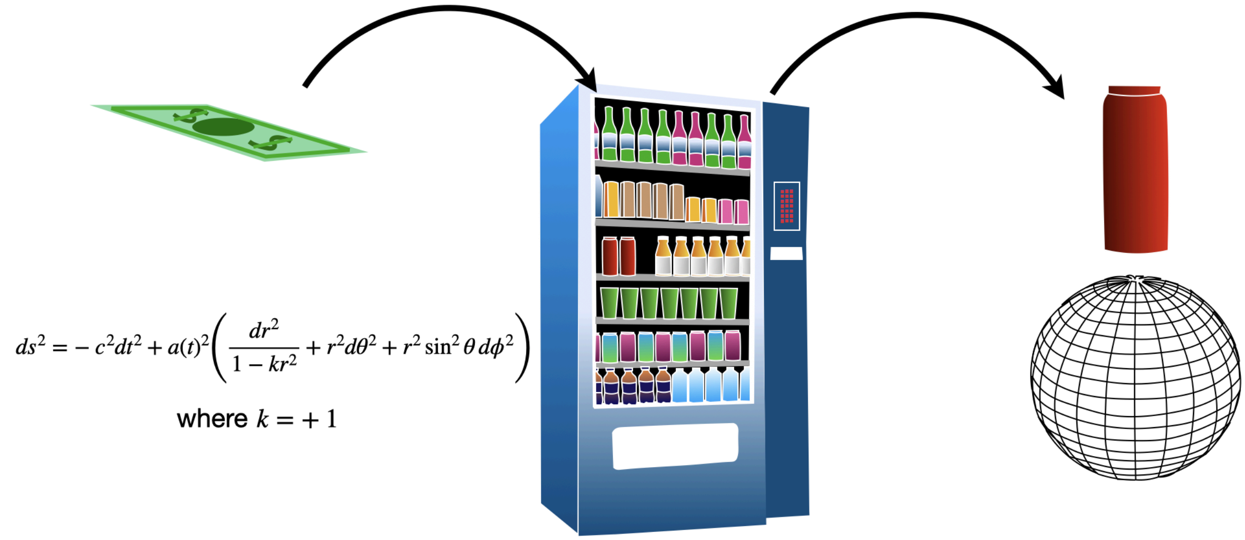
From the mathematics point of view, *General Relativity* (*GR*) is just a combination of *Differential Geometry, Tensor Calculus, Topology, Algebraic Geometry, Lie Groups and Lie Algebras, Functional Analysis* and *Partial Differential Equations* (*PDEs*). Yeah… just that… I know, it is a lot! But here we will focus on a specific solution of this nonlinear PDE, namely the *Schwarzschild solution*:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\vee$$

$$g = \begin{bmatrix} -\left(1 - \frac{r_s}{r}\right) c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

A solution of this PDE is a metric tensor that defines the geometry of the manifold (spacetime). A differential equation can be seen, in a very simplistic way, as a soda machine. A soda machine works this way: you give it an *input* (usually money, or a strong kick, depending on your mood) and it "spits out" an *output* (usually the desired soda can, depending on its mood). A PDE, like the one we are interested in, works in a similar way. The input in this case is one of its correct mathematical solutions, and the output might be a geometrical interpretation of the result in the manifold we are working with. In the specific case of GR, this manifold is *4*-dimensional, such that *3* dimensions represent *space* and *1* represents *time*. In GR, also, this geometrical interpretation (or the output in our analogy) is a specific shape of the spacetime manifold. So, each input, or solution of Einstein's equation, gives a different output, or geometry of spacetime.



