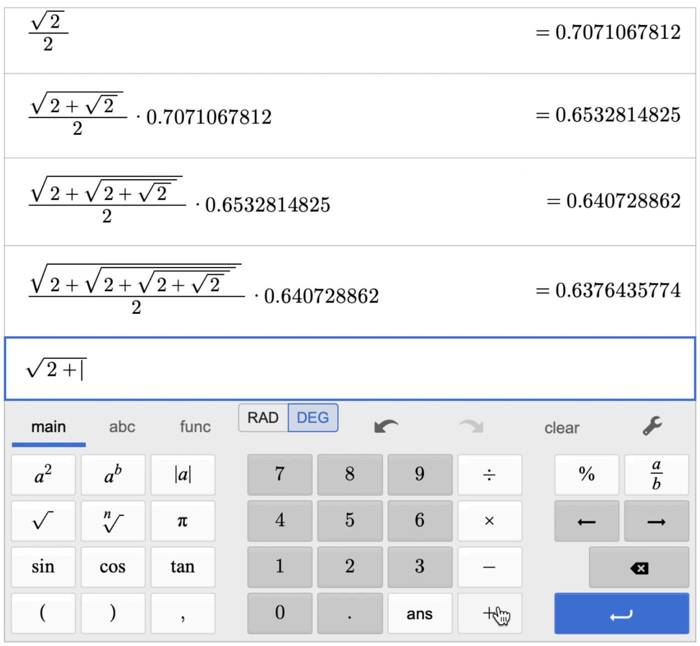
Why Some Formulas for π Are Faster Than Others

by DiBeos

Open a calculator and type: $\sqrt{2}$. Now divide it by 2 and save the result. Then, type $\sqrt{2}$ plus 2. Calculate the square root of all of it, and then divide it by 2. Now, multiply by the number you saved before. Continue this process over and over again.



You will notice that after each iteration the result gets closer and closer to $0.63661977...$ Actually, if we could keep on multiplying terms infinite times, this would converge exactly to $\frac{2}{\pi} \approx 0.63661977...$

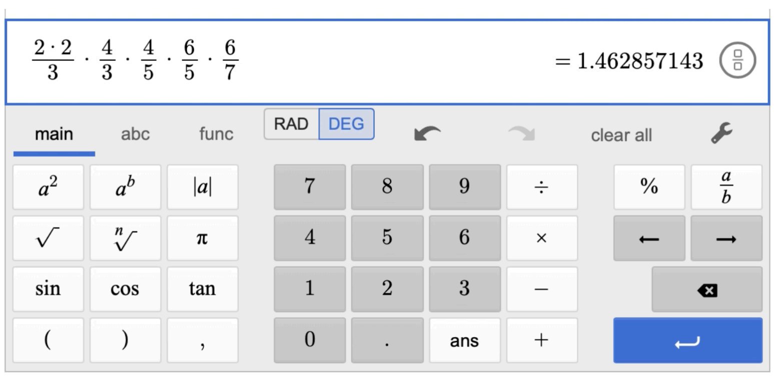
$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \quad \cdots$$

This formula was discovered in 1593 by the French mathematician François Viète.

It is a pretty shocking infinite product because it shows that π can be calculated just from the number 2 by a succession of additions, divisions, multiplications and square roots. But the really surprising part of it (at least for the sixteenth century) was the 3 dots at the end of it. This was the first time that an infinite process was explicitly expressed as a mathematical formula.

Soon after, in 1650, the English mathematician John Wallis discovered another one involving π .

Open your calculator again. Type 2 times 2, divided by 3, times 4, divided by 3, times 4, divided by 5, times 6, divided by 5, times 6, divided by 7, and so on… Basically pairs of even numbers multiplied together, divided by pairs of odd numbers multiplied together. The result gets closer and closer to $1.570796...$ Or, more precisely, $\frac{\pi}{2}$.



$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}$$

By the way, it was Wallis who proposed this symbol ∞ for infinity.

We will show the detailed proofs of these formulas at the end of this document.

We can plot a graph of the first formula as a function of each of the terms $x_1, x_2, x_3, \ldots$

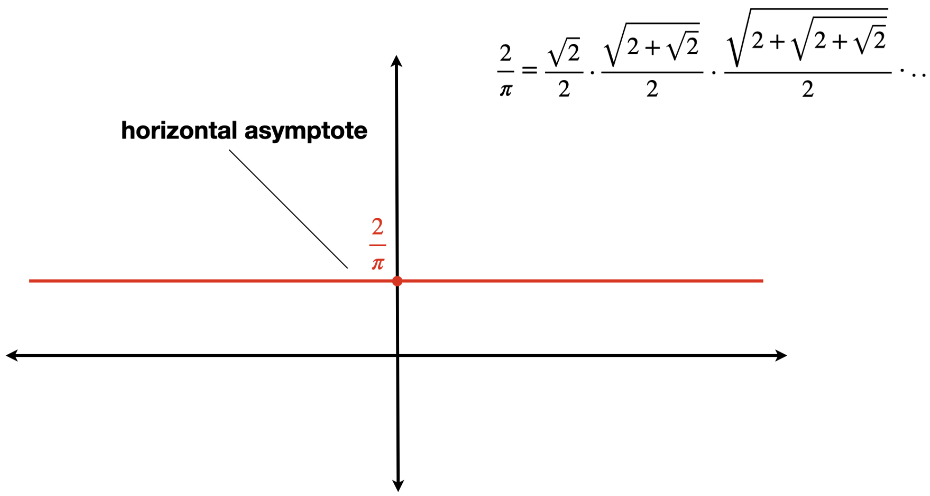
"last" (actually, there is no last term — it is just a representation)

