The Riemann-Stieltjes Integral

Last time we saw 33 types of integrals. Let's study the Riemann-Stieltjes integral today, and find a very nice visualization of it.

Without further ado, let's get into it! Let *f* and *g* be functions from a real interval *[a,b]* to the real line. Then this integral is the Riemann-Stieltjes integral of *f* w.r.t. *g*:

$$f : [a,b] \longrightarrow \mathbb{R} \qquad g : [a,b] \longrightarrow \mathbb{R}$$

$$\boxed{\int_a^b f(x)\, dg(x)}$$

is the Riemann-Stieltjes integral of f w.r.t. g.

with respect to

Let's construct a simple example before defining it rigorously. We will choose 2 functions *f(x)* and *g(x)* such that *g(x)* is not simply *x* or a *constant*, otherwise this example would reduce to the regular Riemann integral. In the case *g* is a constant the integral is just zero.

$$g(x) \neq x$$

$$\implies \int_a^b f(x)\, dg(x) = \int_a^b f(x)\, \overbrace{d(x)}^{dx} = \int_a^b f(x)\, dx$$

$$g(x) \neq constant$$

$$\implies \int_a^b f(x)\, dg(x) = \int_a^b f(x)\, \overbrace{d(constant)}^{zero} = 0$$

The first non-trivial example is *f(x)=x* and *g(x)=x²* over the interval *[0,1]*. *'I'*, here, will be defined as the Riemann-Stieltjes integral from *zero* to *1* of *xd(x²)*. The final result is ⅔.

① $f(x) = x \;\; \wedge \;\; g(x) = x^2 \;\; over \; [0, 1]:$

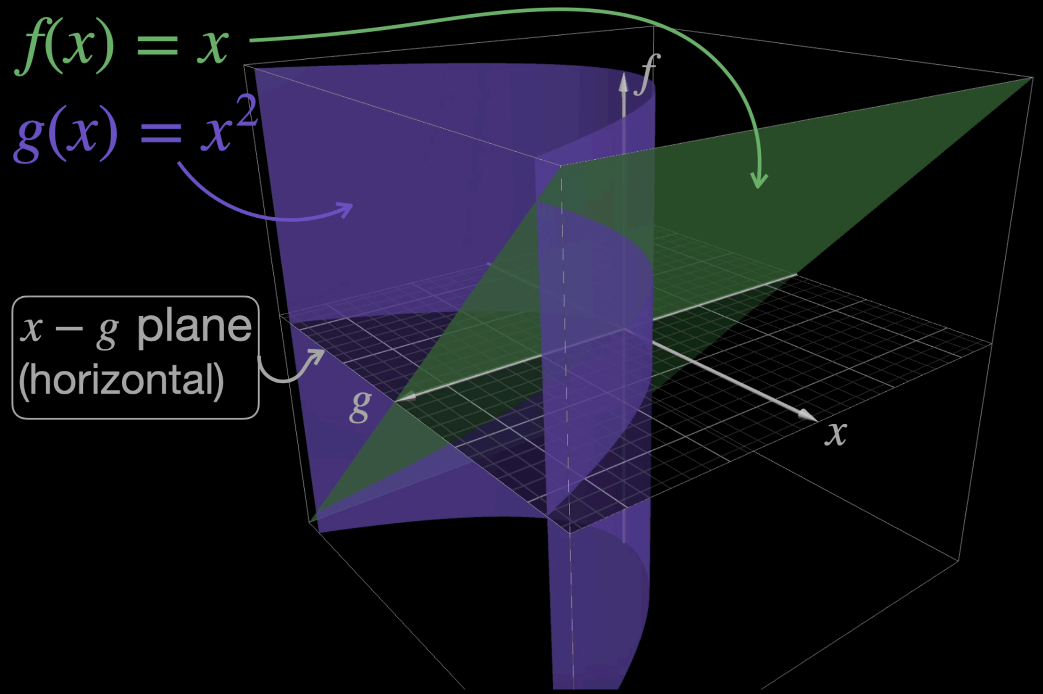
$$I := \int_0^1 x\, d\left(x^2\right) = \int_0^1 x \cdot (2x)\, dx =$$

$$= 2\int_0^1 x^2 dx = 2 \cdot \left(\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}$$

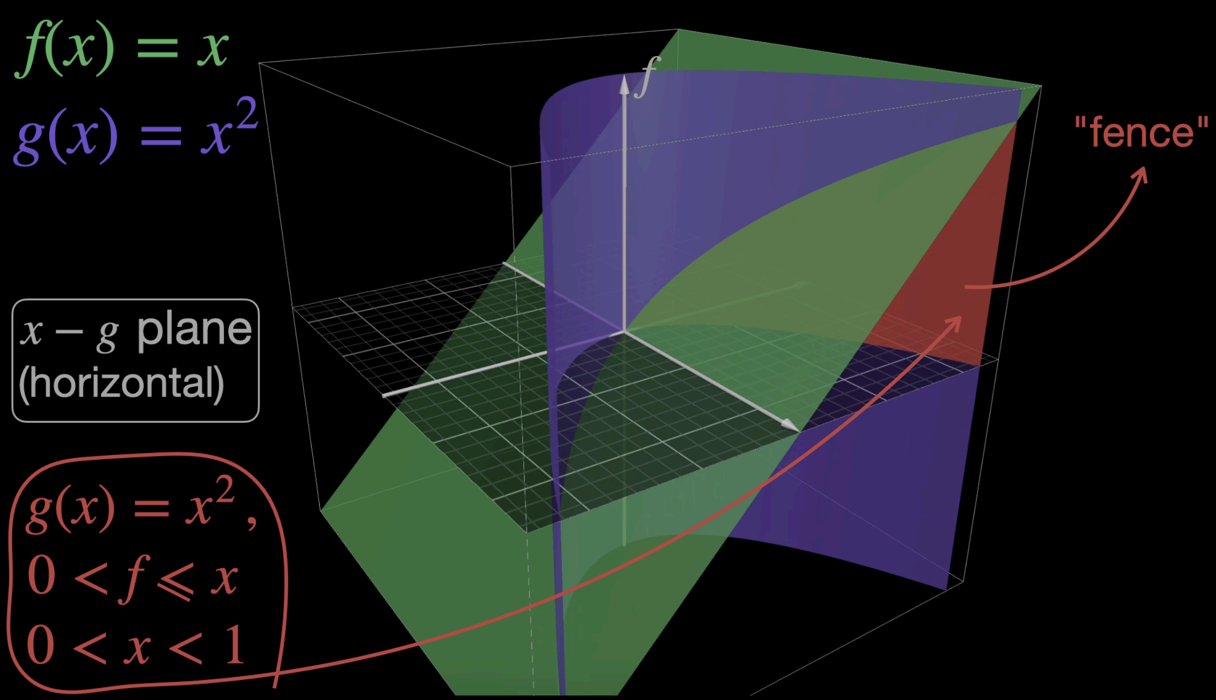
But how can we interpret this result geometrically?

(BTW, consider becoming a member of the channel!) Thanks!