$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

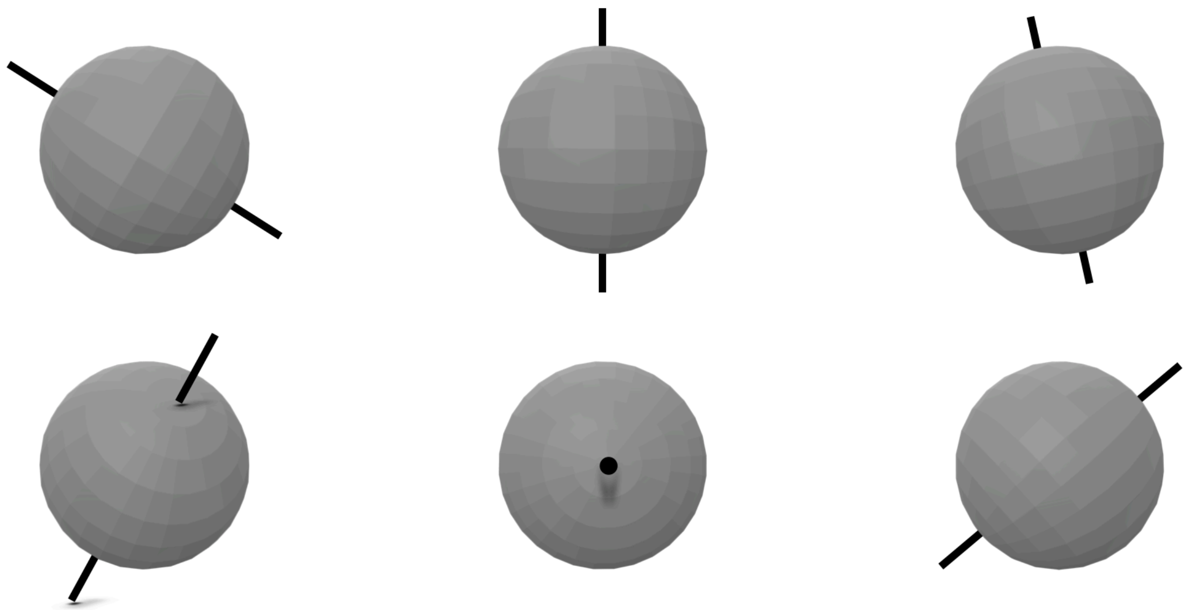
The Schwarzschild solution is just one of these possible inputs. The very first one to be discovered. We will not prove that the metric shown above is indeed a solution of the Einstein field equations, because the calculations are really long, but we will analyse and interpret each term. Before anything, though, let's see the assumptions used by Schwarzschild to find this specific solution:

(I) Spherical symmetry:

The manifold is assumed to be invariant under the group of rotations $SO(3)$ (for more details on this group check out the last pages of this document).

Group of Actions SO(3): This is the set of all possible rotations in 3D space. Imagine spinning a ball around any axis — SO(3) describes all the ways you can do that without changing the ball's size or shape.

SO(3)



This means that the metric is unchanged under transformations corresponding to any rotation in *3D*.

The metric coefficients $g_{\mu\nu}$ can depend only on the radial coordinate r (i.e. the distance from the center), except for the angular part, which includes a dependence on $sin^2\theta$ arising from the spherical coordinate system.

So the metric has a specific form:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + \boxed{r^2 d\theta^2 + r^2\, sin^2\theta\, d\phi^2}$$

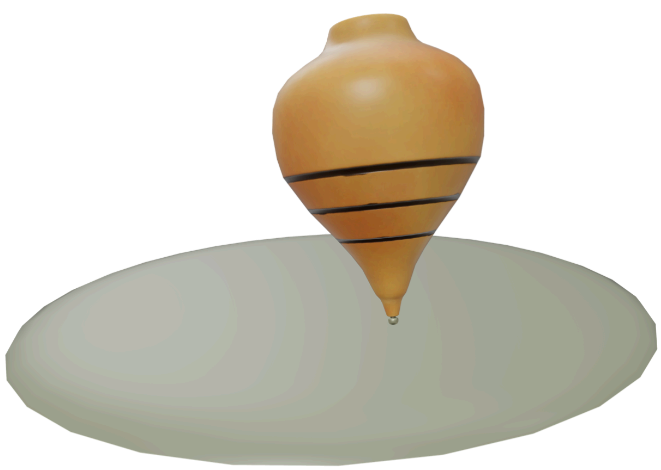
These terms reflect the spherical symmetry