$$\frac{2}{\pi} = \underbrace{\frac{\sqrt{2}}{2}}_{x_1} \cdot \underbrace{\frac{\sqrt{2+\sqrt{2}}}{2}}_{x_2} \cdot \underbrace{\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}_{x_3} \cdot \ldots \underbrace{\cdot}_{x_n} \, x_\infty = f(x_\infty)$$

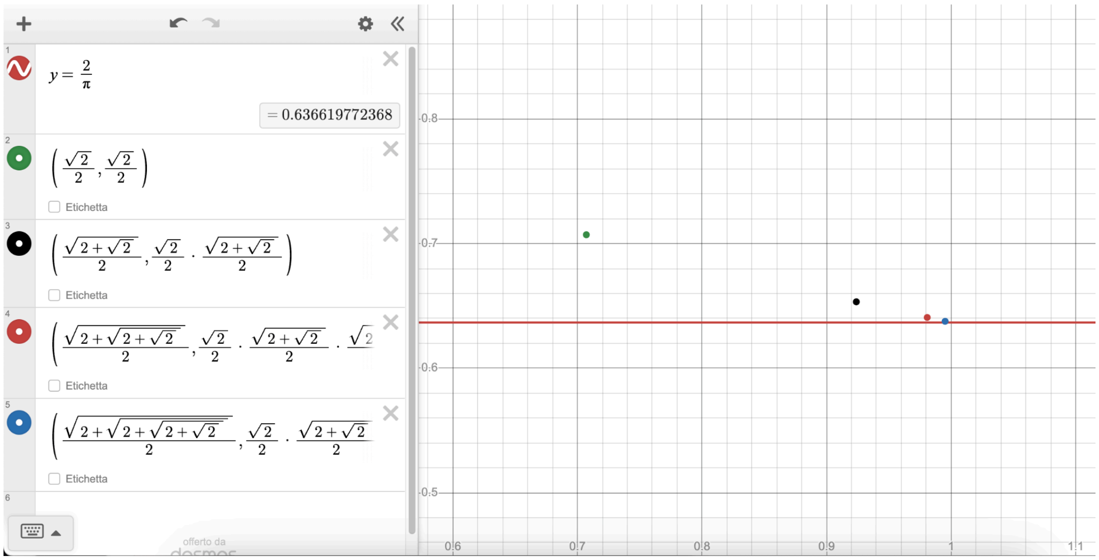
We will call it $f(x_\infty)$. But why x_∞ ? Well, because if we were to continue this process of multiplying terms x_1 , x_2 , x_3 , and so on to infinity, the "last" term would be x_∞ , of course this point does not exist because we would never get to it, but it is just a representation. Now, another way of representing this infinite process is just up to *n iterations*, which is:

$$f(x_n) = \underbrace{\frac{\sqrt{2}}{2} \cdot \ldots \cdot \frac{\sqrt{2+\ldots+\sqrt{2}}}{2}}_{n}$$

Now, let us plot the graph. Since we want to approximate $\frac{2}{\pi}$ we started by drawing this horizontal red line: $y = \frac{2}{\pi}$. Our approximation must get closer and closer to this line after each iteration. This line is called a *horizontal asymptote*.



The first input is $x_1 = \frac{\sqrt{2}}{2}$, and its output is $f(x_1) = \frac{\sqrt{2}}{2}$, since the first term is not yet multiplied by anything. Ok, the next iteration has input $x_2 = \frac{\sqrt{2+\sqrt{2}}}{2}$, and the output is $f(x_2) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2}$. Interesting, we notice that the second iteration is closer to the asymptote $y = \frac{2}{\pi}$. Next, $x_3 = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$, and the result is $f(x_3) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$. And yeah, we can clearly see a pattern at this point. After each iteration we are indeed getting a better and better approximation of the number we want, in this case $\frac{2}{\pi}$. We could continue this process and we would see something like this (image below), which is the characteristic signature of a convergent sequence of numbers.



Now, I need to confess, it is pretty tempting to try and connect these points together in order to form a curve that tends to the asymptote here, but this would have no meaning in the context of discrete sequences. This is so because there is nothing between x_1 and x_2 , for example. At least not in the formula we are using here:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots$$

Remark:

Maybe there is a generalization of this formula for the continuous case, but I don't know… If there is, though, my guess is that it would involve an *infinite continuous product*.

$$\prod_{\frac{\sqrt{2}}{2}}^{\infty} \left(1 + f(x)\right)^{dx}$$

This is a generalization of the (discrete) product operator $\prod$, and it is called the *continuous product* or *product integral*. It is to products what integrals are to sums.