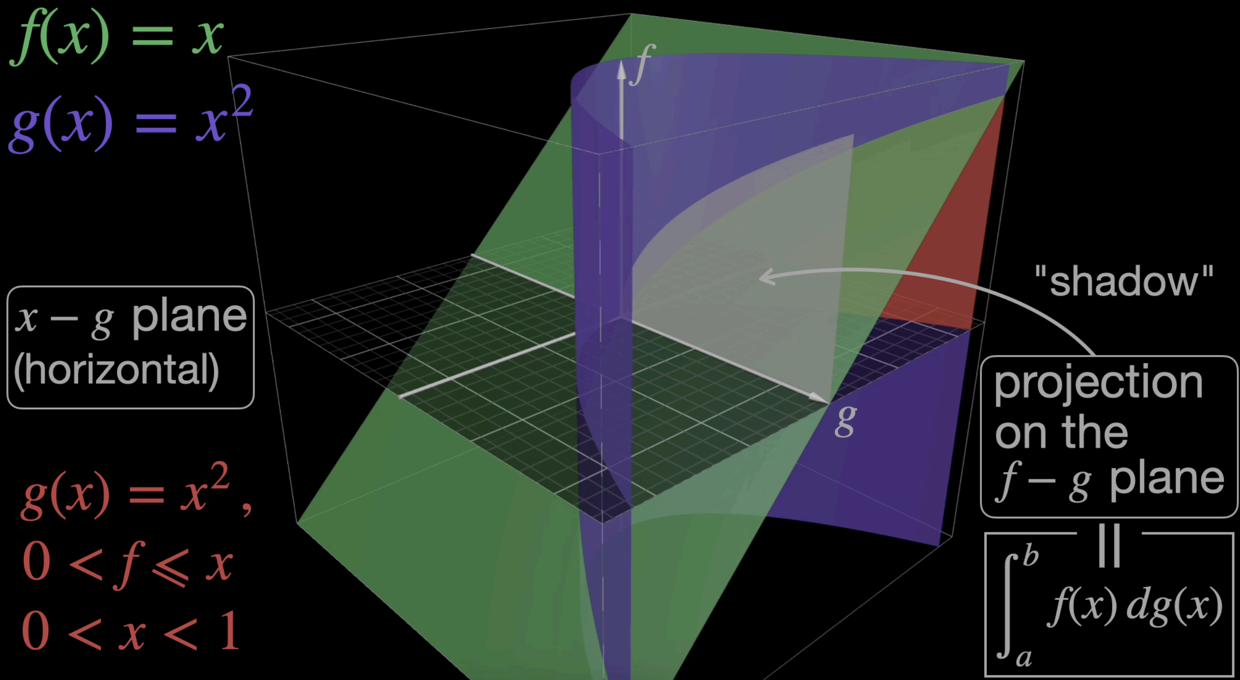
We start by defining these 3 axes: f, g and x. This horizontal plane is formed by x and g. Observe them attentively in order to fully grasp what's going on here.



We're going to paint a piece of one of these surfaces in orange.

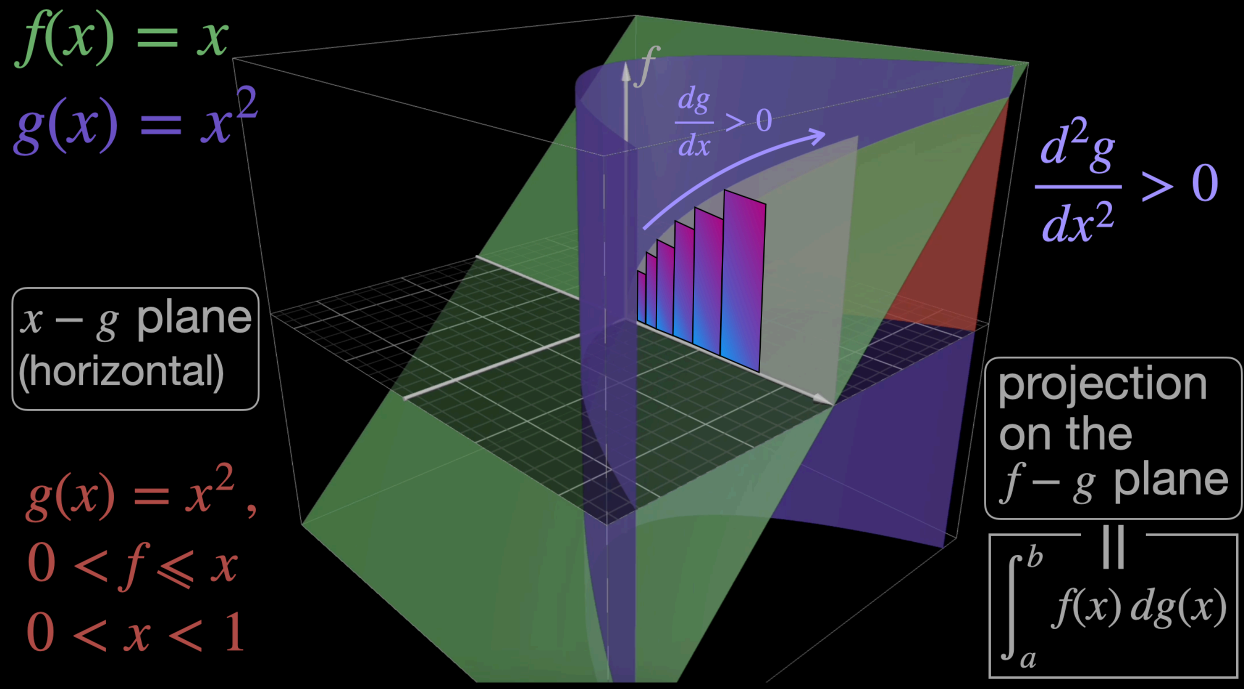


It is important that you have a very good visual understanding of how we defined this “fence”, because the area of one of its “shadows” will be the result of the Riemann-Stieltjes integral.



This is the fence’s projection on the *f*-*g* plane. In other words, that’s the “shadow” that will give us the result of the Riemann-Stieltjes integral. Try to understand how this projection was made before moving on.

Now, this integral is not like the Riemann integral, and the first difference is that the widths of each rectangle involved in the sum are different depending on the variation (or derivative) of g. In this case, as we move away from the origin the derivative of g is positive, so the widths of the rectangles are not zero, and the second derivative of g is positive as well, which means that the widths of these rectangles become larger and larger.



We will repeat that using different words, since that's literally the key point for fully understanding the Riemann-Stieltjes integral. We are integrating *f(x)* with respect to *g(x)*, meaning that instead of taking the usual uniform spacing along the *x*-axis, the "widths" of the subintervals are determined by changes in *g(x)* (i.e., the derivative of *g*). When *g* grows faster than *x*, the intervals are "stretched" more for larger *x*. Geometrically, we are still summing areas of rectangles with heights given by *f(x)*, but the widths of these rectangles are no longer uniform, and instead, they grow according to the derivative of *g(x)*. This gives a different kind of "weighted" area accumulation compared to the regular Riemann integral.

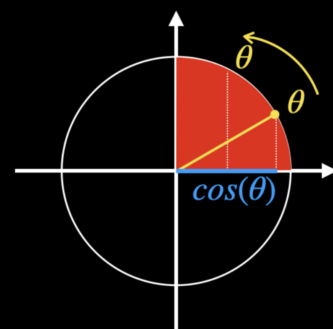
The second example is the case in which *f(x)=sin(x)* and *g(x)=cos(x)*, and we want to perform the integral over the interval *[0, pi/2]*.

② $f(x) = sin(x) \;\wedge\; g(x) = cos(x)$ *over* $\left[0, \frac{\pi}{2}\right]$:

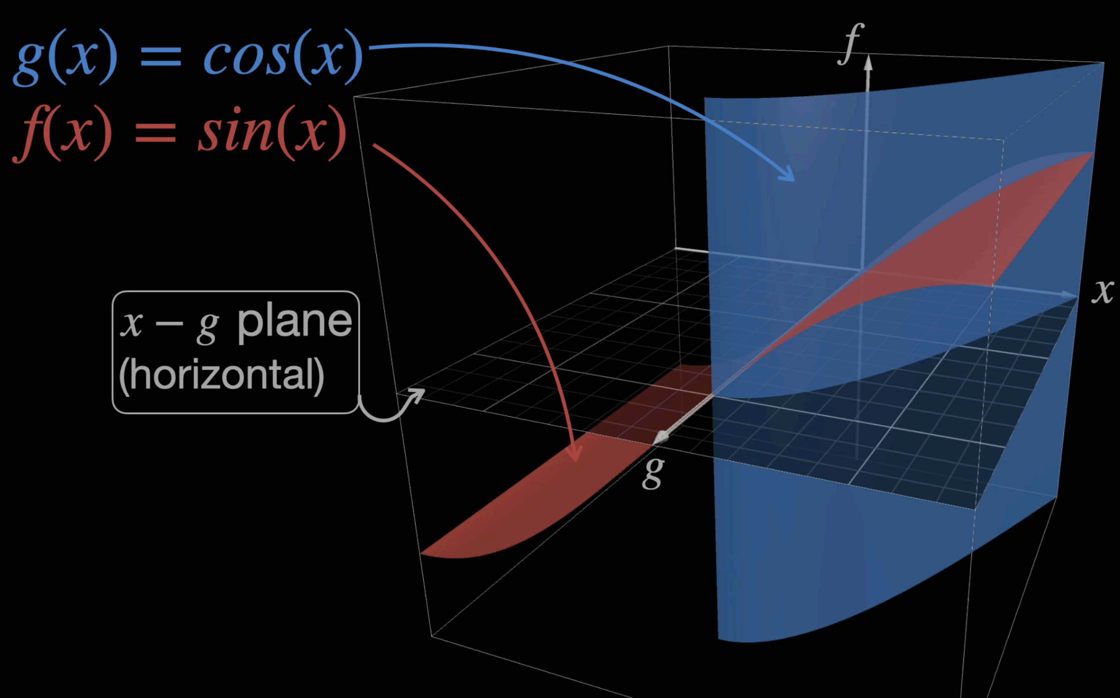
$$I := \int_0^{\frac{\pi}{2}} sin(x)\, d\left(cos(x)\right) = \int_0^{\frac{\pi}{2}} sin(x)\cdot\left(-sin(x)\right) dx =$$

$$= \int_{\frac{\pi}{2}}^{0} sin^2(x)\, dx = -\frac{\pi}{4}$$

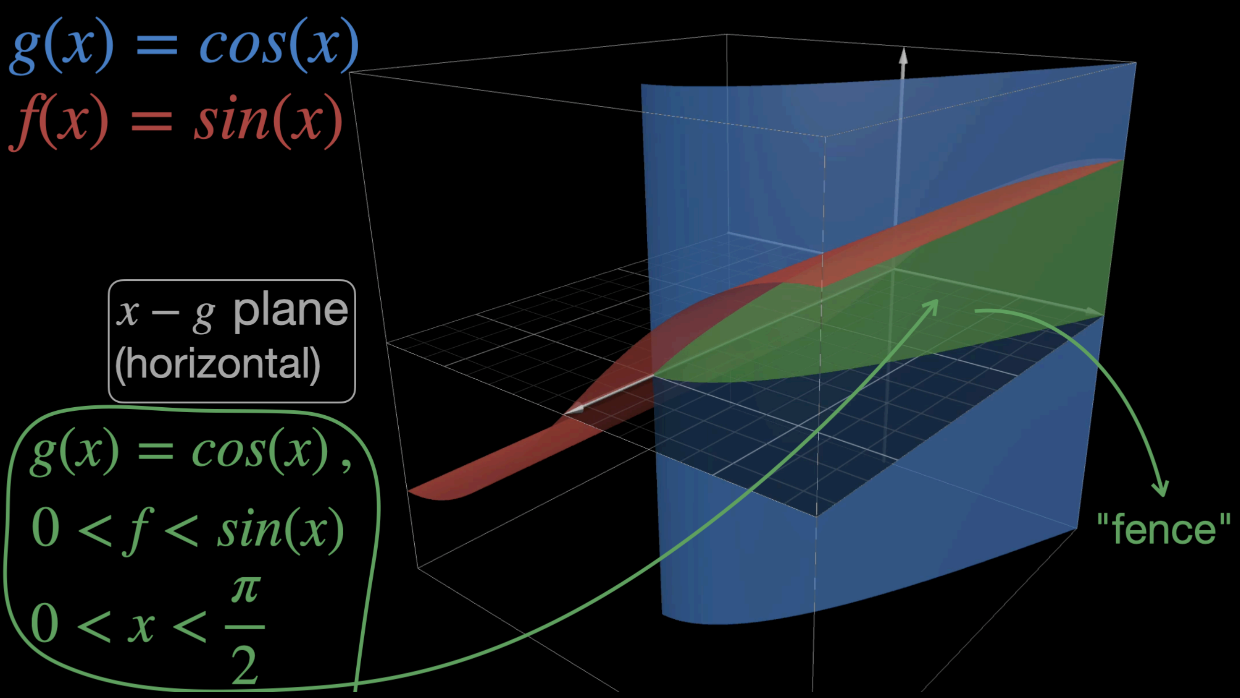
why???

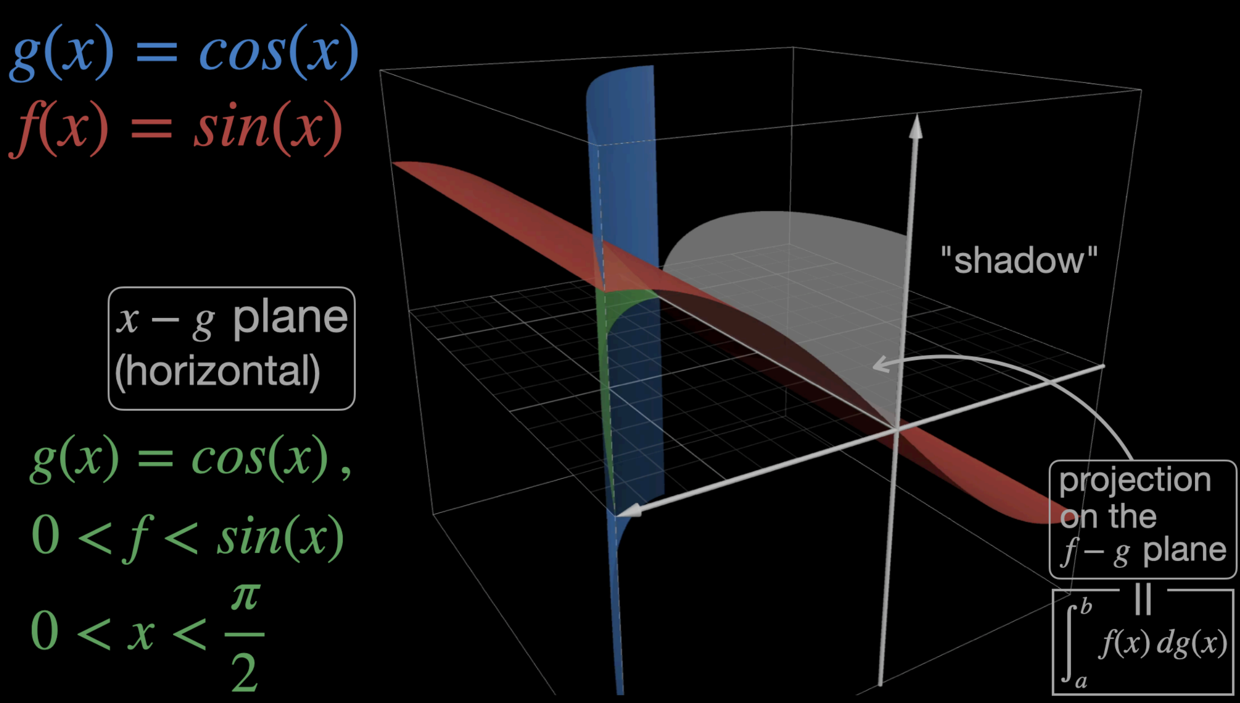


But wait a second! How come the integral is negative?! How can the area of the "shadow" cast by this part of the surface be negative? The minus sign comes from the direction in which the surface (or "fence" in our previous analogy) is being projected onto the *f-g* plane, and it is directly related to whether the function *g(x)* is increasing or decreasing. In this case, *g(x)=cos(x)*, which is decreasing in the first quadrant, which is our interval of integral, *[0, pi/2]*.

