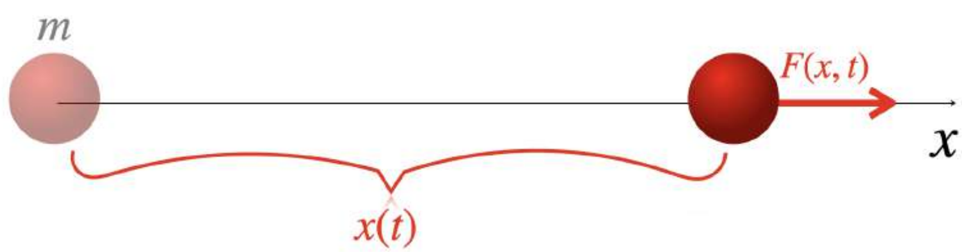
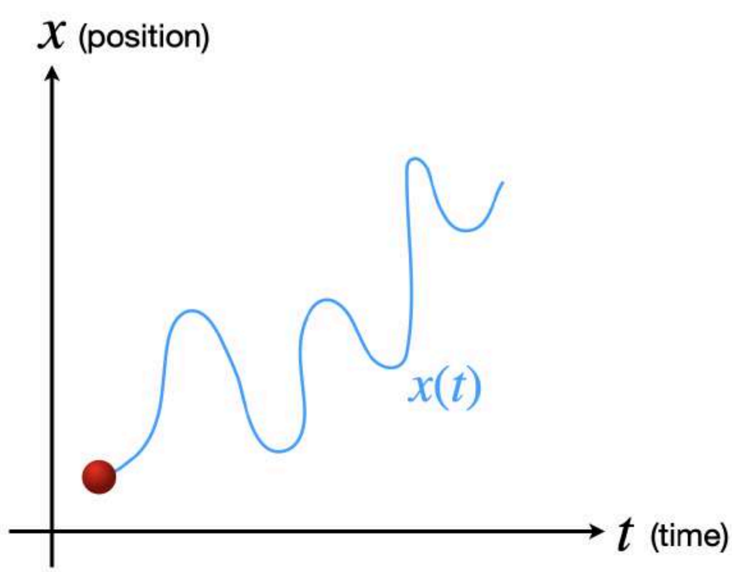
The Intuition Behind Quantum Mechanics

by DiBeos

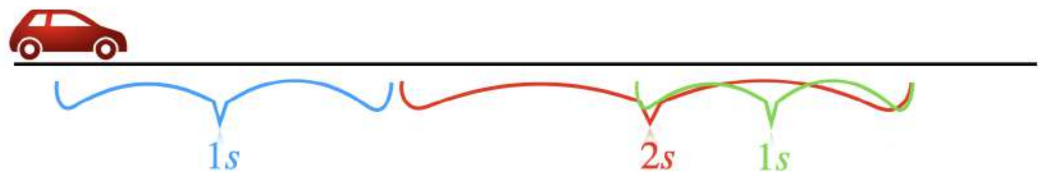
Imagine a particle of mass m, constrained to move along a one dimensional space, like a line. Let's call it the x-axis. This particle is subject to some force $F(x, t)$ that is a function of its position x and time t.



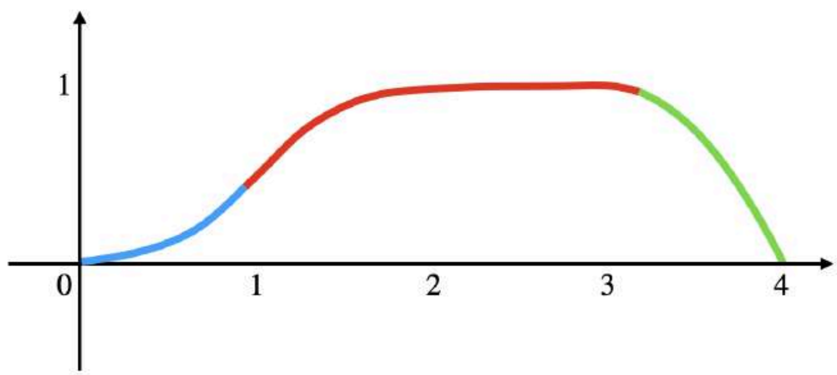
In order to understand what the goal of QM (quantum mechanics) is we first need to take a step back and talk about what *classical mechanics* is all about. In classical mechanics we aim to determine the position of the particle at any given time. We call it $x(t)$.



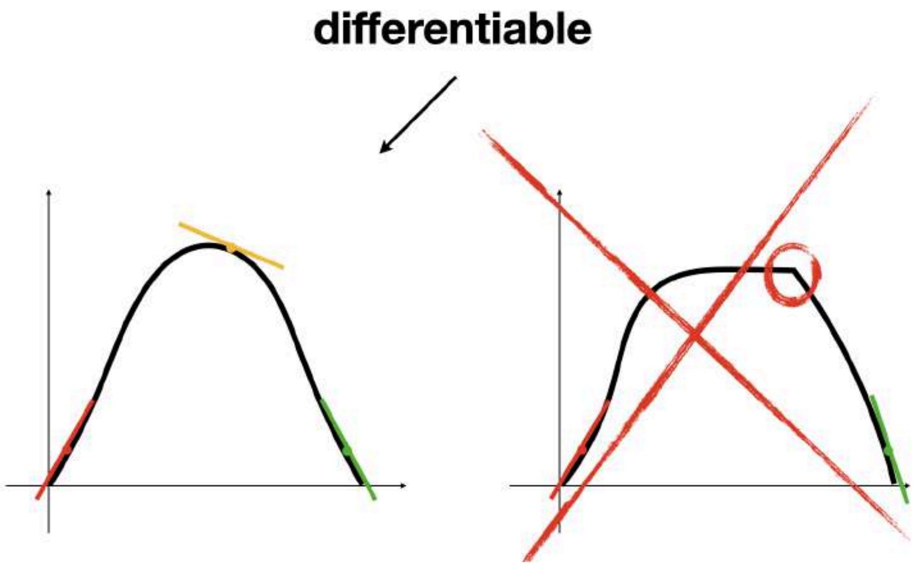
For example, imagine a car that accelerates for 1 second, then starts to slow down until it stops 2 seconds later, and finally reverses gear while speeding up for 1 second. This kind of motion can be described using a piecewise function. The first part happens from 0 to 3 seconds, and the second takes only 1 second.



$$x(t) = \begin{cases} \dfrac{1}{1+e^{-4(t-1)}} - \dfrac{4e^{-8}}{(1+e^{-8})^2}(t-3), & 0 \le t \le 3 \\ -(t-3)^2 + \dfrac{1}{1+e^{-8}}, & 3 < t \le 4 \end{cases}$$



This function is differentiable everywhere, including at the moment $t = 3$. If you do not believe me let’s check it out. In order to be differentiable the function needs to be *continuous* at this point and the *derivative from the RHS* (right-hand side) *must be equal to the derivative on the LHS* (left-hand side).



Continuity:

At $t = 3$:

- Left-hand limit: $\frac{1}{1+e^{-4(3-1)}} - \frac{4e^{-8}}{(1+e^{-8})^2}(3-3) = \frac{1}{1+e^{-8}}$.
- Right-hand limit: $-(3-3)^2 + 1 = 1$.

The function is continuous at $t = 3$.

Differentiability:

At $t = 3$:

- Left-hand derivative:

$$x'(3) = \frac{4e^{-8}}{(1+e^{-8})^2} - \frac{4e^{-8}}{(1+e^{-8})^2} = 0.$$

- Right-hand derivative:

$$x'(3) = -2(3-3) = 0.$$

Since the derivatives match, the function is differentiable at $t = 3$.

Anyway, in classical mechanics we are very interested in this expression of the position as a function of time, because from it we can calculate the velocity at all points by taking its derivative:

For $t \leq 3$, the function is:

$$x(t) = \frac{1}{1 + e^{-4(t-1)}} - \frac{4e^{-8}}{(1 + e^{-8})^2}(t - 3).$$

First Term:

$$f(t) = \frac{1}{1 + e^{-4(t-1)}}.$$

The derivative is calculated using the chain rule.