If $f(x)$ is a continuous function, the product integral (in our case) is defined as:

$$\prod_{\frac{\sqrt{2}}{2}}^{\infty} \left(1 + f(x)\right)^{dx} := exp\left(\int_{\frac{\sqrt{2}}{2}}^{\infty} ln\left(1 + f(x)\right) dx \right)$$

This is particularly useful in areas like *continuous compounding* in finance, where you have growth over continuous intervals, or in certain branches of *statistics and probability* (e.g. continuous random variables).

Let us see now how the second sequence converges. But, be ready cause this one will be a little weirder…

$$f\left(x_\infty\right) = \frac{\pi}{2} = \frac{\underset{x_1}{\frac{2}{1}} \cdot \underset{x_2}{\frac{2}{3}} \cdot \underset{x_3}{\frac{4}{3}} \cdot \underset{x_4}{\frac{4}{5}} \cdot \underset{x_5}{\frac{6}{5}} \cdot \underset{x_6}{\frac{6}{7}} \cdot \ldots}{}$$

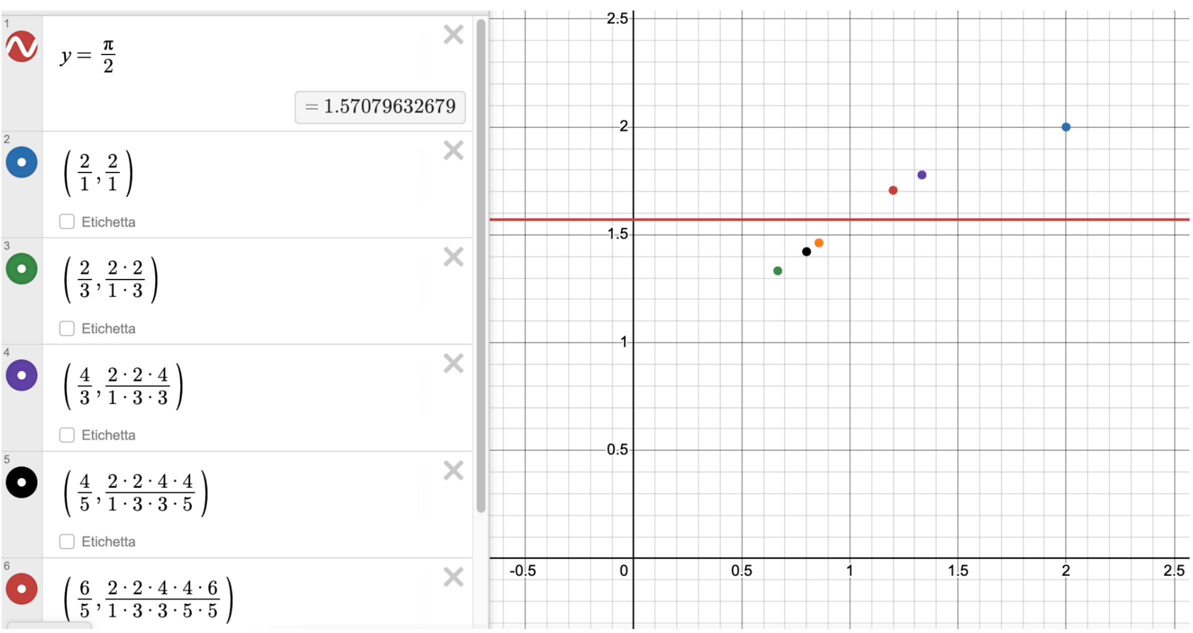
$$f\left(x_\infty\right) = \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}$$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6$

We will plot its graph as a function of each of these terms $\left(x_1, x_2, x_3, \ldots\right)$, just as before. We will call it $f(x_\infty)$ once again. As a consequence, its representation up to *n iterations* is this:

$$f\left(x_n\right) = \frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots} \cdot x_n$$

Now let us plot the graph. Since we want to approximate $\frac{\pi}{2}$ we drew the horizontal asymptote: $y = \frac{\pi}{2}$.



The first input is $x_1 = \frac{2}{1}$, and its output is simply $f(x_1) = \frac{2}{1}$ (blue point in the above graph) , no surprises so far. The next iteration has input $x_2 = \frac{2}{3}$, and the output is $f(x_2) = \frac{2}{1} \cdot \frac{2}{3}$. Look at the location of each of these points in the graph. Interesting! The second iteration (green point in the above graph) is indeed closer to the asymptote line, but weirdly it is located below the line now. Hm… Let us see the next one, $x_3 = \frac{4}{3}$, and the result is $f(x_3) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3}$. Yeah, it went back up (purple point in the above graph). We can continue this process and we would see that the points alternate from being above and under the asymptote line, but the most important thing is that after each iteration these points are indeed getting closer and closer to the line. So this infinite product indeed converges to $\frac{\pi}{2}$.

A good question to ask at this point is: why would somebody want to do such a thing? I mean, ok, it is pretty nice to see that such infinite products produce such non-intuitive results, especially involving π . In principle there is no reason for π to show up here. But, apart from the mere curiosity behind it, why is it useful? Well, it turns out that the real usefulness of these formulas lies in the fact that we can actually approximate π:

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \ldots$$

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}$$

As you may have heard, π is an irrational number, which means that there is no way to write its value in a finite way. It has infinite digits after the point. So, you may need an approximation of π , the precision of which depends on your purpose and intent. If you just want to estimate the area of a circle, $\pi \approx 3.14$ might be enough. If you need to use it to design wheels and gears in mechanical engineering, or to perform calculations involving resonant circuits, you might approximate $\pi \approx 3.14159$. However, if you need to calculate the trajectory of a spacecraft, or of a GPS satellite system, you might even have to use approximations of π involving 15 to 20 digits! Some websites claim that at NASA most calculations involve 15 digits of π. So, it just depends on the context, really.