$$\vee$$

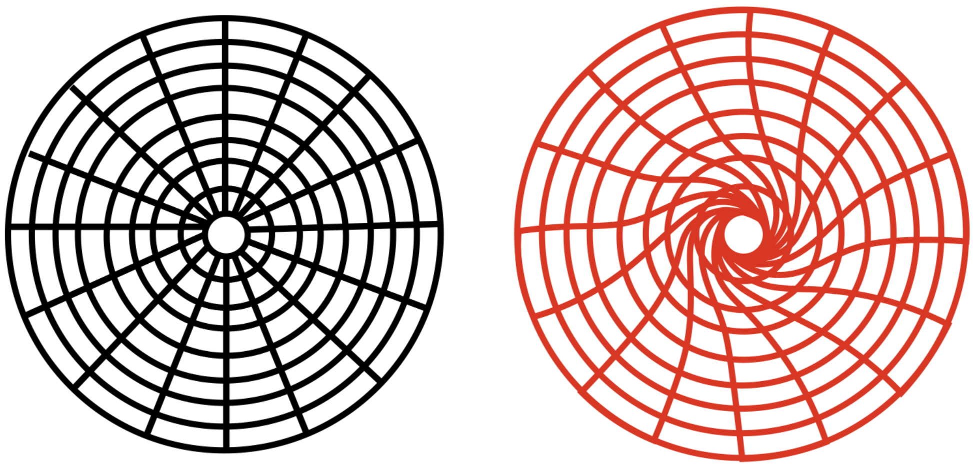
$$g = \begin{bmatrix} -\left(1 - \frac{r_s}{r}\right) c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\, sin^2\theta \end{bmatrix}$$

(II) <u>No rotation</u>:

To talk about this one, we need to define Angular Momentum first. This is the "spin" of an object. If you spin a top or a planet rotates, it has angular momentum — it's the way the object keeps spinning unless something stops it.



In this metric the angular momentum is zero, which is illustrated in the fact that all the terms not in the diagonal of the matrix are zero, such as $g_{t\phi}$, for example. If this was not the case, there would be something called *frame dragging*, which by the way, is present in the famous *Kerr solution*. We will talk more about it later on.



$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \frac{1}{1 - \frac{r_s}{r}}dr^2 + \boxed{r^2d\theta^2 + r^2\,sin^2\theta\,d\phi^2}$$

no angular velocity ω

Counterexample:
(Flat rotating cylindrical coordinates)

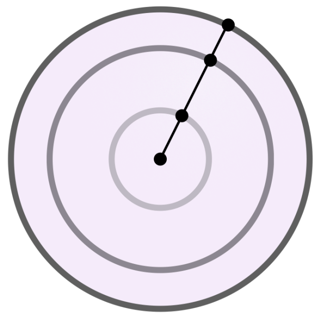
$$ds^2 = \ldots + \boxed{\frac{r^2}{1 - \omega^2\frac{r^2}{c^2}}d\phi^2} + \ldots \implies \textit{rotation!}$$

(III) Radial symmetry:

This is a consequence of the spherical symmetry. It means that the metric depends only on the magnitude of r , not on direction.