1. Let $u(t) = 1 + e^{-4(t-1)}$, so:

$$f(t) = u(t)^{-1}.$$

2. The derivative of $u(t)^{-1}$ is:

$$\frac{df}{dt} = -u(t)^{-2} \cdot \frac{du}{dt}.$$

3. Calculate $\frac{du}{dt}$:

$$u(t) = 1 + e^{-4(t-1)}, \quad \frac{du}{dt} = -4e^{-4(t-1)}.$$

4. Substitute $u(t)$ and $\frac{du}{dt}$:

$$\frac{df}{dt} = -\frac{1}{(1 + e^{-4(t-1)})^2} \cdot (-4e^{-4(t-1)}).$$

5. Simplify:

$$\frac{df}{dt} = \frac{4e^{-4(t-1)}}{(1 + e^{-4(t-1)})^2}.$$

Second Term:

$$g(t) = -\frac{4e^{-8}}{(1+e^{-8})^2}(t-3).$$

The derivative is:

$$\frac{dg}{dt} = -\frac{4e^{-8}}{(1+e^{-8})^2}.$$

Combine Terms:

The derivative for $t \leq 3$ is:

$$\boxed{v(t) = \frac{4e^{-4(t-1)}}{(1+e^{-4(t-1)})^2} - \frac{4e^{-8}}{(1+e^{-8})^2}.}$$

For $t > 3$, the function is:

$$x(t) = -(t-3)^2 + 1.$$

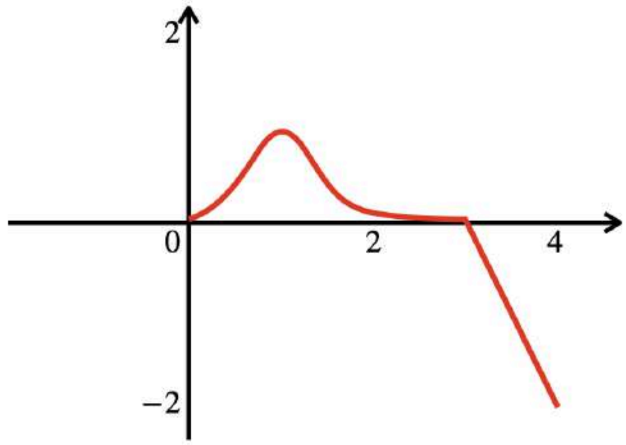
1. Expand $x(t)$:

$$x(t) = -\left(t^2 - 6t + 9\right) + 1 = -t^2 + 6t - 8.$$

2. Differentiate:

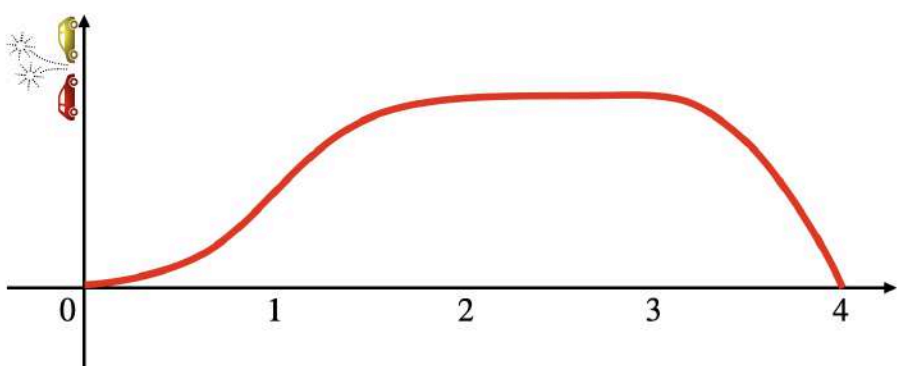
$$\boxed{v(t) = \frac{dx}{dt} = -2(t-3).}$$

$$v(t) = \frac{dx}{dt} = \begin{cases} \frac{4e^{-4(t-1)}}{(1+e^{-4(t-1)})^2} - \frac{4e^{-8}}{(1+e^{-8})^2}, & 0 \le t \le 3 \\ -2(t-3), & 3 < t \le 4 \end{cases}$$



We notice here that the function $v(t)$ is continuous but has a “sharp corner” at $t = 3$, and therefore it is not differentiable at this point. Since acceleration is proportional to the force applied to an object, this could mean, for instance, that the car was abruptly hit by another one that produced a change in its acceleration in a discontinuous way.

$$a \propto F$$



Let’s calculate its derivative.

Function $v(t)$:

$$v(t) = \begin{cases} \frac{4e^{-4(t-1)}}{(1+e^{-4(t-1)})^2} - \frac{4e^{-8}}{(1+e^{-8})^2}, & t \leq 3, \\ -2(t-3), & t > 3. \end{cases}$$

For $t \leq 3$, the function is:

$$v(t) = \frac{4e^{-4(t-1)}}{(1+e^{-4(t-1)})^2} - \frac{4e^{-8}}{(1+e^{-8})^2}.$$

The first term depends on t, while the second term is constant.

First Term:

$$f(t) = \frac{4e^{-4(t-1)}}{(1+e^{-4(t-1)})^2}.$$

We'll use the quotient rule.

1. Let $u(t) = 4e^{-4(t-1)}$ and $w(t) = (1+e^{-4(t-1)})^2$. Then:

$$f(t) = \frac{u(t)}{w(t)}.$$

2. Differentiate $u(t)$:

$$u(t) = 4e^{-4(t-1)}, \quad \frac{du}{dt} = -16e^{-4(t-1)}.$$

3. Differentiate $w(t)$:

$$w(t) = (1 + e^{-4(t-1)})^2, \quad \frac{dw}{dt} = 2(1 + e^{-4(t-1)}) \cdot (-4e^{-4(t-1)}) = -8e^{-4(t-1)}(1 + e^{-4(t-1)}).$$

4. Apply the quotient rule:

$$\frac{df}{dt} = \frac{\frac{du}{dt} \cdot w(t) - u(t) \cdot \frac{dw}{dt}}{w(t)^2}.$$

5. Substitute:

$$\frac{df}{dt} = \frac{-16e^{-4(t-1)}(1 + e^{-4(t-1)})^2 - 4e^{-4(t-1)} \cdot [-8e^{-4(t-1)}(1 + e^{-4(t-1)})]}{(1 + e^{-4(t-1)})^4}.$$

6. Simplify numerator:

$$-16e^{-4(t-1)}(1 + e^{-4(t-1)})^2 + 32e^{-8(t-1)}(1 + e^{-4(t-1)}).$$

7. Factor:

$$\frac{df}{dt} = \frac{-16e^{-4(t-1)}(1 + e^{-4(t-1)}) + 32e^{-8(t-1)}}{(1 + e^{-4(t-1)})^3}.$$

Second Term:

$$g(t) = -\frac{4e^{-8}}{(1 + e^{-8})^2}.$$

This term is constant, so:

$$\frac{dg}{dt} = 0.$$

Combine:

The derivative for $t \leq 3$ is:

$$a(t) = \frac{-16e^{-4(t-1)}(1 + e^{-4(t-1)}) + 32e^{-8(t-1)}}{(1 + e^{-4(t-1)})^3}.$$

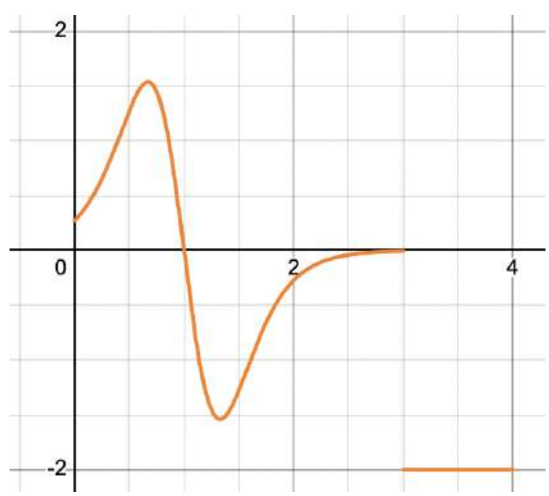
For $t > 3$, the function is:

$$v(t) = -2(t - 3).$$

1. Differentiate:

$$a(t) = \frac{dv}{dt} = -2.$$

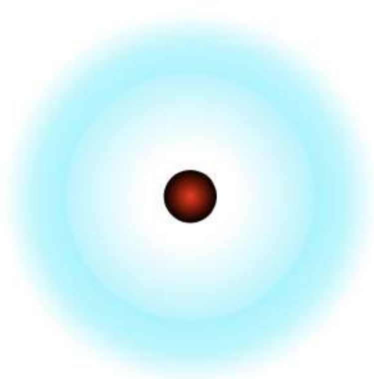
$$a(t) = \begin{cases} \frac{-16e^{-4(t-1)}(1+e^{-4(t-1)})+32e^{-8(t-1)}}{(1+e^{-4(t-1)})^3}, & t \leq 3, \\ -2, & t > 3. \end{cases}$$