A better question to ask though is: Which one of these infinite products converges faster to π? In other words, which one takes me less iterations in order to get us a "satisfactory approximation" of π ? Which one is more computationally efficient?

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \ldots$$

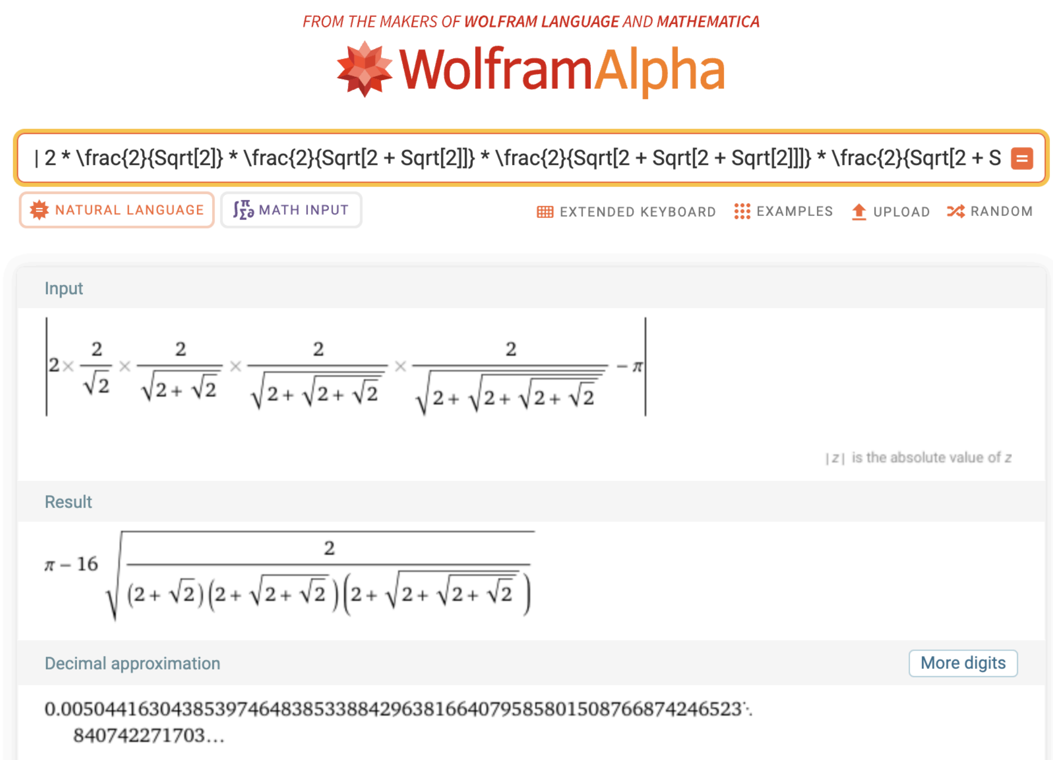
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}$$

Well, as we already pointed out, a "satisfactory approximation" depends on the context, but we can still compare the two formulas by calculating the error associated with each one of them. The error is calculated as the difference between my approximation and the real value of π, all of it in absolute value.

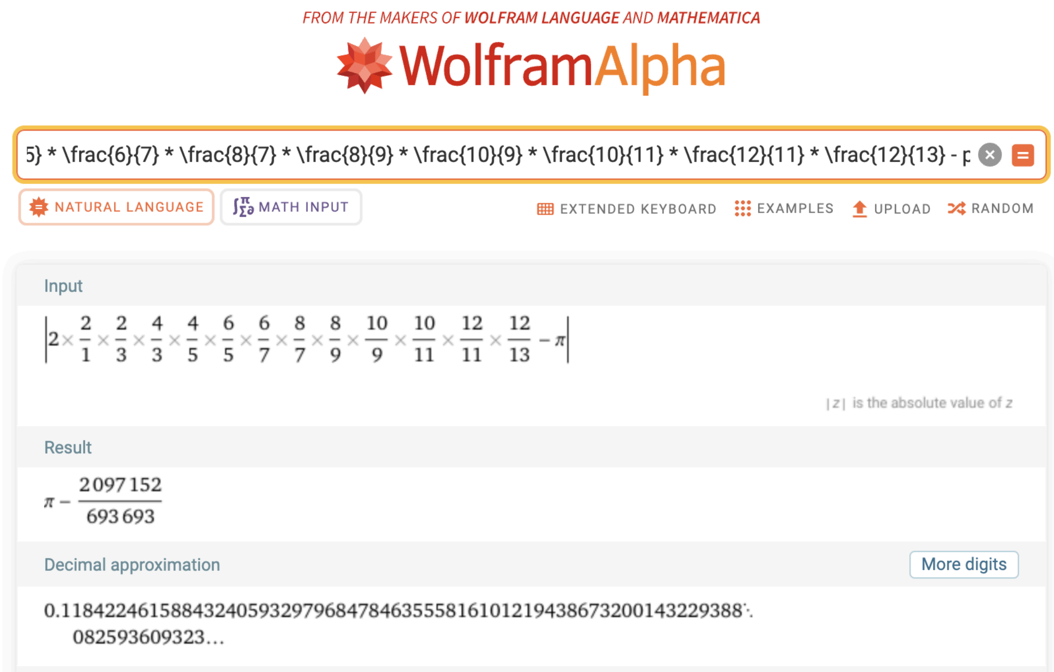
$$\text{ERROR} = \left| \text{approx.} - \pi \right|$$