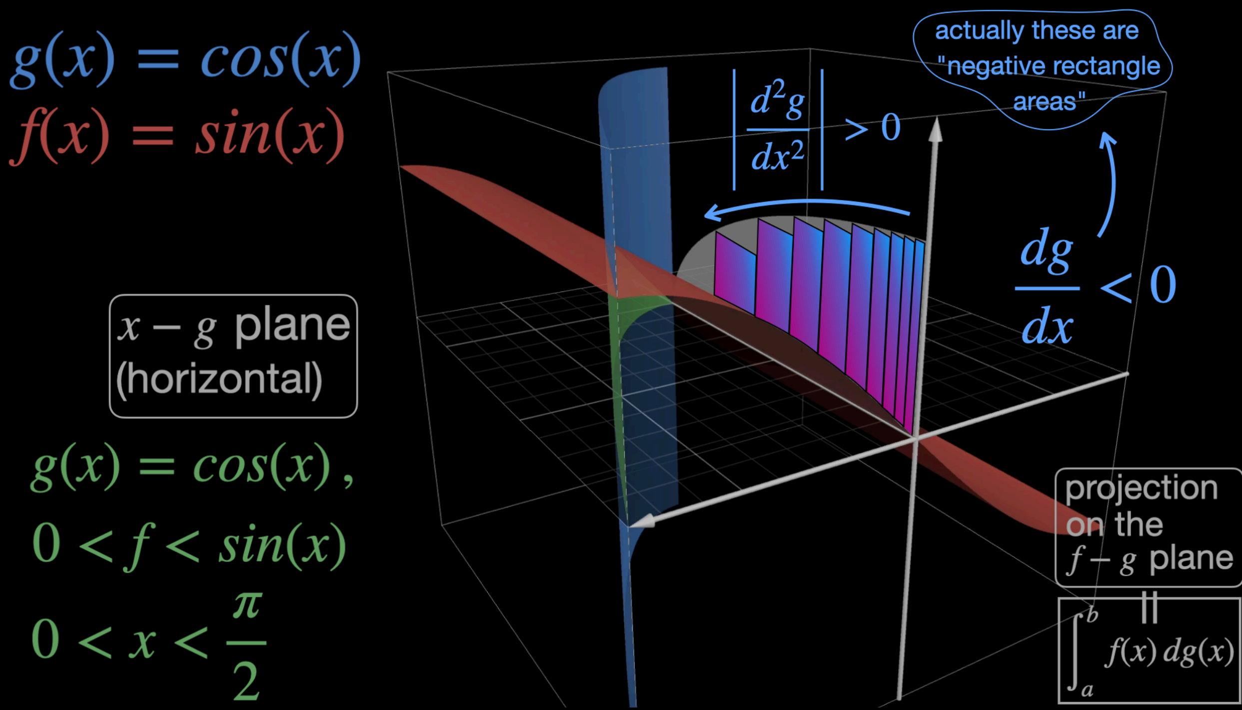


Our “fence” will be the green piece of surface below, which is just *g(x)* restricted by the intervals *[0 , sin(x)]* for the *f* axis, and *[0 , pi/2]* for the *x* axis. Again, take a close look at it so that you can understand what kind of projection we will make.





Notice that here as we move away from the origin the widths of the rectangles increase, since the second derivative of g is non-zero. The first derivative is non-zero as well, which means that the widths of the rectangles will never be zero in this interval. In fact, its first derivative is negative, which implies that actually these are negative rectangle areas.

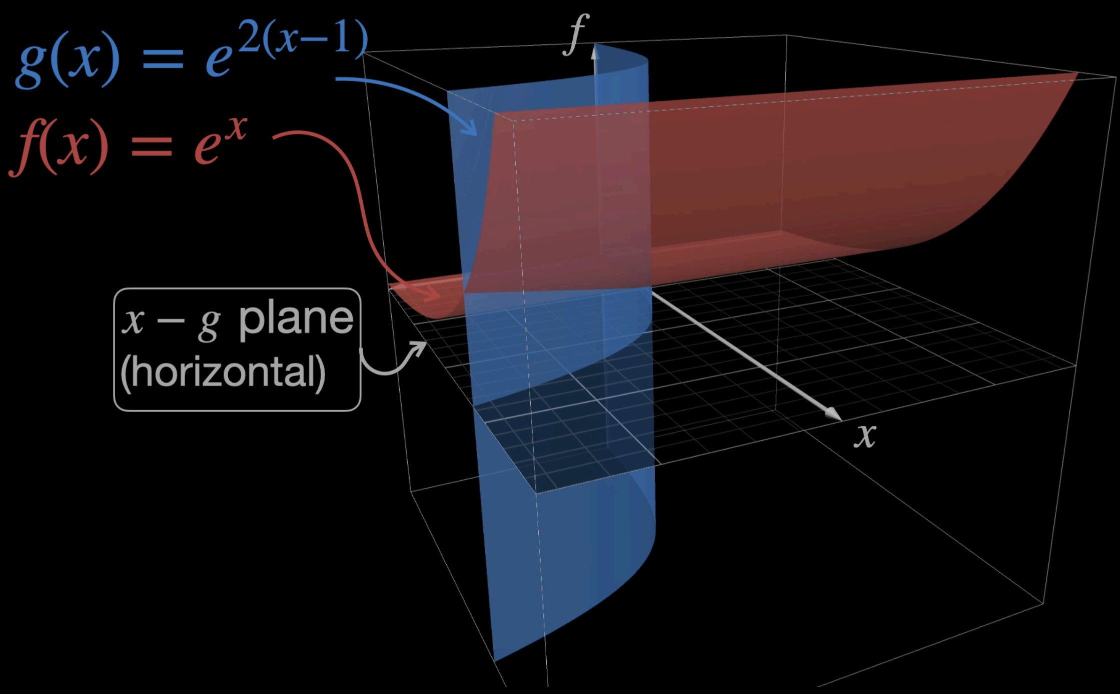


Let's move to the third example now:

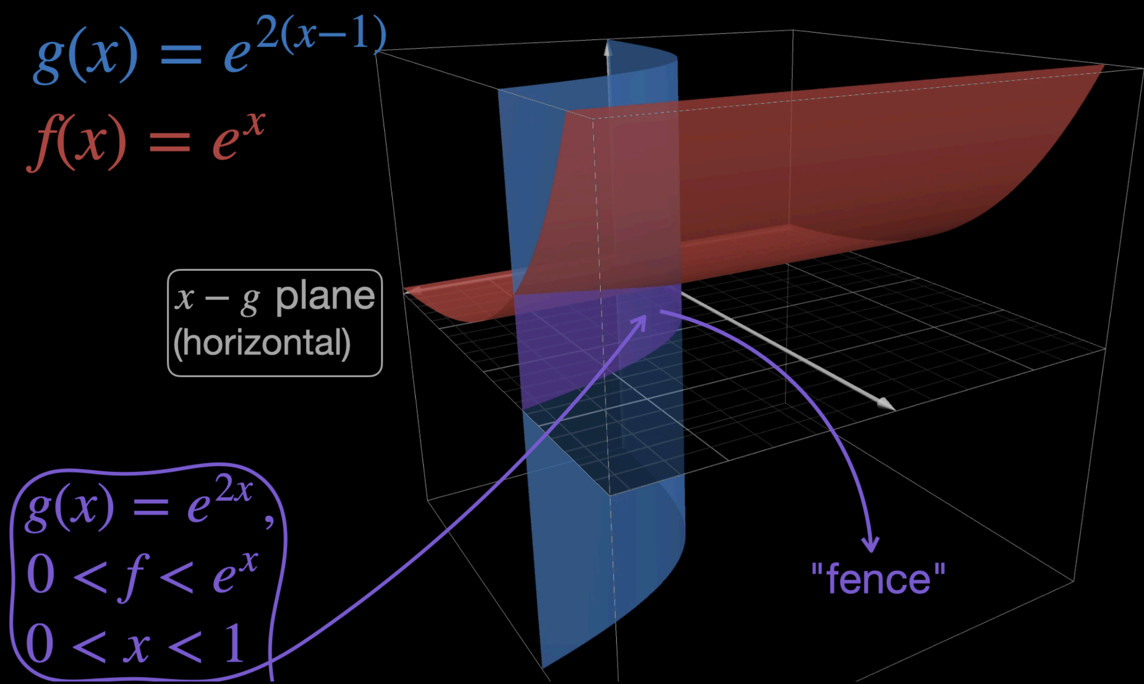
③ $f(x) = e^x \ \wedge \ g(x) = e^{2(x-1)}$ *over* $[0, 1]$:

$$I := \int_0^1 e^x \, d\left(e^{2(x-1)}\right) = \int_0^1 e^x \cdot 2\, e^{2(x-1)} dx =$$

