Constructible Trigonometry: Numerical Trigonometry over a Field

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ABSTRACT: One of the primary objects in trigonometry is the angle, a transcendental quantity which can typically only be approximated. By instead using the slope at vertices, many new formulae and discoveries can be found, and even computed by hand. This paper presents a reformulation of trigonometry based entirely on constructible numbers, using slope instead of angle as the fundamental quantity. This approach not only provides exact solutions where traditional methods require approximation, but also reveals new solvable cases in triangle construction. Most notably, we present the first complete solution to the Inradius-Side-Side (ISS) case, which surprisingly yields cubic equations. This connection to cubic equations provides a natural bridge from constructible numbers to the broader field of algebraic numbers, offering rich opportunities for teaching both mathematics history and number theory. The pedagogical advantages of this approach include stronger connections to early algebra concepts, delayed introduction of transcendental numbers, and natural progression from elementary to advanced mathematical ideas.

Keywords: trigonometry, field, construct, cubic

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Introduction

Trigonometry is the application of numbers to the field of geometry. As such, it requires careful consideration of number theory. We will begin by briefly tracing the historical development of the discipline.

Historical Context

Remark. The historical tension between approximate astronomical measurement and exact geometric construction directly motivates our approach. While the Babylonian degree system prevails in practice, the Greek constructive tradition offers pedagogical advantages unexplored in modern education.

Trigonometry's mixed beginnings continue to shape the field today. On the one hand, astronomical observation and prediction certainly came first. The Babylonians studied the heavens and introduced the degrees, minutes, and seconds of arc we use to this day. On the other hand, the Greeks valued proofs and constructions. Euclid never once used degrees to describe an angle in his *Elements*. The incommensurate nature of diagonal lengths (hypotenuses) troubled the Greeks. All ancient mathematicians had approximations for these quantities, but the radical symbol wouldn't be invented until the 13th century with Fibonacci (Leonardo of Pisa), who initially used a stylized 'r' (for 'radix'). Our symbol first appeared in Christoph Rudolff's *Arithmetica* (1522) [7] (not in Christopher Clavius's 1608 algebra textbook, as is widely claimed).

Today, all students are taught planar trigonometry and memorize the Unit Circle's major, constructible points. Only a few experts delve into spherical trigonometry, though it is constantly in use for navigation and GPS. While both originated in the ancient world, these two disciplines were rigorously developed by Arabic scientists (who were carrying forward the work of Indian mathematicians). The other major branch of trigonometry — hyperbolic trig — awaited the link between logarithms and standard trigonometric functions by Borelli (1608-1679) and Newton. Later, Lobachevsky and Bolyai formalized the connection between the hyperbolic functions and the new, non-Euclidean geometry [11]. All forms of trigonometry play major roles in analysis today.

Pedagogically, logs and trig mark the end of our student's time without calculators. No one can exhibit the sine of $23°$ exhaustively. No trig-table can contain all the digits of even one trigonometric function at even one angle. Floating point arithmetic machines — hand-held calculators — become necessary from the moment angles become arbitrary. The jump from constructible, algebraic solutions to transcendental ones is a violent lurch to most learners, and is seldom explained or justified. Engineers and other professionals work with approximations and truncations, without typically contemplating exact solutions. Mathematic preports to only use well-defined objects, but this area sees a lot of hand-waving in that regard.

A better, more historically grounded approach is possible. In this paper, we will advocate for a trigonometry that progresses from constructibles to transcendentals over an entire course, analogous to the slow adaptation from degrees to radians, or radicals to rational exponents. We will argue that the genre of irrational numbers is too vast, and must be broken down into stages for the learner.

1 Foundations

In this section, we will develop constructible trigonometry, a natural bridge from introductory algebra to analysis.

1.1 From Slope to Trigonometry

The fundamental object of introductory algebra is the basic equation of a line.

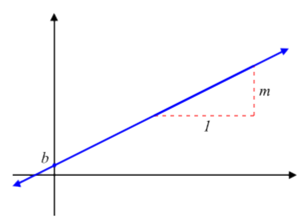

Figure 1: The principle object of early mathematics education is the line. Some classes also stress *direct variation*, usually in the form $y = kx$.