The absolute value is there because we are not interest in knowing if the error is *negative* (which would mean that our approximation is *lower* than the actual value π), or if the error is *positive* (which would mean that our approximation is *higher* than the actual value of π). We just want to know how far off we are from the actual value of π . Another important thing to say is that we consider “the actual value of π” as the best known approximation of it so far.

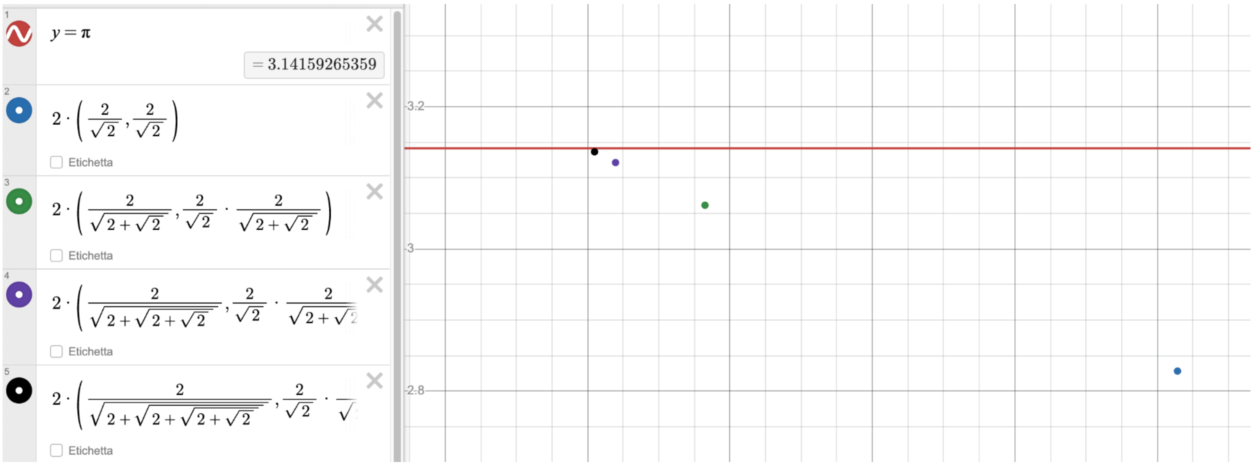
In order to check which formula wins the race, I used the website *wolframalpha.com*. It turns out that after performing 4 iterations using the first formula we get an accuracy of 10^{-2}. But what does *accuracy* mean? It means that the **error is** $\approx$ 0.00504... So, up to the second digit after the point my approximation (with just 4 iterations) is correct. But, after the second digit the error is not zero anymore, which means that my approximation has an accuracy of 0.01 or 10^{-2} .