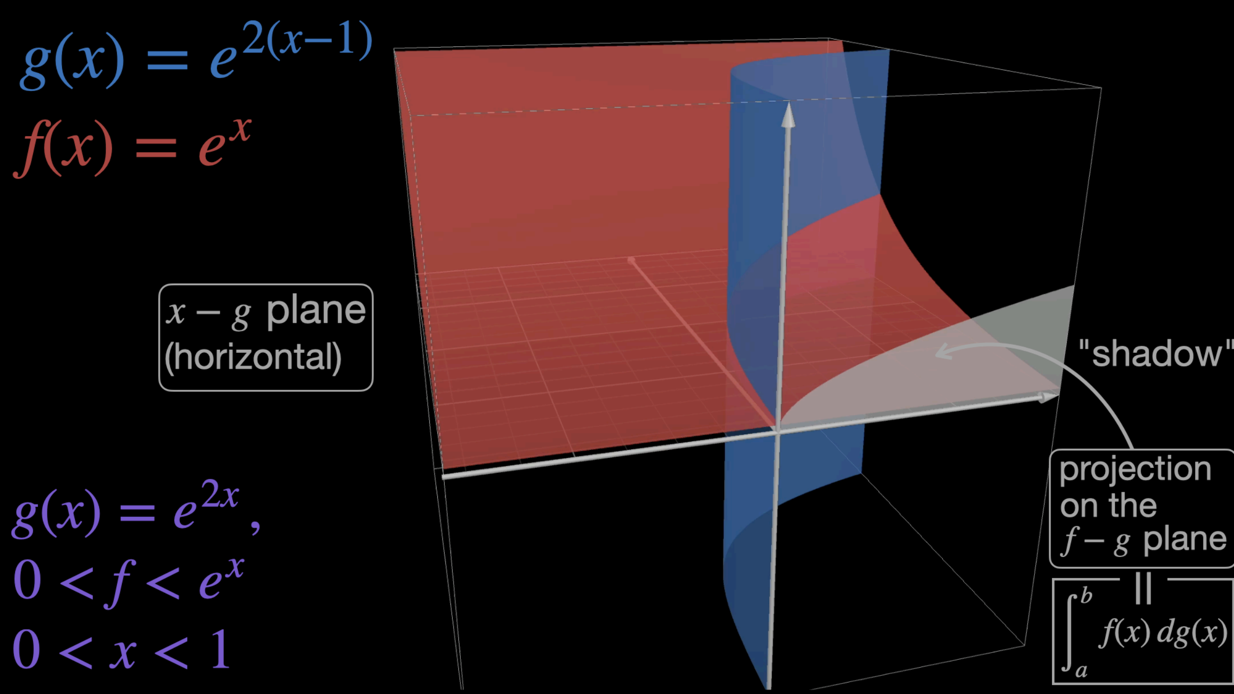
$$= 2\int_0^1 e^{3x-2}\, dx = 2 \cdot \left[\frac{e^{3x-2}}{3}\right]\Bigg|_0^1 = \frac{2}{3}\left(e - \frac{1}{e^2}\right)$$



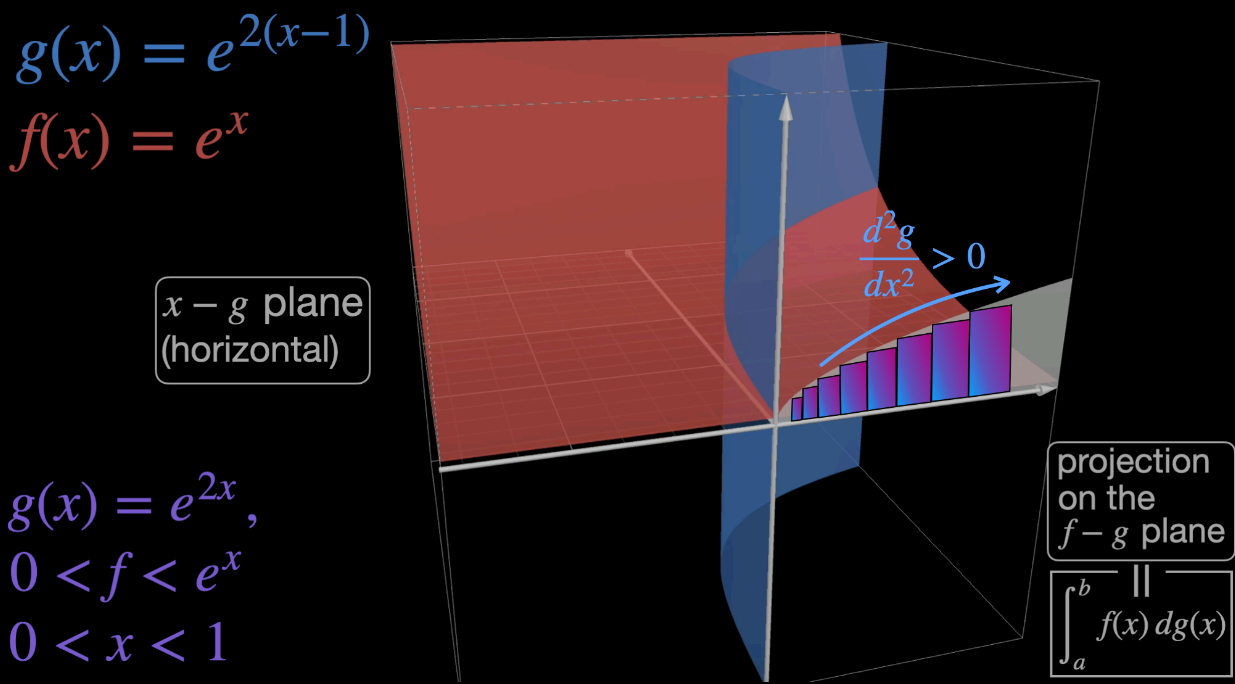
Once again we see that this purple surface below is the "fence" that we are interested in. Its "shadow", or projection, is the gray one. The widths of each rectangle increase since the second derivative of g is positive. In this case, we do have positive areas.



$g(x) = e^{2(x-1)}$
$f(x) = e^{x}$
$x - g$ plane (horizontal)
$g(x) = e^{2x}$, $0 < f < e^{x}$, $0 < x < 1$
"fence"



$g(x) = e^{2(x-1)}$
$f(x) = e^{x}$
$x - g$ plane (horizontal)
$g(x) = e^{2x}$, $0 < f < e^{x}$, $0 < x < 1$
"shadow"
projection on the $f - g$ plane
$\int_a^b f(x)\, dg(x)$



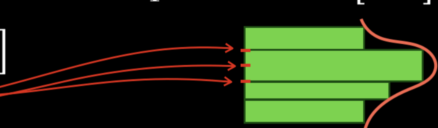
Now that we have a good grasp of how the Riemann-Stieltjes integral works, its geometrical interpretation, and how to reduce it to a simple Riemann integral, let's see its rigorous definition:

Definition: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a real-value function, and $g : [a, b] \longrightarrow \mathbb{R}$ be a function of bounded variation on $[a, b]$, then the *Riemann-Stieltjes integral* is:

$$\int_a^b f(x)\, dg(x) = \lim_{\|\mathscr{P}\| \to 0} \sum_{i=1}^{n} f(\xi_i)\left[g(x_i) - g(x_{i-1})\right]$$

where $\mathscr{P} := \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$ is a partition of $[a, b]$,

$\|\mathscr{P}\| := \max_{i \in \{\overline{1,n}\}} (x_i - x_{i-1})$, $\xi_i \in [x_{i-1}, x_i]$

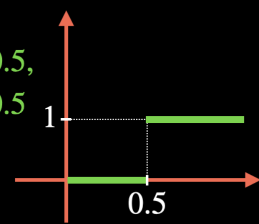


The fact that g is a function of bounded variations on *[a, b]* means that g may not have a derivative everywhere, but it is well-behaved enough for the integral to make sense in this interval. The Riemann-Stieltjes integral of *f(x) dg(x)* is defined as the limit of what are called "Riemann-Stieltjes sums", provided that this limit exists. *P* is a partition of the interval *[a, b]*, i.e. a set with points that divide the interval *[a, b]* into subintervals. The norm of the partition *P* is defined as the length of the largest subinterval, and \xi$_i$ is any point in the subinterval $[x_{i-1}, x_i]$, usually chosen to be the left endpoint, the right endpoint or the midpoint, depending on the convention.