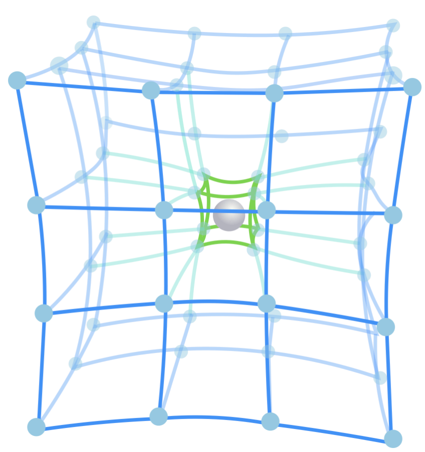
$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \boxed{\frac{1}{1-\frac{r_s}{r}}}dr^2 + r^2d\theta^2 + r^2\, sin^2\theta\, d\phi^2$$

no dependence on θ or ϕ , only on r



(IV) <u>Vacuum</u>:

This one may sound a little weird at first, since vacuum is a situation in which there is no matter involved, however the spherical geometry of our manifold is the direct consequence of the spherical geometry of a body at its center (like a planet or a star). So, it is important to notice that this metric solution describes only the curvature outside the body, not with respect to a point in its interior.



Mathematically, the energy-momentum tensor $T_{\mu\nu}$ is *zero* everywhere. This condition simplifies a lot the EFE:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi \cancelto{0}{T_{\mu\nu}} \implies$$

$$\implies \quad \boxed{2R_{\mu\nu} = Rg_{\mu\nu}}$$

In terms of Differential Geometry, the spacetime is *Ricci-flat*:

$$\boxed{R_{\mu\nu} = 0}$$

Proof:

Let's take the trace of both sides of the equation $R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}$ *:*

$$\underbrace{g^{\mu\nu}R_{\mu\nu}}_{= R} = \frac{1}{2}R\underbrace{g^{\mu\nu}g_{\mu\nu}}_{= 4} \implies$$

$$\implies R = \frac{1}{2}R \cdot 4 \implies R = 2R \implies$$

$$\implies R = 0 \implies R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} = 0 \quad \square$$

(V) <u>No electrical charge</u>:

As a consequence, the metric $g_{\mu\nu}$ remains purely gravitational, and there is no *electromagnetic field tensor* $F_{\mu\nu}$.

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

So, let us see once again the full version of the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

spherical coordinates

$$g = \begin{array}{cc} \begin{array}{cccc} \quad dt \quad & \quad dr \quad & \; d\theta \; & \quad d\phi \quad \end{array} & \\ \begin{bmatrix} -\left(1 - \frac{r_s}{r}\right) c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{r_s}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} & \begin{array}{c} dt \\ dr \\ d\theta \\ d\phi \end{array} \end{array}$$

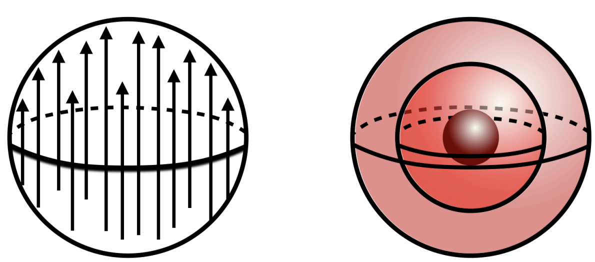
It is important to understand that in Differential Geometry the metric's components determine the *distances*, *angles* and *causal structures*. When the metric tensor $g_{\mu\nu}$ has only diagonal terms (just like in the Schwarzschild solution) it implies certain symmetries and simplifications, both mathematically and physically. Cases with non-zero off-diagonal terms reflect more complex phenomena, such as rotation, non-static spacetimes, or the presence of additional fields, like an electromagnetic field, for example.