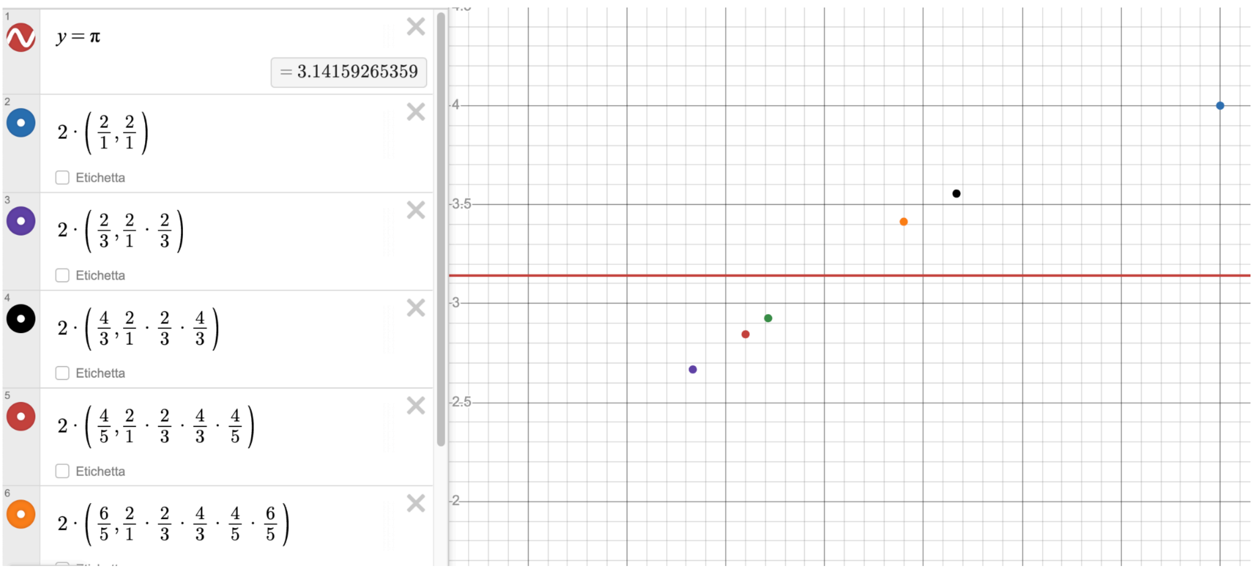


Now, that’s where things get interesting! If we try to do the same with the second formula we will notice very quickly how inefficient it is. Actually, I tried to compute as many iterations as I could, way more than just 4 as we did for the first formula. But then I found out that *wolframalpha.com* has a limit of characters, and so I had to stop at iteration number 12, and the error was still way greater than the one we saw for the first formula.



After doing some research online, I found out that it would take us around **78 iterations** of the second formula in order to get to an approximation of π with **accuracy of** 10^{-2}, which took us just **4 iterations** of the first formula to get to.





This concludes the experiment, and we can confidently say that the first formula is way more efficient than the second one. I honestly think that it is a pity because the second formula looks much more elegant than the first one, but that's just my opinion…

Dive deeper: (Proofs)

We will prove both of these formulas. The first one is easier, or maybe I should say: less complex. Let us start by noticing that there is a recurrent process here, and we can call each term 'a' with an index 'k'.

The general formula is:

$$a_k := \sqrt{2 + a_{k-1}}$$
$$a_0 := 0$$

So, for example:

$$a_1 = \sqrt{2}$$
$$a_2 = \sqrt{2 + \sqrt{2}}$$
$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$
$$a_n = \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}$$

This infinite product can be rewritten as

$$P_n := \prod_{k=1}^{n} \frac{a_k}{2}$$

This is just the symbol of the *product operator*, which means that each term is multiplied by each other from $k = 1$ to n . Our goal here is to show that P_n tends to $\frac{2}{\pi}$. In order to do that we will have to use two auxiliary formulas:

$$1) \quad a_k = 2\cos\left(\frac{\pi}{2^{k+1}}\right) = \sqrt{2 + a_{k-1}}$$

$$2) \quad \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$$

Now let us prove that P_n indeed tends to $\frac{2}{\pi}$:

$$(1)\ a_k = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$$

$$P_n = \prod_{k=1}^{n} \frac{a_k}{2} = \prod_{k=1}^{n} \frac{\cancel{2}}{\cancel{2}} \cos\left(\frac{\pi}{2^{k+1}}\right) \xrightarrow{n\to\infty} \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}}\right) =$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi} \qquad \square$$

$$(2)\ \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$$

The first formula (1) can be shown to hold by induction. Notice that in the base case (so, $k = 1$) we have that:

$$a_1 \overset{def}{=} \sqrt{2 + a_{1-1}} = \sqrt{2 + a_0} = \sqrt{2}$$

$$a_1 = 2\cos\left(\frac{\pi}{2^{1+1}}\right) = 2\cos\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$(1)\ a_k = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$$

In the inductive step we assume that the formula is true for $k = n$, and we prove it for $k = n + 1$.

$$a_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

$$a_{n+1} \overset{def}{=} \sqrt{2 + 2a_n} = \sqrt{2 + 2\ cos\left(\frac{\pi}{2^{n+1}}\right)} = \sqrt{2} \cdot \sqrt{1 + cos\left(\frac{\pi}{2^{n+1}}\right)} =$$

$$= \sqrt{2} \cdot \sqrt{1 + cos\left(\frac{2\pi}{2^{n+2}}\right)} = \sqrt{2} \cdot \sqrt{1 + cos^2\left(\frac{\pi}{2^{n+2} - 1}\right)} =$$

$$cos\left(2 \cdot \left(\frac{\pi}{2^{n+2}}\right)\right) = 2\ cos^2\left(\frac{\pi}{2^{n+2}}\right) - 1$$

$$= \sqrt{2} \cdot \sqrt{2} \cdot cos\left(\frac{\pi}{2^{n+2}}\right) = 2\ cos\left(\frac{\pi}{2^{n+2}}\right) \qquad \square$$

The second formula (2) is more complicated, but nothing too crazy.

$$(2)\quad \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$$

Its proof comes from the fact that:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) \implies \cos(\theta) = \frac{\sin(2\theta)}{2\sin(\theta)}$$

$$\implies \cos(\theta) = \frac{\sin(2\theta)}{2\sin(\theta)} \implies \prod_{k=1}^{\infty}\cos\left(\frac{x}{2^{k+1}}\right) = \prod_{k=1}^{\infty}\frac{\sin\left(2\cdot\frac{x}{2^{k+1}}\right)}{2\sin\left(\frac{x}{2^{k+1}}\right)} = \prod_{k=1}^{\infty}\frac{\sin\left(\frac{x}{2^{k}}\right)}{2\sin\left(\frac{x}{2^{k+1}}\right)} =$$