Important conditions:

<u>Ex.</u>:

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0.5, \\ 0 & \text{if } x < 0.5 \end{cases}$$

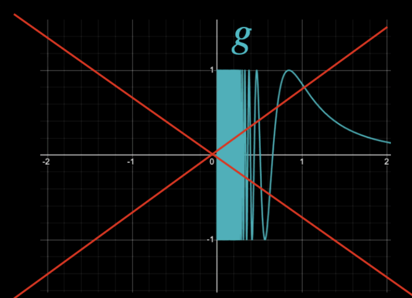


→ *f must be bounded on* $[a, b]$;

→ *f* ~~*continuous*~~ , *but discontinuities on the sum vanishes as the partition is refined* ;

→ *g must be of bounded variation on* $[a, b]$.

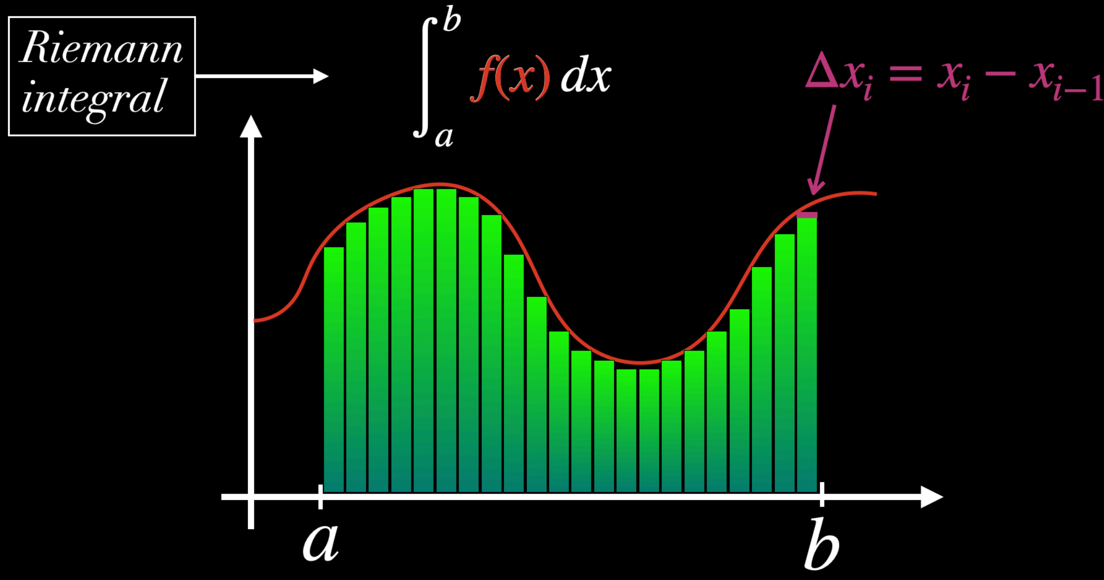
$$V_a^b(g) = \sup_{\mathscr{P}} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| < \infty$$

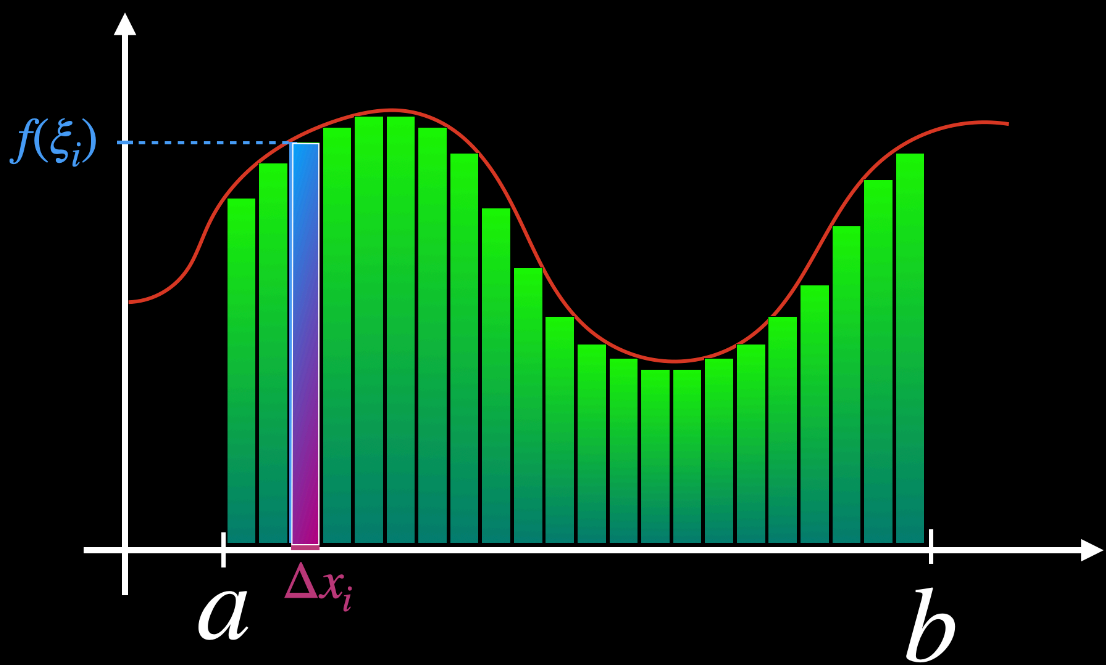


f doesn't need to be continuous, but the integral only exists if the effect of discontinuities on the sum vanishes as the partition is refined. *g* must be of bounded variation on *[a, b]*. This condition ensures that the function *g* doesn't oscillate too wildly. A function is of <u>bounded variation</u> if the total variation of *g*, which is defined as the supremum of the sum of absolute differences, is finite over the interval *[a, b]*.

What is the intuition behind the Riemann-Stieltjes integral? In the classical Riemann integral, the increments, $x_i - x_{i-1}$, represent small changes in *x*, and these are multiplied by the function's height *f(*\xi$_i$*)* , evaluated at a sample point \xi$_i$ in the subinterval $[x_{i-1}, x_i]$.

Intuition?



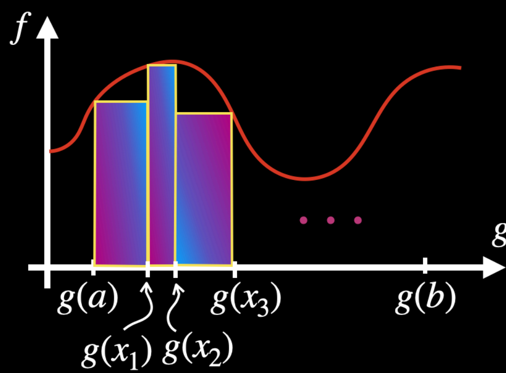


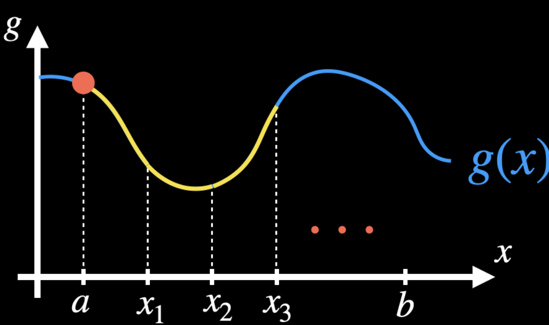
In the Riemann-Stieltjes integral the increments, $g(x_i) - g(x_{i-1})$, represent changes in the function $g(x)$. This allows for more flexibility in what is used as the "differential" (so, the small increments in the integration process). The function $g(x)$ could be non-linear, stepwise, or even more complex than that.