The fact that the Schwarzschild solution is a diagonal metric indicates that the coordinate basis vectors (so, ∂_t, ∂_r, ∂_θ and ∂_ϕ) are *orthogonal* to each other. Another alternative metric that satisfies EFE, and that is diagonal, is the Friedmann-Lemaître-Robertson-Walker metric for cosmology, which assumes a *homogeneous* and *isotropic* universe:

$$ds^2 = -\,dt^2 + a(t)^2\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right)\right]$$

$$\vee$$

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & a(t)^2\, r^2 & 0 \\ 0 & 0 & 0 & a(t)^2\, r^2\, sin^2\theta \end{bmatrix}$$



homogeneous **isotropic**

Off-diagonal terms imply that the coordinate basis vectors are no longer orthogonal. For example, the Kerr solution has a non-zero term $\left(g_{t\phi}\right)$, which introduces coupling between the time and angular components:

Gravitational constant $\left(\approx 6{,}67 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2}\right)$

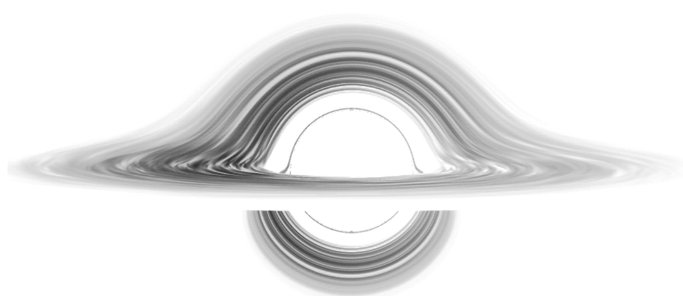
Mass of the blackhole

Specific angular momentum (spin parameter) of the blackhole: $a = \frac{J}{M}$, where J is the angular momentum

$$\boxed{g_{t\phi} = -\frac{2GMa\sin^2\theta}{c^2 r}}$$

Speed of light in vacuum $\left(\approx 3 \cdot 10^8\ m/s\right)$

The Kerr metric describes a rotating blackhole.



The discovery of what we now know as the Schwarzschild solution was very surprising at the time because the EFE are a set of complex, highly nonlinear

PDEs. Just to illustrate how complex these equations are, notice the “Matryoshka dolls effect” of trying to express it just in terms of the metric tensor $g_{\mu\nu}$:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu}} \qquad \begin{array}{c} \left(\mu, \nu \in \{0, 1, 2, 3\}\right) \\ \lor \\ \left(\mu, \nu \in \{t, r, \theta, \phi\}\right) \end{array}$$

$$\boxed{\partial_\mu := \frac{\partial}{\partial \mu}}$$

Ricci tensor:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu}$$

Stress-energy tensor:

$$T_{\mu\nu} = \begin{bmatrix} \text{Energy density} & \text{Energy flux} \\ \text{Momentum flux} & \text{Stress} \end{bmatrix}$$

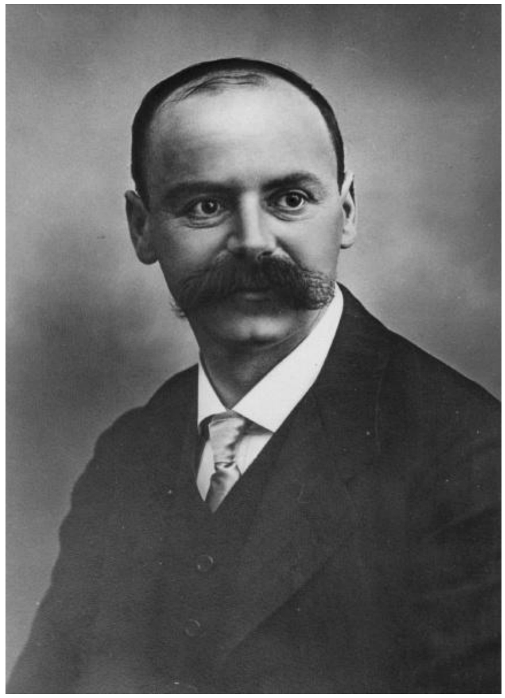
Riemann curvature tensor:

$$R^{\lambda}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\mu\rho} - \partial_\rho \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\sigma}_{\mu\rho} \Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\lambda}_{\sigma\rho}$$

Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

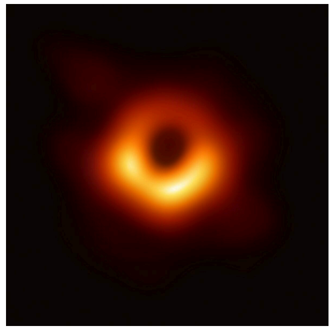
Karl Schwarzschild discovered an exact solution just a few months after Einstein published the final form of his equations.