("θ" $= \frac{x}{2^{k+1}}$)

$$= \prod_{k=1}^{\infty}\frac{\sin\left(2\cdot\frac{x}{2^{k+1}}\right)}{2\sin\left(\frac{x}{2^{k+1}}\right)} = \prod_{k=1}^{\infty}\frac{\sin\left(\frac{x}{2^{k}}\right)}{2\sin\left(\frac{x}{2^{k+1}}\right)} = \frac{\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2^2}\right)}\cdot\frac{\sin\left(\frac{x}{2^2}\right)}{2\sin\left(\frac{x}{2^3}\right)}\cdot\frac{\sin\left(\frac{x}{2^3}\right)}{2\sin\left(\frac{x}{2^4}\right)}\cdots =$$

$$= \frac{\sin\left(\frac{x}{2}\right)}{\lim\limits_{k\to\infty} 2^k\cdot\sin\left(\frac{x}{2^{k+1}}\right)} = \frac{\sin\left(\frac{x}{2}\right)}{\lim\limits_{k\to\infty} 2^k\cdot\frac{x}{2^{k+1}}} = \frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}$$

$$\sin\left(\frac{x}{2^{k+1}}\right) \sim \frac{x}{2^{k+1}} \quad \left(\text{as } \frac{x}{2^{k+1}} \xrightarrow{k\to\infty} 0\right)$$

When $x = \pi$: $\prod_{k=1}^{\infty}\cos\left(\frac{\pi}{2^{k+1}}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)}$

This was a good warm up. Let's face the monster now!!!

$$\boxed{\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot4\cdot6\cdot6\cdot\ldots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot\ldots}}$$

Right off the bat we notice that this formula that we want to prove can actually be written as:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$$

Why is that? Well, let's expand this expression:

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} = \prod_{n=1}^{\infty} \frac{2n}{(2n-1)} \cdot \frac{2n}{(2n+1)} =$$

$$= \left[\frac{2 \cdot 1}{(2 \cdot 1 - 1)} \cdot \frac{2 \cdot 1}{(2 \cdot 1 + 1)}\right] \cdot \left[\frac{2 \cdot 2}{(2 \cdot 2 - 1)} \cdot \frac{2 \cdot 2}{(2 \cdot 2 + 1)}\right] \cdot \left[\frac{2 \cdot 3}{(2 \cdot 3 - 1)} \cdot \frac{2 \cdot 3}{(2 \cdot 3 + 1)}\right] \cdot \ldots =$$

$$= \left[\frac{2}{1} \cdot \frac{2}{3}\right] \cdot \left[\frac{4}{3} \cdot \frac{4}{5}\right] \cdot \left[\frac{6}{5} \cdot \frac{6}{7}\right] \ldots = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}$$

Therefore, if we find a way of proving that $\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$, then we also prove that:

$$\boxed{\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot \ldots}}$$

In order to do so, let us use 2 concepts, one of which is called *Wallis' integral* and the other *the reduction formula*:

Wallis' integral: $\boxed{I_n = \int_0^{\pi} sin^n(x)\, dx}$

Reduction formula: $\boxed{I_n = \left(\frac{n-1}{n}\right) I_{n-2}}$

Wallis' integral is a definition, and thus we do not need to prove it. The reduction formula, on the other hand, needs to be proven. It is just a consequence of the Walli's integral. So, let us try to calculate it explicitly:

$$\vdash\ I_n = \left(\frac{n-1}{n}\right) I_{n-2} \quad \textbf{where} \quad I_n = \int_0^{\pi} \sin^n(x)\, dx$$

$$I_n = \int_0^{\pi} \sin^n(x)\, dx = \int_0^{\pi} \sin(x) \cdot \sin^{n-1}(x)\, dx =$$

Integration by parts:

$$u = \sin^{n-1}(x) \implies du = (n-1)\ \sin^{n-2}(x) \cdot \cos(x)\, dx$$

$$dv = \sin(x)\, dx \implies v = -\cos(x)$$

$\cos^2(x) = 1 - \sin^2(x)$

0

$$= \left[\sin^{n-1}(x) \cdot \left(-\cos(x)\right)\right]\Big|_0^{\pi} - \int_0^{\pi} \left(-\cos(x)\right) \cdot (n-1)\sin^{n-2}(x)\cos(x)dx =$$

$$= (n-1)\int_0^{\pi} \sin^{n-2}(x)\cdot\left(1-\sin^2(x)\right)\,dx =$$

$$= (n-1)\left[\underbrace{\int_0^{\pi}\sin^{n-2}(x)dx}_{I_{n-2}} - \underbrace{\int_0^{\pi}\sin^{n}(x)dx}_{I_n}\right] =$$

$$= (n-1)\cdot\left(I_{n-2}-I_n\right) \implies I_n = (n-1)\cdot\left(I_{n-2}-I_n\right) \implies$$

$$\implies \quad I_n = \left(\frac{n-1}{n}\right) I_{n-2} \quad \square$$

Now that we proved the reduction formula we are "allowed" to use these 2 formulas. We will split the problem in 2 cases: one of them *even*, the other one *odd*.