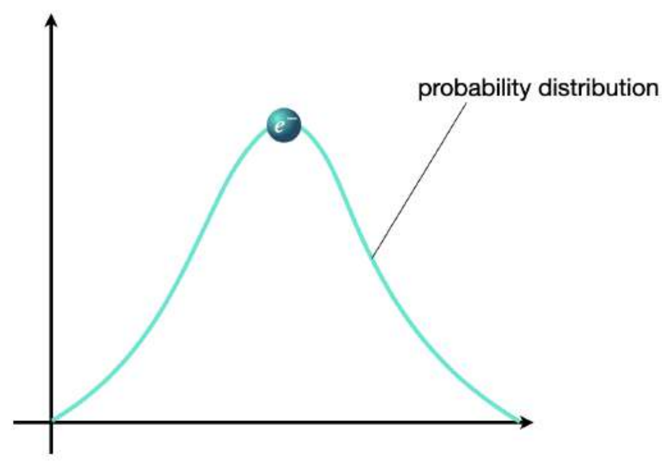


We could go on and calculate the *momentum*, which is defined as $p = m \cdot v$. We could calculate its *kinetic energy* $T = \frac{1}{2}m \cdot v^2$. Or any other dynamical variable of interest.

Ok, but usually we do not get the function $x(t)$ of a mechanical process on a silver platter, as we have just given you guys here. So, how do we usually go about determining $x(t)$? We apply Newton's second law: $F = m \cdot a$. For conservative systems, the force can be expressed as the derivative of a *potential energy function*, $F =- \frac{\partial V}{\partial x}$, with respect to space, and Newton's law becomes this: $m\frac{d^2x}{dt^2} =- \frac{\partial V}{\partial x}$. This a partial differential equation, and together with the appropriate initial conditions (which are usually the position and velocity at time $t = 0$), we can determine the position $x(t)$ for all moments of time t.

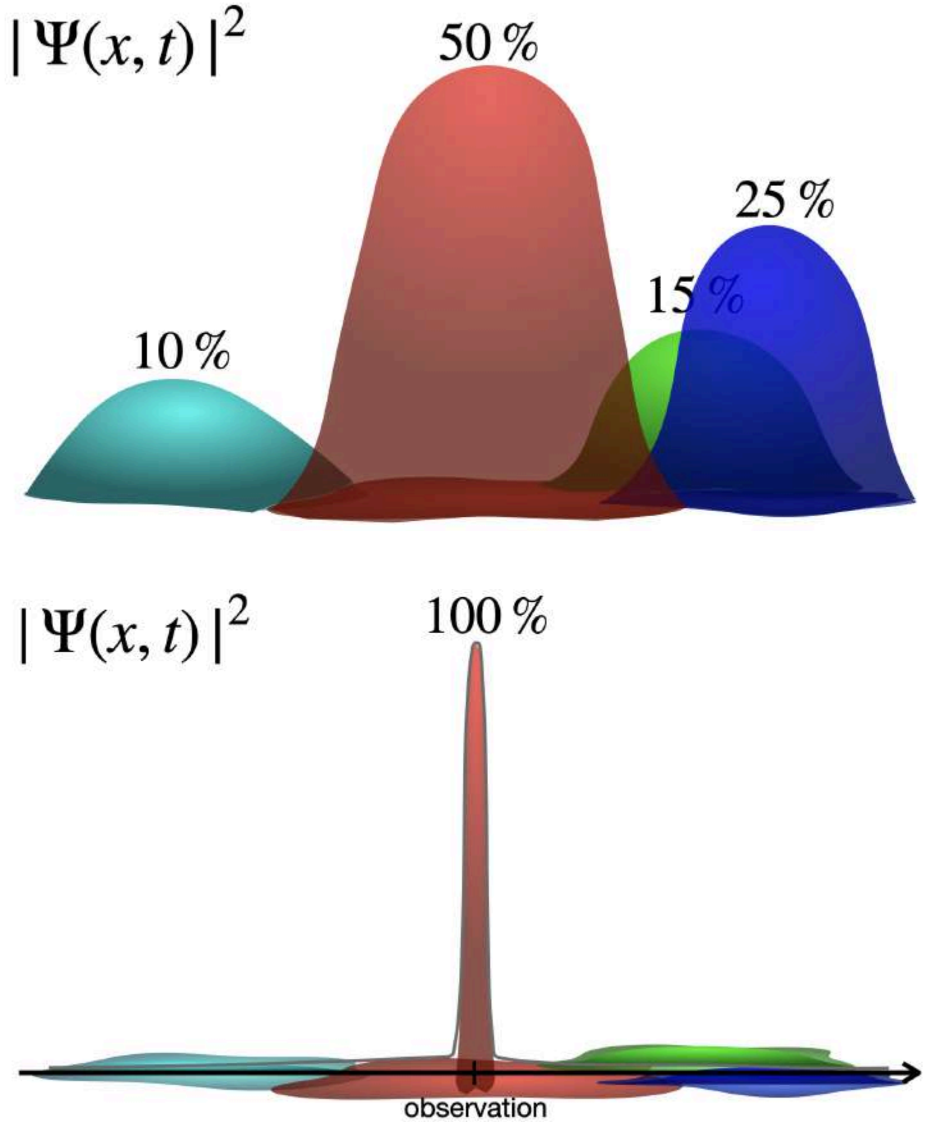
Quantum mechanics approaches the same problem, but very differently. First of all, for most particles smaller than 10^{-10} m (so not for a car, but instead for an electron, for example), classical mechanics fails to describe their behavior. A *deterministic* function $x(t)$ does not exist anymore, and instead, the particle's behavior must be described by a *wavefunction*, which gives us the *probability distribution* for its position. QM was discovered in order to explain the odd behaviors of microscopic particles.





The particle is no longer thought of as a point in space, but rather as a wave, as if the particle were broken into many different pieces and spread out all over the place. Of course, this is just a nice mental picture of the situation, just the intuition behind it. What really happens though, from the mathematical and physical points of view, is more complex. But this wave, which is described by the particle's *wavefunction* $\Psi(x, t)$, represents the *probability amplitude* of finding the particle at a particular position and time. The particle's behavior is therefore governed not by deterministic paths like in classical mechanics, but by probabilities, where the wavefunction's squared magnitude,

$|\Psi(x,t)|^2$, gives the likelihood of locating the particle in a specific region. In this view, the particle simultaneously exists in many possible states or locations, until a measurement is made, which produces the effect of collapsing the wavefunction to a definite outcome.