Intuition?

Riemann-Stieltjes integral →

$$\int_a^b f(x)\,dg(x) \qquad \Delta g_i = g(x_i) - g(x_{i-1})$$



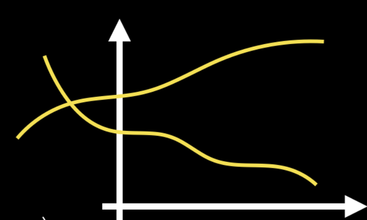


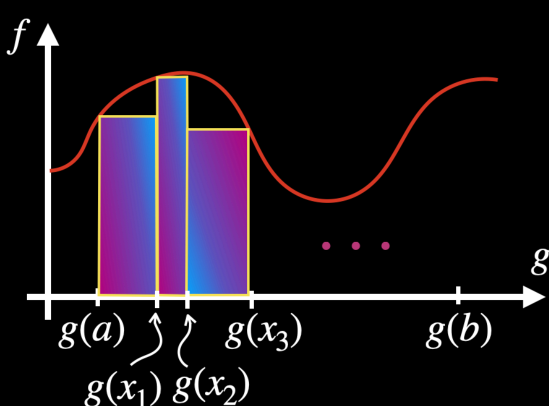
Now, let's talk about **when** the Riemann-Stieltjes integral exists:

When the *Riemann-Stieltjes integral* exists:

→ f ~~*continuous*~~ ;

→ g *monotonic* ; ($\implies$ *bounded variation*)





The Riemann-Stieltjes integral exists if *f* is continuous and *g* is of bounded variation on the interval *[a, b]*. Again, the continuity of *f* is not necessary, but more often than not it is a common condition. Another common condition is to require *g* to be monotonic (so, *g* is either increasing or decreasing, which implies that *g* is also of bounded variation).

Now we are going to do something **really** cool! The first example of this video was the calculation of the Riemann-Stieltjes integral for $f(x)=x$ and $g(x)=x^2$ over the interval $[0, 1]$.

$$\text{①}\quad f(x) = x \;\wedge\; g(x) = x^2 \quad \textit{over}\ [0,1]:$$

$$I := \int_0^1 x\, d\left(x^2\right) = \int_0^1 x \cdot (2x)\, dx =$$

$$= 2\int_0^1 x^2 dx = 2 \cdot \left(\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}$$

Let’s rewrite this integral using our more precise (and rigorous) definition.

$$① \quad f(x) = x \;\wedge\; g(x) = x^2 \quad \textit{over } [0\,,1]:$$

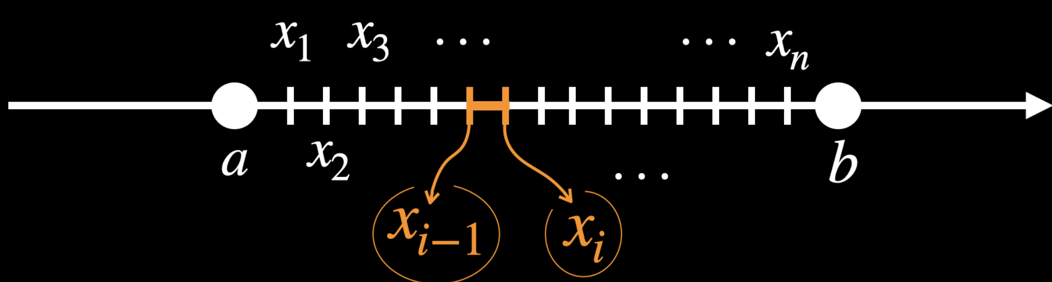
$$\int_0^1 x\,d\left(x^2\right) = \lim_{\|\mathscr{P}\|\to 0} \sum_{i=1}^{n} x_i \left(x_i^2 - x_{i-1}^2\right)$$

$g(x_i)$, $g(x_{i-1})$

$f(\xi_i)\,,\ \xi_i = x_i$

And the really cool thing that we are about to do is to solve this Riemann-Stieltjes sum limit expression analytically, step by step.

STEP 1 : *Interpret the sum*



$$\left(x_i^2 - x_{i-1}^2\right) \ \textit{is the change in} \ g(x) = x^2 \ \textit{over} \ \left[x_{i-1}\,, x_i\right]$$

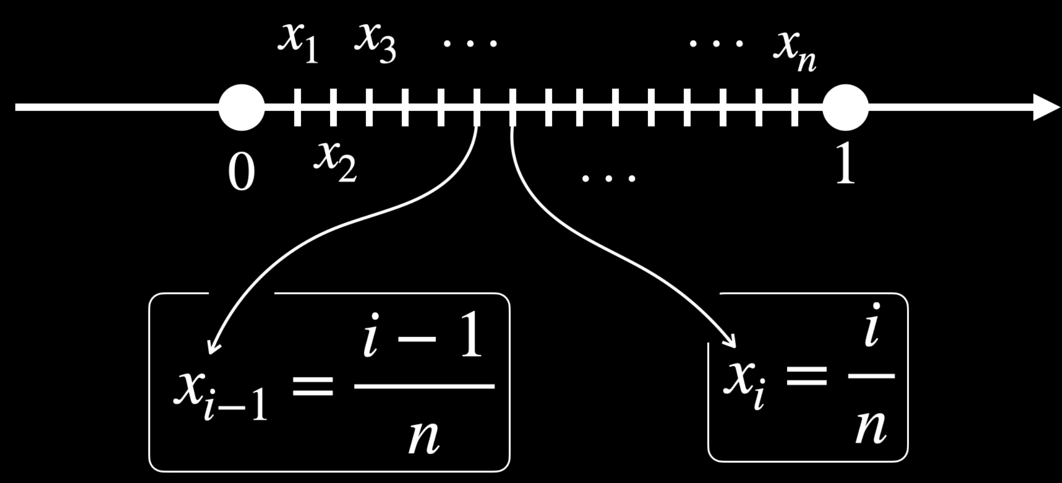
We can write it as:

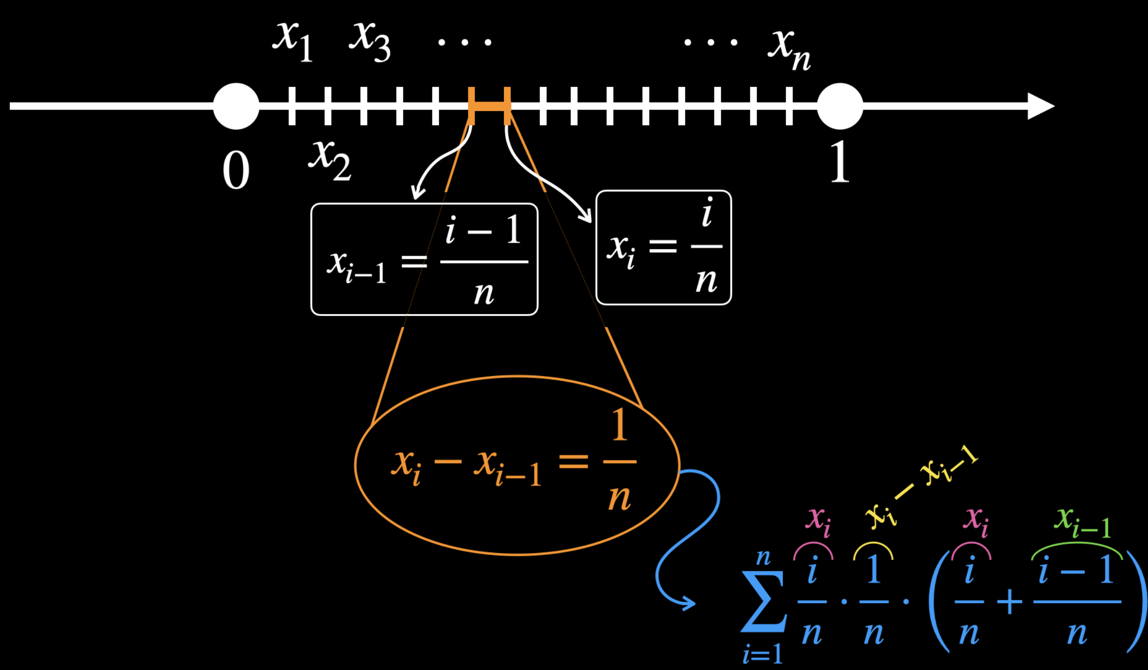
$$\left(x_i^2 - x_{i-1}^2\right) = \left(x_i - x_{i-1}\right)\left(x_i + x_{i-1}\right)$$