This solution revealed a *singularity* at $r_s = \frac{2GM}{c^2}$, which later became known as the *Schwarzschild radius*.

A singularity is a point where something breaks down completely—like dividing by zero in math. In spacetime, a singularity is a place where gravity becomes so strong that the usual rules of physics stop working, like at the center of a black hole.

At the time, the interpretation of this singularity was not understood. The idea that spacetime could “end” or that a region could exist where nothing, not even light, could escape, was weird. Today, that’s what we call a blackhole.



The Schwarzschild radius defines the boundary of a blackhole (the so-called *event horizon*). This means that if an object's radius is smaller than r_s, then it becomes a blackhole.

The Schwarzschild solution also shows that clocks run slower near massive objects. This can be concluded from the following limit:

$$\lim_{r\to r_s} g_{tt} = \lim_{r\to r_s} -\left(1 - \frac{2GM}{c^2 r}\right) = 0$$

This implies that time essentially "stops" for an outside observer.

SO(3) – Special orthogonal group in 3 dimensions: *(just a few words…)*

This is the set of all rotations in 3D space about the origin. Since the elements of this set satisfy the properties shown below, this is more than a set – it is a *Lie group*, which combines the structures of both a group and a smooth manifold:

1. Group properties:

(a) *Closure*: The product of any two elements R_1, $R_2 \in SO(3)$ (which are rotation matrices) is also an element, so another rotation matrix in $SO(3)$.

$$R_1 R_2 \in SO(3)$$

(b) *Identity*: There is a rotation matrix that "does nothing", called the *identity matrix*: $I \in SO(3)$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) *Inverse*: Each rotation matrix $R \in SO(3)$ has an inverse, which in this case is its *transpose*:

$$R^{-1} = R^T$$

(d) *Associativity*: Let R_1, R_2, $R_3 \in SO(3)$, then:

$$\left(R_1 R_2\right) R_3 = R_1 \left(R_2 R_3\right)$$

2. <u>Lie group properties</u>:

(a) *Smooth manifold*: $SO(3)$ is not just a discrete set of matrices – it forms a smooth $3D$ surface (manifold) in the $9D$ space of all 3×3 matrices.

(b) *Differentiable structure*: The group operations (so, matrix multiplication and inversion) are smooth, which means that they can be differentiated.

Beyond that, $SO(3)$ is *special* in the sense that, for any $R \in SO(3)$, we have that $det(R) = 1$.

As said earlier, $SO(3)$ is the group of rotations, and we use it to model rotations in physics. In the day-to-day life when you rotate a pencil in the $3D$ space around it, the pencil does not grow or shrink in length. Mathematically, this is expressed by the fact that rotations $R \in SO(3)$ preserve the Euclidean norm of vectors:

$$\left|\left| R\vec{v} \right|\right| = \left|\left| \vec{v} \right|\right| \quad , \forall\ vector\ \vec{v}$$

Let us finish with a concrete example involving and element of $SO(3)$.

We want to rotate a $3D$ vector

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

by an angle θ around the z-axis. In order to achieve that we will use one of the elements of $SO(3)$, namely the rotation matrix:

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

N.B.: This matrix satisfies the properties of $SO(3)$*:*

(I) $R^T R = I$

(II) $det(R) = 1$

(check it on your own!!! 😀)

For example, to rotate the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

by $\theta = 90°$ we "act" $R_z(90°)$ on $\vec{v}$:

$$\vec{v}' = R_z(90°)\,\vec{v} = \begin{bmatrix} cos(90°) & -sin(90°) & 0 \\ sin(90°) & cos(90°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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