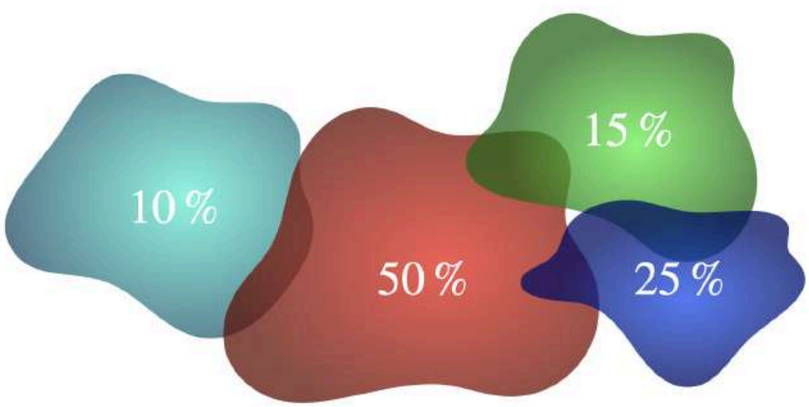
In this new framework, Newton's second law is replaced by the Schrödinger equation:

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}$$

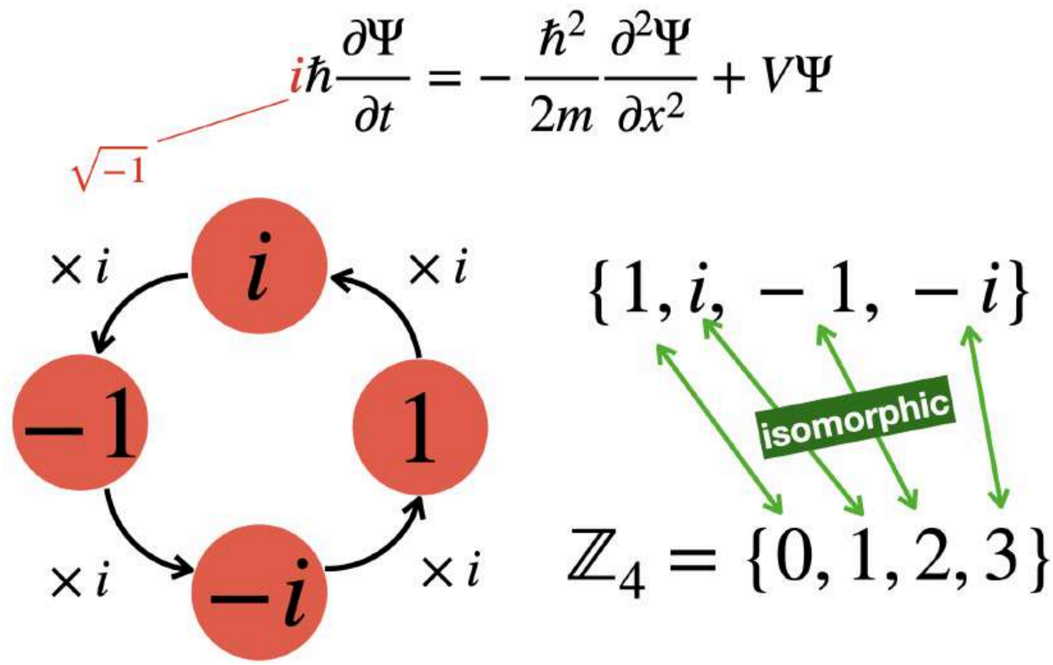
$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Its solutions, so the functions $\Psi(x, t)$ that satisfy this equation, will then allow us to predict the probability of locating a particle in a specific region.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$



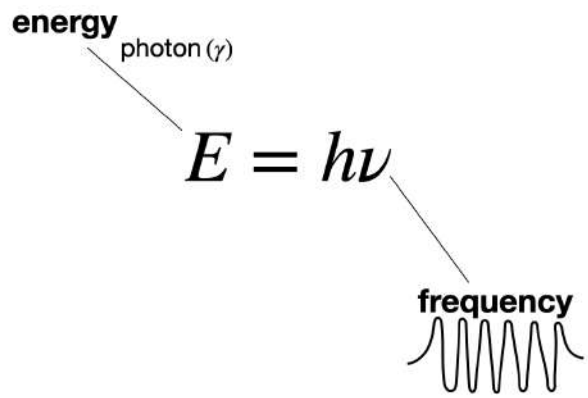
i here is the imaginary unit $\sqrt{-1}$. It is pretty convenient to use imaginary numbers, since they have nice properties. For example, the set of powers of the imaginary unit i ($\{1, i, -1, -i\}$) form a cyclic group of order 4 under multiplication. This set is created by multiplying each element by i at each step. The structure of this set is similar to the additive cyclic group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. In mathematical terms, we say these 2 sets are *isomorphic*.



ħ is Planck's constant $h \approx 6.62... \times 10^{-34} J \cdot s$ divided by 2π $\left(i.e.\ \hbar = \frac{h}{2\pi}\right)$. The Planck's constant h represents the smallest energy "chunk" that can exist in a quantum system. It plays a crucial role in QM because it sets the scale at which quantum effects become significant. It is present in many equations in QM, for example:

1) Energy and Frequency Relation:

E is the energy of a photon (for instance) and ν its frequency.



Remember, particles are waves in QM. This shows that energy is quantized and comes in discrete packets (called *quanta*).

2) Wave-Particle Duality:

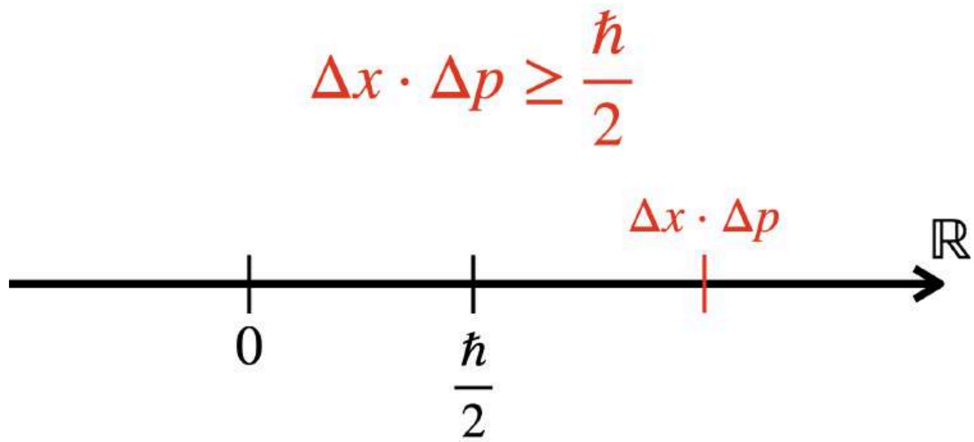
$$\lambda = \frac{h}{p}$$

This equation links the wavelength λ of a particle to its momentum p through the *de Broglie relation*. It shows how particles can behave like waves.

3) Heisenberg Uncertainty Principle:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

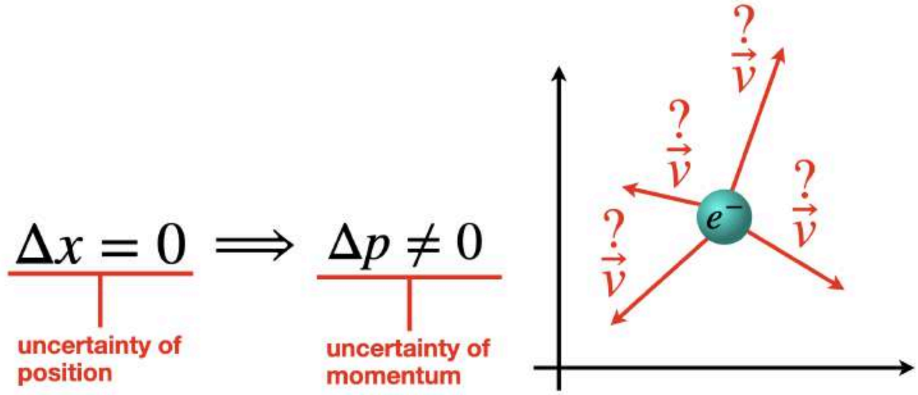
As you might know, Δ in physics usually means “variation”. So, the LHS of this inequality tells us that “the variation of position times the variation of momentum of a particle is always greater than a fixed positive number, which is $\frac{\hbar}{2}$ ” . A better way of putting it, is that “the uncertainty in the particle’s position multiplied by the uncertainty in the particle’s momentum (which is actually the uncertainty in the particle’s velocity, since its momentum is just mass times velocity, and usually we are certainty about the particle’s mass, but not about its velocity), this number cannot be zero or negative”.