$$\Downarrow$$

$$\boxed{\sum_{i=1}^{n} x_i \cdot \left(x_i^2 - x_{i-1}^2\right) = \sum_{i=1}^{n} x_i \cdot \left(x_i - x_{i-1}\right)\left(x_i + x_{i-1}\right)}$$

$\boxed{\text{STEP 2}}$: *Refine the partition*





STEP 3 : *Simplify the expression*

$$\frac{i}{n} \cdot \frac{1}{n} \cdot \left(\frac{i}{n} + \frac{i-1}{n} \right) = \frac{i}{n^2} \cdot \left(\frac{2i-1}{n} \right) = \frac{i\,(2i-1)}{n^3}$$

$$\Downarrow$$

$$\boxed{\sum_{i=1}^{n} \frac{i\,(2i-1)}{n^3}}$$

$\boxed{\text{STEP 4}}$: *Simplify the general expression*

$$\sum_{i=1}^{n} \frac{i\,(2i-1)}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i\,(2i-1) = \frac{1}{n^3}\sum_{i=1}^{n}\left(2i^2 - i\right) =$$

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right)$$

$$\frac{1}{n^3}\left(\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right)$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n =: S \quad \Longrightarrow$$

$$\Longrightarrow \quad S = n + (n-1) + (n-2) + \ldots + 1 \quad \Longrightarrow$$

$$\Longrightarrow \quad S + S = \left[n + (n-1) + (n-2) + \ldots + 1\right] +$$

$$+ \left[1 + 2 + 3 + \ldots + n\right]$$

$$S + S = [n + (n-1) + (n-2) + \ldots + 1] +$$
$$+ [1 + 2 + 3 + \ldots + n]$$

$$S + S = [n + (n-1) + (n-2) + \ldots + 1] +$$
$$+ [1 + 2 + 3 + \ldots + n]$$

$$2S = (n+1) + \big((n-1) + 2\big) + \big((n-2) + 3\big) + \ldots + (n+1)$$
$$2S = (n+1) + (n+1) + (n+1) + \ldots + (n+1)$$

n

$$2S = n \cdot (n+1) \implies S = \frac{n \cdot (n+1)}{2}$$

$$\therefore \sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

$$\frac{1}{n^3}\left(\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right)$$

$$\sum_{i=1}^{n} 2i^2 = 2\sum_{i=1}^{n} i^2 = 2\cdot\left(\overbrace{1^2 + 2^2 + 3^2 + \ldots + n^2}^{=:\,S}\right)$$

We want to prove that $S = \frac{n\cdot(n+1)\cdot(2n+1)}{6}$

Mathematical *induction* :

Base case $\xrightarrow{(n=1)}$ $S = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6} =$

$$= \frac{1\cdot 2\cdot 3}{6} = 1 = \sum_{i=1}^{1} i^2$$

$\therefore$ *The base case holds.*

Inductive step

→ assume that $S_n = \dfrac{n \cdot (n+1) \cdot (2n+1)}{6}$ holds.

We want to prove *that*, *for* $n = k+1$.

Notice: $S_{k+1} = \sum_{i=1}^{k+1} i^2 = 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 \implies$

$$\implies S_{k+1} - S_k = \left[1^2 + 2^2 + \ldots + k^2 + (k+1)^2\right] - \left[1^2 + \ldots + k^2\right]$$

$$S_{k+1} - S_k = \left[\cancel{1^2} + \cancel{2^2} + \ldots + \cancel{k^2} + (k+1)^2\right] - \left[\cancel{1^2} + \cancel{\ldots} + \cancel{k^2}\right]$$

$$S_{k+1} - S_k = (k+1)^2 \implies \boxed{S_{k+1} = S_k + (k+1)^2}$$

Substitute the inductive hypothesis S_k *into this* *equation* :

$$S_{k+1} = \frac{k \cdot (k+1) \cdot (2k+1)}{6} + (k+1)^2 =$$

$$= \frac{k \cdot (k+1) \cdot (2k+1) + 6\,(k+1)^2}{6} =$$

$$= \frac{(k+1) \cdot \left[k \cdot (2k+1) + 6k + 6\right]}{6} =$$

$$S_{k+1} = \frac{k \cdot (k+1) \cdot (2k+1)}{6} + (k+1)^2 =$$

$$= \frac{k \cdot (k+1) \cdot (2k+1) + 6\,(k+1)^2}{6} =$$

$$= \frac{(k+1) \cdot \left[k \cdot (2k+1) + 6k + 6\right]}{6} = \frac{(k+1) \cdot \left[2k^2 + k + 6k + 6\right]}{6} =$$

$$= \frac{(k+1) \cdot \left[2k^2 + 7k + 6\right]}{6} = \frac{(k+1) \cdot (2k+3) \cdot (k+2)}{6} \implies$$

$$\implies \boxed{S_{k+1} = \frac{(k+1) \cdot \left((k+1)+1\right) \cdot \left(2 \cdot (k+1) + 1\right)}{6}}$$

Formula for S_{k+1}

$$\boxed{\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}} \qquad \boxed{\sum_{i=1}^{n} 2i^2 = 2\,\frac{n \cdot (n+1) \cdot (2n+1)}{6}}$$

$$\sum_{i=1}^{n} \frac{i\,(2i-1)}{n^3} = \frac{1}{n^3}\left(\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right) =$$

$$= \frac{1}{n^3}\left(2\,\frac{n \cdot (n+1) \cdot (2n+1)}{6} - \frac{n \cdot (n+1)}{2}\right) =$$

$$= \frac{1}{n^3}\left(\cancelto{1}{2}\,\frac{n\cdot(n+1)\cdot(2n+1)}{\cancelto{3}{6}} - \frac{n\cdot(n+1)}{2}\right) =$$

$$= \frac{\cancelto{1}{n}\,(n+1)}{\cancelto{n^2}{n^3}}\left(\frac{2n+1}{3} - \frac{1}{2}\right) = \frac{(n+1)}{n^2}\left(\frac{4n+2-3}{6}\right) =$$

$$= \frac{(n+1)}{n^2}\left(\frac{4n-1}{6}\right) = \frac{(n+1)\cdot(4n-1)}{6\,n^2}$$

STEP 5 : *Take the limit as* $n \to \infty$

$$\lim_{n\to\infty} \frac{(n+1)\cdot(4n-1)}{6\,n^2}$$

$$\lim_{n\to\infty} \frac{(n+1)\cdot(4n-1)}{6\,n^2} = \lim_{n\to\infty} \frac{4n^2+3n-1}{6\,n^2} =$$

$$= \lim_{n\to\infty}\left(\frac{4}{6} + \frac{3}{6n} - \frac{1}{6n^2}\right) = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \int_0^1 x\, d\left(x^2\right) = \lim_{\|\mathscr{P}\|\to 0} \sum_{i=1}^{n} x_i \left(x_i^2 - x_{i-1}^2\right) = \frac{2}{3}$$

So, ⅔. Just as we've calculated before in the beginning of the video. Q.E.D.

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