Let us see the even case first:

even

$$I_{2n} = \left(\frac{2n-1}{2n}\right) \cdot I_{2n-2} = \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-2-1}{2n-2}\right) \cdot I_{2n-2-2} =$$

$2(n-1)$

$$= \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot I_{2n-4} =$$

$2(n-2)$

$$= \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-4-1}{2n-4}\right) \cdot I_{2n-4-2} =$$

$$= \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot I_{2n-6} =$$

$2(n-3)$

$$= \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot I_{2(n-3)} =$$

$$= \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot \ldots \cdot I_{2(n-n)} =$$

0

$$= \left(\frac{2n-1}{n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot \ldots \cdot I_0$$

$$I_2 = \left(\frac{2-1}{2}\right) I_0 = \frac{1}{2} \cdot I_0$$

Let us calculate the first term of the recursive rule with even index:

$$I_0 = \int_0^{\pi} \sin^0(x)dx = \int_0^{\pi} 1 \cdot dx = \pi$$

$I_0 = \pi$

$$\therefore I_{2n} = \left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot \ldots \cdot I_0$$

even

Therefore we can write that I_{2n} is the product of all these terms until π .

Now we analyse the odd case:

odd

$$\begin{aligned} I_{2n+1} &= \left(\frac{2n+1-1}{2n+1}\right) \cdot I_{2n+1-2} = \left(\frac{2n}{2n+1}\right) \cdot I_{2n-1} = \\ &= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-1-1}{2n-1}\right) \cdot I_{2n-1-2} = \\ &= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot I_{2n-3} = \\ &= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-3-1}{2n-3}\right) \cdot I_{2n-3-2} = \end{aligned}$$

$$= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-4}{2n-3}\right) \cdot I_{2n-5} =$$

$$= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-4}{2n-3}\right) \cdot \ldots \cdot I_{2n-(2n-1)} =$$

$$= \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-4}{2n-3}\right) \cdot \ldots \cdot I_1$$

$$I_3 = \left(\frac{3-1}{3}\right) I_1 = \frac{2}{3} I_1$$

Let us calculate the first term of the recursive rule with odd index:

$$I_1 = \int_0^{\pi} \sin^1(x)dx = \int_0^{\pi} \sin(x)dx = \left(-\cos(x)\right)\Big|_0^{\pi} = 1 + 1 = 2$$

$$I_1 = 2$$

$$\therefore I_{2n+1} = \left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-4}{2n-3}\right) \cdot \ldots \cdot 2$$

odd

Therefore we can write that I_{2n+1} is the product of all these terms until 2.

The next step is to show that $\frac{I_{2n+1}}{I_{2n}} = \frac{2}{\pi} \cdot \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$:

$$\frac{I_{2n+1}}{I_{2n}} = \frac{\left(\frac{2n}{2n+1}\right) \cdot \left(\frac{2n-2}{2n-1}\right) \cdot \left(\frac{2n-4}{2n-3}\right) \cdot \ldots \cdot 2}{\left(\frac{2n-1}{2n}\right) \cdot \left(\frac{2n-3}{2n-2}\right) \cdot \left(\frac{2n-5}{2n-4}\right) \cdot \ldots \cdot \pi} =$$

$$= \left[\frac{2n}{2n+1} \cdot \frac{2n}{2n-1}\right] \cdot \left[\frac{2n-2}{2n-1} \cdot \frac{2n-2}{2n-3}\right] \cdot \left[\frac{2n-4}{2n-3} \cdot \frac{2n-4}{2n-5}\right] \cdot \ldots \cdot \frac{2}{\pi} =$$

$$= \left[\frac{4n^2}{4n^2-1}\right] \cdot \left[\frac{\overbrace{(2n-2)^2}^{4(n-1)^2}}{\underbrace{(2n-1)\cdot(2n-3)}_{4(n-1)^2-1}}\right] \cdot \left[\frac{\overbrace{(2n-4)^2}^{4(n-2)^2}}{\underbrace{(2n-3)\cdot(2n-5)}_{4(n-2)^2-1}}\right] \cdot \ldots \cdot \frac{2}{\pi} =$$

$$= \frac{2}{\pi} \cdot \ldots \cdot \left[\frac{4(n-2)^2}{4(n-2)^2-1}\right] \cdot \left[\frac{4(n-1)^2}{4(n-1)^2-1}\right] \cdot \left[\frac{4n^2}{4n^2-1}\right] =$$

$$= \frac{2}{\pi} \cdot \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \qquad \square$$

Once we convince ourselves of that $\frac{I_{2n+1}}{I_{2n}} = \frac{2}{\pi} \cdot \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$ we can evaluate the limit, for $n \to \infty$, on both sides.

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{2}{\pi} \cdot \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} = 1$$

The polynomials in the numerator $\left(4k^2\right)$ and the denominator $\left(4k^2 - 1\right)$ have the same degree (2), so their growth rates are comparable (the same) as $k \to \infty$.

Furthermore, since the coefficients of the highest-degree terms are equal (both are 4), the ratio $\frac{4k^2}{4k^2-1}$ approaches 1, and the product converges to a limit scaled by $\frac{2}{\pi}$, which evaluates to 1.

$$\frac{2}{\pi} \cdot \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = 1$$

$$\Downarrow$$

$$\boxed{\prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \frac{\pi}{2}}$$

With this we finish the proof that:

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 6 \cdot 7 \cdot \ldots} = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \frac{\pi}{2}$$

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