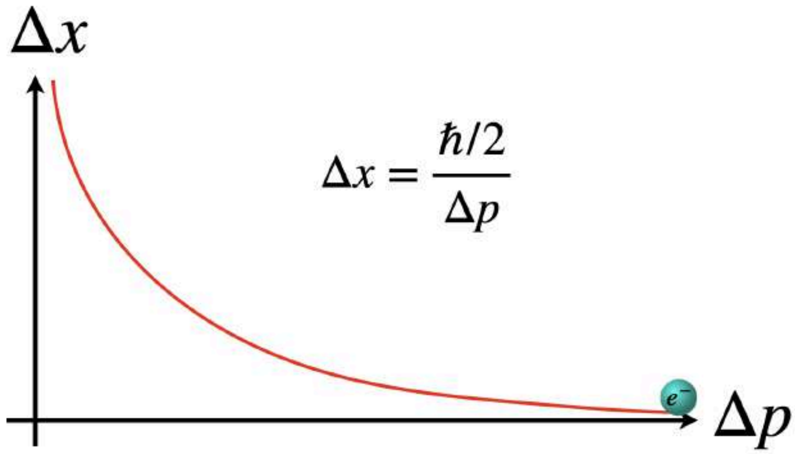


What would happen if it were zero? Well, arithmetics tells us that this implies that either $\Delta x = 0$ or $\Delta p = 0$, or maybe both. Let's analyse each case:

$$\Delta x \cdot \Delta p = 0 \quad \Rightarrow \quad \boxed{\Delta x = 0} \text{ or/and } \boxed{\Delta p = 0}$$

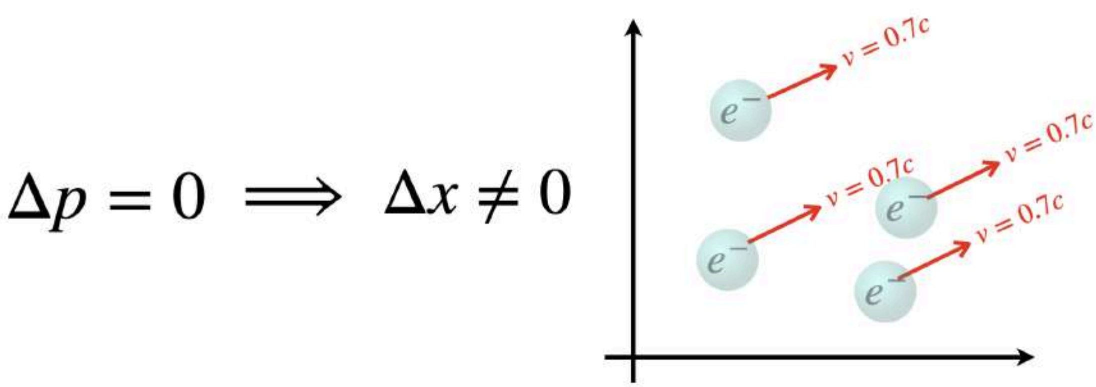
<u>Case 1</u>: $\Delta x = 0$. Experiments in QM show us that this implies that $\Delta p \neq 0$. In other words, if your uncertainty about the particle's position is zero, namely $\Delta x = 0$, then you know for sure where the particle is located. However, QM tells us that this implies that your uncertainty about the particle's momentum is infinite.

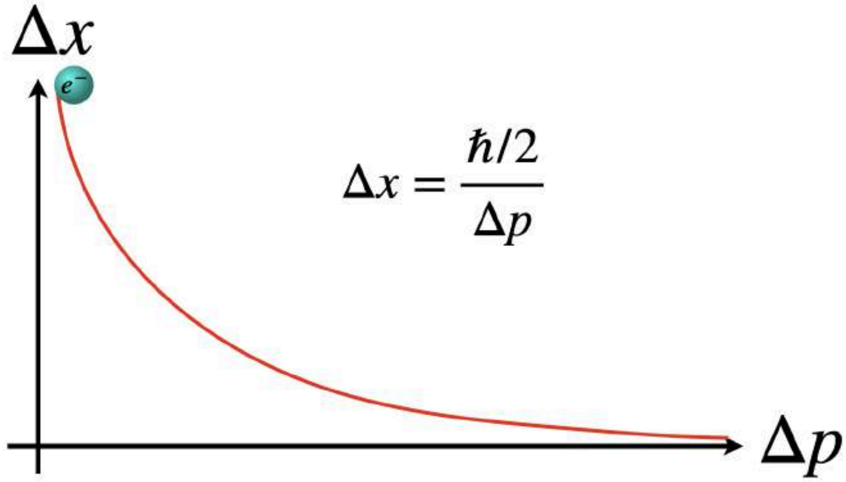




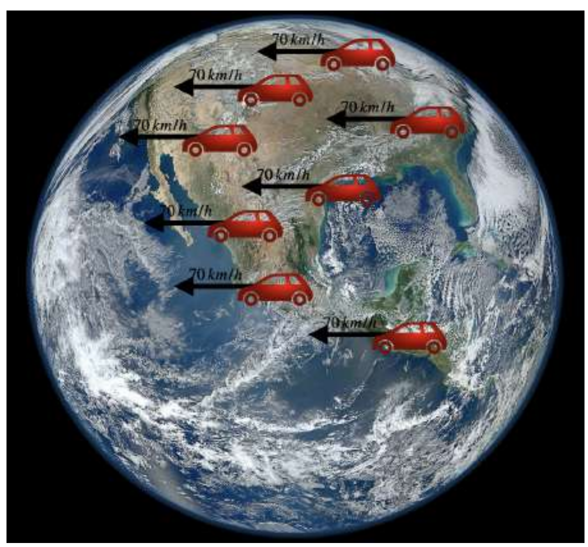
So, you have no clue about the particle's velocity, which is a big problem if you are trying to predict where it will be next, and how the system will evolve.

<u>Case 2</u>: $\Delta p = 0$. QM says that, as a consequence, $\Delta x \neq 0$. In this case, if your uncertainty about the particle's momentum (or velocity) is zero, namely if $\Delta p = 0$, then you know for sure where the particle is heading to, and with what speed. However, QM also tells us that this implies that your uncertainty about the particle's position is infinite.

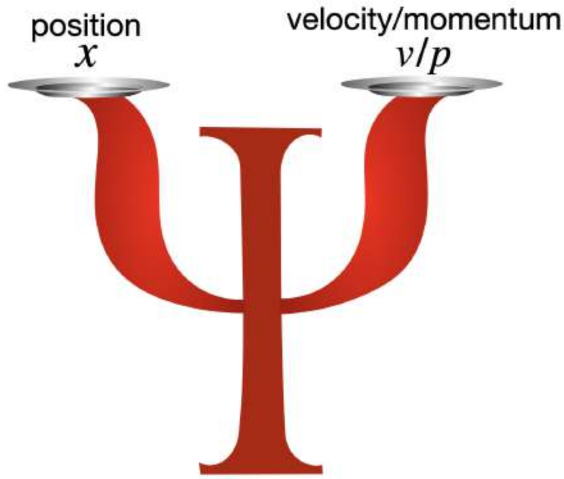




So, imagine I tell you that my car is moving at 70 km/h west, but I do not tell you anything about its position. It could be anywhere in the whole planet Earth. I mean, this information would not help you at all to predict the car's motion. It's useless.

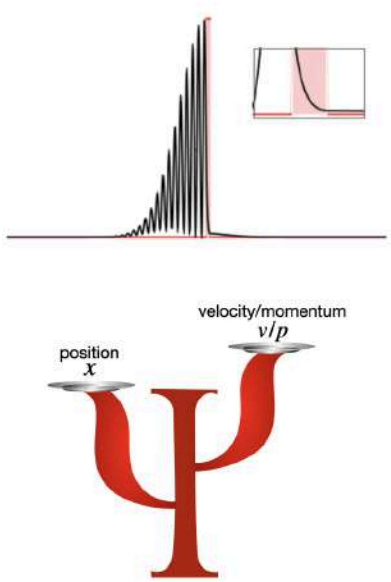


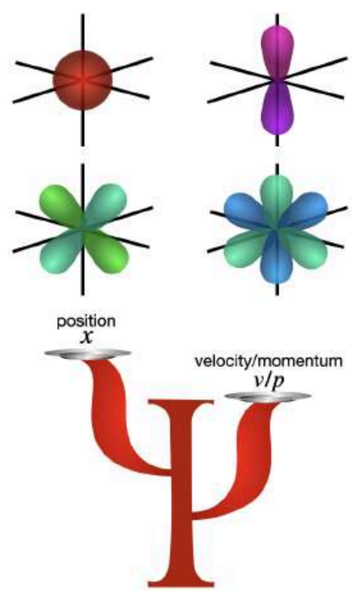
So, what we try to do in QM is find the balance between these two things. In some contexts it may be worth it to sacrifice information about one quantity for the other.



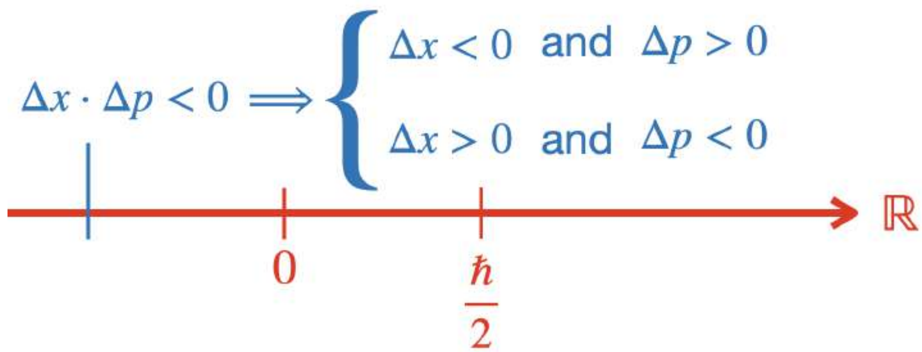
For example, sometimes in QM we would want to know with 80% certainty the particle's position, and have just 20% certainty about the particle's momentum. In other situations we may want to be 30% certain about its position, and 70% sure about its momentum. It really depends on the problem you are trying to solve.

More concrete examples would be in experiments involving *quantum tunneling*. You may care more about the particle's momentum (to calculate its kinetic energy) than its exact position. Conversely, in systems like *atomic orbitals*, the position of an electron may be more relevant than its momentum.

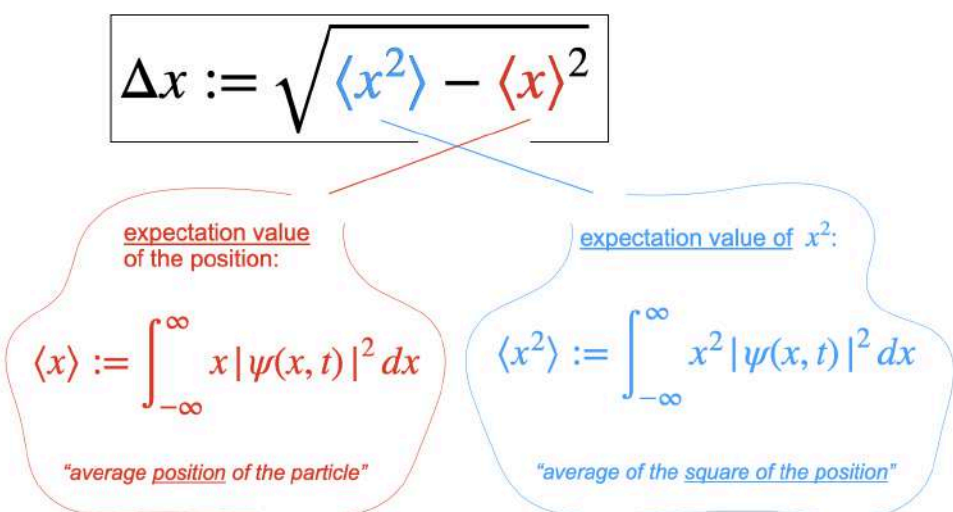


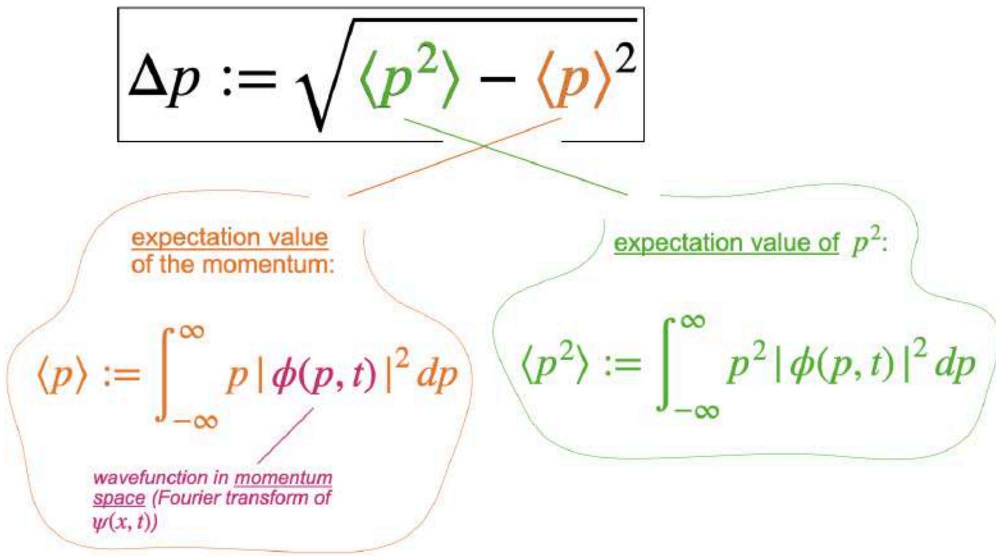


Let's see now what happens if $\Delta x \cdot \Delta p < 0$. Arithmetics tells us that: either $\Delta x < 0$ and $\Delta p > 0$ or $\Delta x > 0$ and $\Delta p < 0$. In any case we have a contradiction.

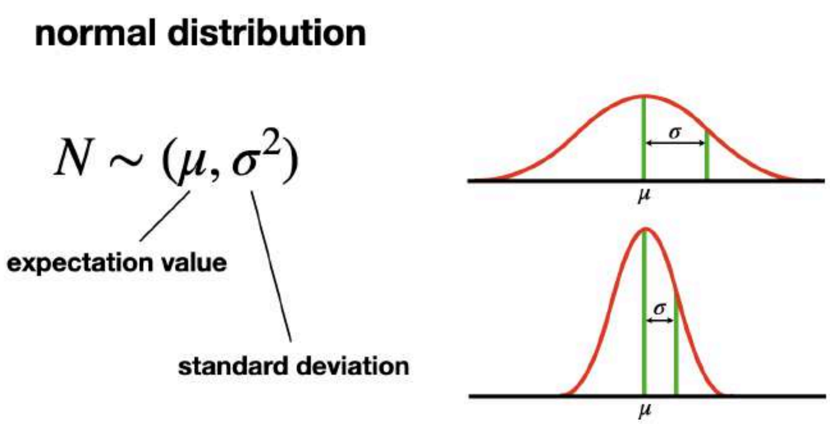


These quantities, Δx and Δp, cannot be negative since these are standard deviations of the position and momentum, respectively. They are defined as:

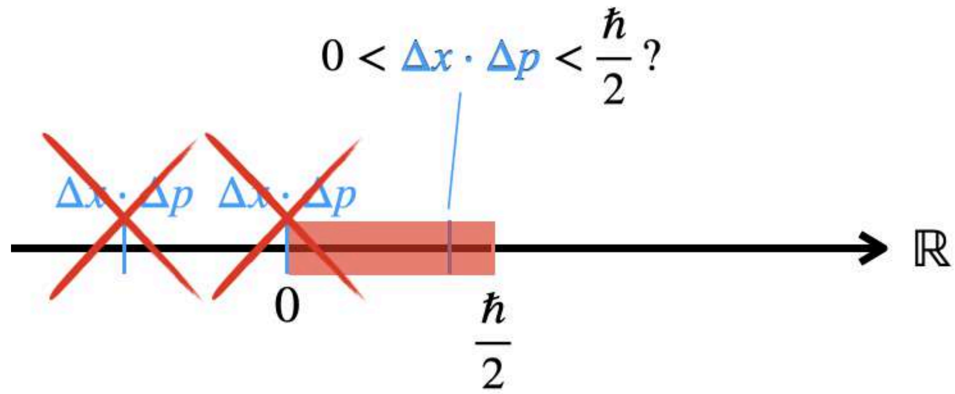




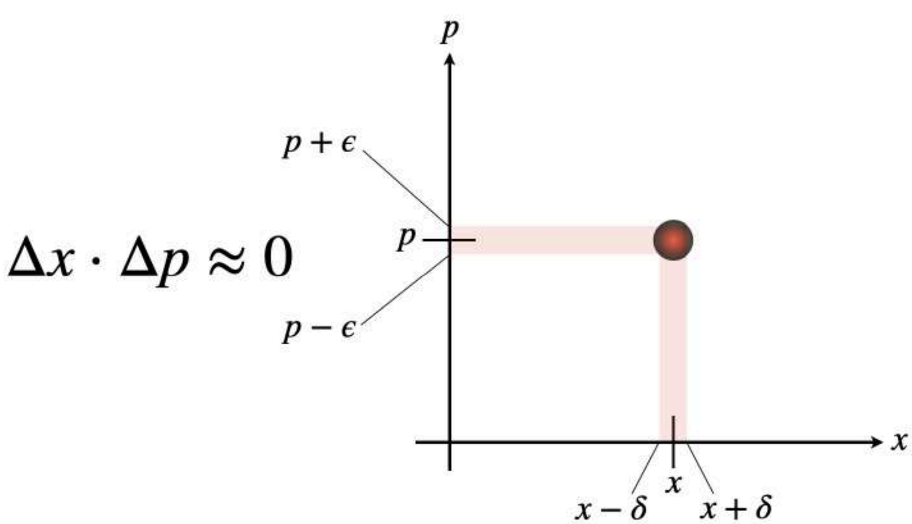
In a normal distribution $N \sim (\mu, \sigma^2)$, μ is the *expectation value* and σ is the *standard deviation*:



Since Δx and Δp are square roots of real numbers, these quantities cannot be negative. So, we already know that the result of $\Delta x \cdot \Delta p < 0$ cannot be less than zero, and cannot be zero.



What about values between zero and $\frac{\hbar}{2}$? So, for $0 < \Delta x \cdot \Delta p < \frac{\hbar}{2}$? Well, it turns out to be impossible as well, but the proof would require us to digress way too much, since its proof involves the *Cauchy-Schwarz Inequality*. However, we can clearly see that there must be a positive lower bound of $\Delta x \cdot \Delta p$, otherwise we could use calculus to get infinitesimally close to zero from the right



without ever touching zero, which for all practical purposes in physics would allow us to be “almost” certain of a particle’s position and velocity simultaneously. Experimentally, no system has ever been observed to violate this bound, as it is a cornerstone of quantum physics.

Going back to the Schrödinger equation now

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

we notice that on the LHS we have the derivative of the wavefunction with respect to time: $\frac{\partial\Psi}{\partial t}$. So, it can be intuitively interpreted as how much the wavefunction Ψ changes with time.

The term

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

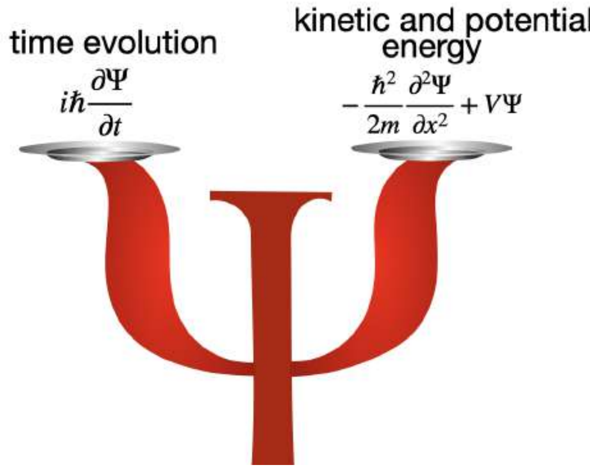
represents the particle's *kinetic energy*.

The term

$$V\Psi$$

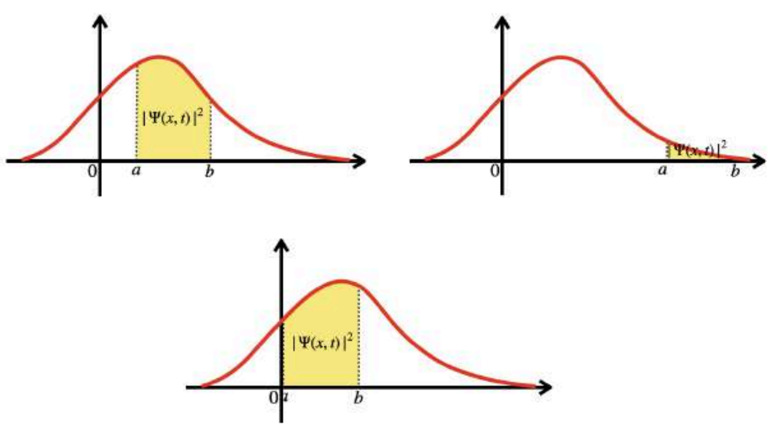
is the *potential energy* and it represents the **stored energy** of a particle due to its position in a system or its interaction with external forces.

The Schrödinger equation describes a balance between the **time evolution** of the wavefunction (left-hand side) and the **effects of kinetic and potential energy** (right-hand side). Intuitively, the kinetic energy term relates to how the wavefunction spreads or oscillates in space. The potential energy term determines how the environment (e.g., a potential well or barrier) influences the wavefunction.

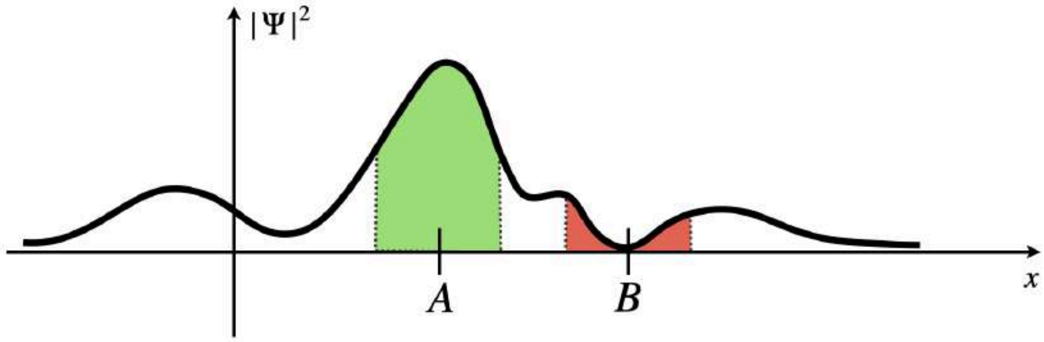


Again, the goal here is to find solutions to this equation. These wavefunctions represent possible scenarios that extend beyond mathematics into the physical world we live in. But a few questions that have probably been lurking in your mind throughout this whole document is: what is the “wavefunction”? What does it actually do once you've found it? How can this mathematical object be said to describe the *state* of a particle?

The answer is provided by *Born's statistical interpretation* of the wavefunction, which states that the modulus squared of the complex wavefunction $\Psi(x,t)$ gives the probability density of finding the particle at position x and time t : $|\Psi(x,t)|^2$. So, the wavefunction $\Psi(x,t)$ on its own has no direct physical meaning (it is purely mathematical). However, when we calculate its modulus squared, it gains a concrete physical interpretation as the probability of finding the particle at a specific point in spacetime.

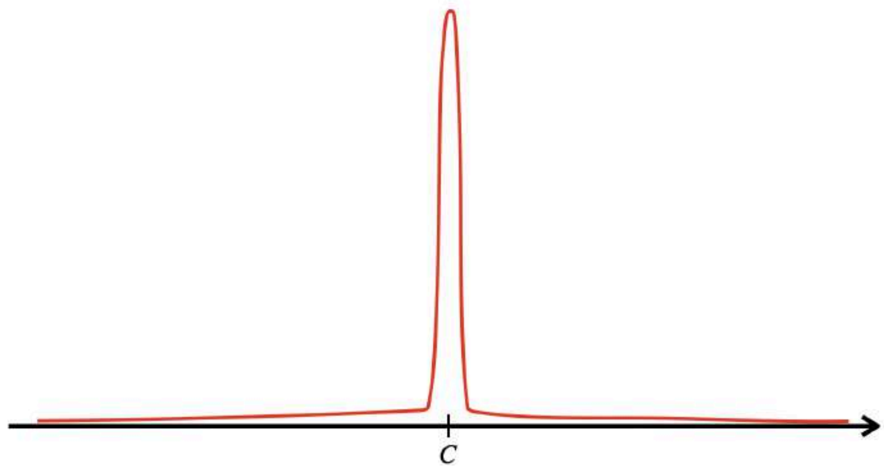


For the wavefunction below, for example, you would be quite likely to find the particle in the vicinity of point A, and relatively unlikely to find it near point B.

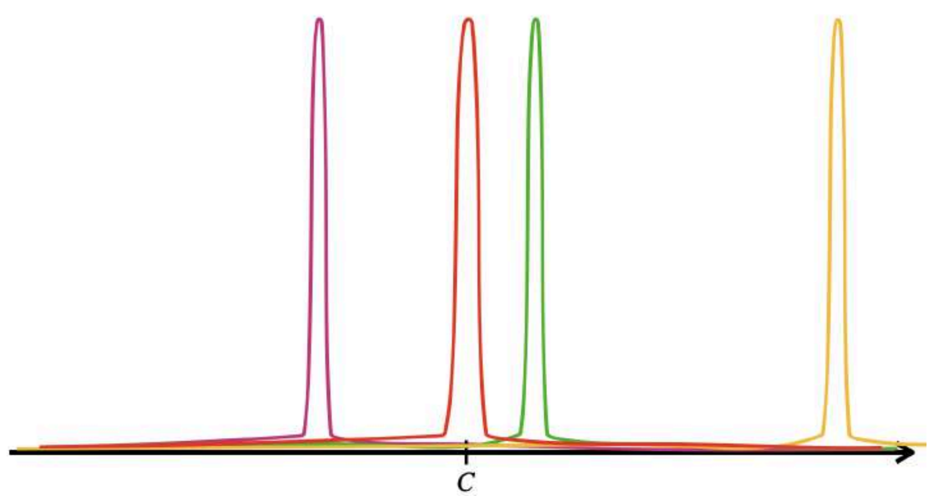


The statistical interpretation introduces a kind of *indeterminacy* into quantum mechanics, for even if you know everything the theory has to tell you about the particle (i.e. its wavefunction), you cannot predict with certainty the outcome of a simple experiment to measure its position — all quantum mechanics has to offer is statistical information about the possible results. This indeterminacy has been very disturbing to physicists and philosophers alike. A classic example is Einstein's opinion that the theory is just incomplete. The question is: Is it a specific feature of nature itself? Is it a deficiency in the theory? Or is it a fault in the measuring apparatus?

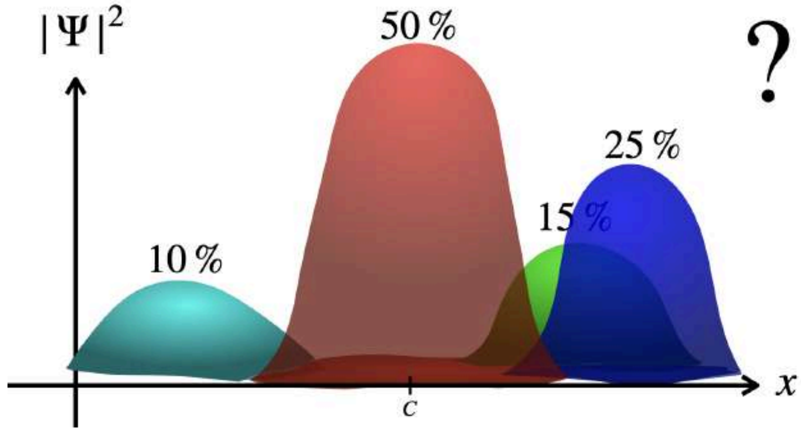
Suppose I do measure the position of the particle, and I find it to be at the point C. Question: Where was the particle just before I made the measurement? There are three plausible answers to this question, and they serve to characterize the main schools of thought regarding quantum indeterminacy.



The first one is the **realist** position: The particle was at C. This certainly seems like a reasonable response, and it is the one Einstein advocated. Note, however, that if this is true then quantum mechanics is an incomplete theory, since the particle really was at C, and yet quantum mechanics was unable to tell us so. To the realist, indeterminacy is not a fact of nature, but a reflection of our ignorance. A realist would say: “Evidently, Ψ is not the whole story — some additional information (known as a *hidden variable*) is needed to provide a complete description of the particle”.



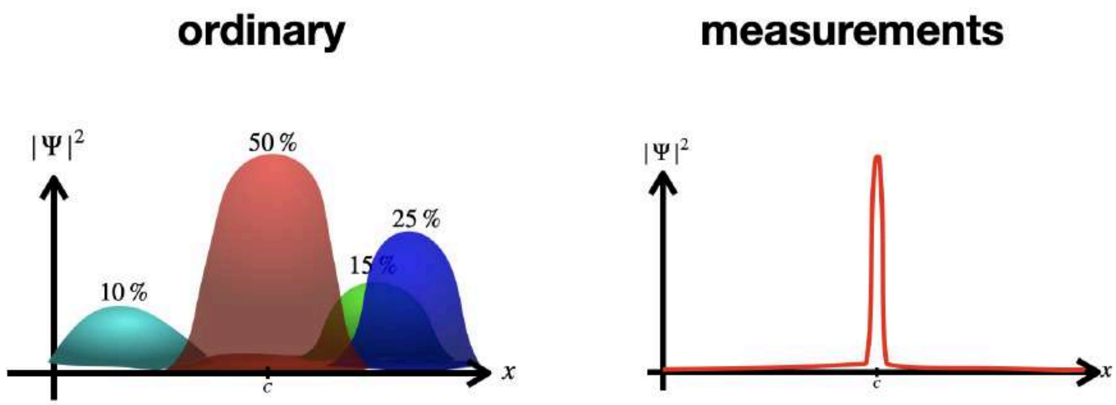
The second school of thought is the **orthodox** position: The particle wasn't really anywhere. It was the act of measurement that forced the particle to "take a stand" (though how and why it decided on the point C we dare not ask). An orthodox would say: "Observations not only disturb what is to be measured, they produce it. We compel the particle to assume a definite position". This view is also called the Copenhagen interpretation, and it is associated with Bohr and his followers. Among physicists it has always been the most widely accepted position. Note, however, that if it is correct there is something very weird about the act of measurement in nature.



And the third is the **agnostic** position: Refuse to answer. I mean, what sense can there be in making assertions about the status of a particle before a measurement, when the only way of knowing whether you were right is precisely to conduct a measurement, in which case what you get is no longer "before the measurement"? It is metaphysics (in the pejorative sense of the word) to worry about something that cannot, by its nature, be tested. For decades this was the "fall-back" position of most physicists: They would try to sell you answer 2, but if you were persistent they would switch to the third one and end the conversation.

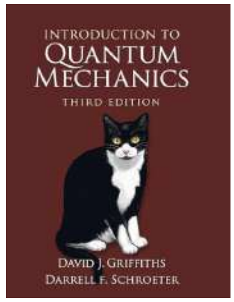
However in 1964 John Bell astonished the physics community by showing that it makes an observable difference if the particle had a precise (though unknown) position prior to the measurement. Bell's discovery effectively eliminated agnosticism as a viable option, and made it an experimental question whether the first or second is the correct choice. Unfortunately we don't have time to go into the math that allowed him to conclude so, but let us know in the comment section if you would like to watch a video dedicated to Bell's theorem. For now, it is enough to say that the experiments have confirmed decisively the orthodox interpretation: A particle simply does not have a precise position prior to measurement. It is the measurement process itself that insists on one particular number, and therefore, in a certain sense, creates the specific result, which is limited only by the statistical weighting imposed by the wavefunction.

But what if I made a second measurement, immediately after the first? Would I get C again, or does the act of measurement spit out some completely new number each time? On this question everyone is in agreement: A repeated measurement (on the same particle) must return the same value. Indeed, it would be tough to prove that the particle was really found at C in the first instance if this could not be confirmed by immediate repetition of the measurement. How does the orthodox interpretation account for the fact that the second measurement is bound to give the value C? Clearly, the first measurement radically alters the wavefunction, so that it is now sharply peaked about C.



We say that the wavefunction collapses upon measurement, to a spike at the point C (Ψ soon spreads out again, in accordance with the Schrödinger equation, so the second measurement must be made quickly). There are, then, two entirely distinct kinds of physical processes: "ordinary" ones, in which the wavefunction evolves in a natural way under the Schrödinger equation, and "measurements", in which Ψ suddenly and discontinuously collapses.

This document was based on the first chapter of the book “Introduction of Quantum Mechanics” by David J. Griffiths:



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