Definition 1.1 (Line in Slope-Intercept Form)**.** A line L in a simple plane is defined by parameters $m, b \in \mathbb{Q}$ such that:

$$L = \{(x, y) \in \mathbb{Q}^2 : y = mx + b\}$$

where m is the slope and b is the y-intercept. The graphical intuition may be seen in Figure 1.

Definition 1.2 (Slope)**.** If two points are given in the form (x_1, y_1) and (x_2, y_2), the slope may be calculated as $m = \frac{y_2 - y_1}{x_2 - x_1}$. This is easily recalled by the mnemonic "rise over run." Other notation-styles include $m = \frac{\Delta y}{\Delta x}$, i.e. change-in-$x$ over change-in-y.

Most student need greater familiarity with slope before they are ready for the transcendental concept of angles [13]. A natural progression from the illustration of lines in this way is to ask about the length of the hypotenuse. This is easily found using the Pythagorean Theorem.

Lemma 1.3 (Length)**.** *The length of a line with slope m is written ℓ and is calculated as $\ell = \sqrt{1 + m^2}$.*

These numbers are strictly constructible (but see below), both in the geometric and algebraic sense. Use of measurable lengths has been shown to increase students' understanding and fluency with trigonometric ratios [3]. Emphasizing ratio before teaching about angles is also highly beneficial to the comprehension of trigonometry [4].

Definition 1.4 (Constructible Numbers)**.** Let K be the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q}$ that is closed under square roots of positive elements. We call K the field of constructible numbers. A real number x is called constructible ($\mathbb{K}$) if and only if there exists a finite sequence of numbers $x_1, \ldots, x_n$ with $x_n = x$ such that:

1. $x_1 = 1$
2. For each $i > 1$, x_i is obtained from previous numbers in the sequence by one of the following operations:
   - Addition or subtraction of two previously constructed numbers
   - Multiplication or division of two previously constructed numbers
   - Taking the positive square root of a previously constructed non-negative number

A key consequence of this definition is that the constructible numbers form a field that is closed under square roots of positive elements. In fact, they form the smallest such field containing the rationals.

An equivalent definition could be given in terms of compass and straightedge constructions, where a number is constructible if and only if its length can be constructed starting from a unit length using only an idealized compass and unmarked straightedge.

1.2 Basic Relationships

Notice that m is equivalent to the tangent of the angle the line makes with the horizontal. Length is equivalent to the secant. This means this system is homomorphic to standard trigonometry, only without the use of angles.

$$\begin{aligned}
\sin x &= m_x/\ell_x \\
\cos x &= 1/\ell_x \\
\tan x &= m_x \\
\sec x &= \ell_x \\
\cot x &= 1/m_x \\
\csc x &= \ell_x/m_x
\end{aligned}$$

Since we're working in the field of constructible numbers, we inherit several crucial properties:

1. Any expression we can build using only addition and subtraction, multiplication and division (apart from zero), square roots of non-negative elements; will give us another constructible number.

2. Therefore, m_x and ℓ_x must be constructible whenever our angle x corresponds to a constructible angle.

3. The closure properties mean that all our ratios will be constructible for constructible angles.

This connects to some famous impossible results. Since trisection of arbitrary angles is impossible with compass and straightedge, there must be many constructible angles x where $\cos x/3$ is not constructible. Since we can't duplicate the cube, there must be some constructible length L where $\sqrt[3]{L}$ is not constructible. We will return to this in the discussion about cubic equations. Our representation makes some of these limitations more transparent, since we can directly see we're limited to square roots in our construction process.

The biggest cost to our (temporary) abandonment of the transcendental quantity of angles is the complication of stacking slopes and length. As seen in Figure 2, the formulas are much more complicated than simple addition. However, these are part of standard trigonometry too, so they are not a new burden.

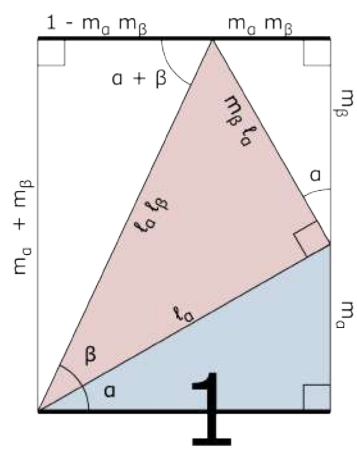

Figure 2: This picture is easy to built from scratch with students. Take a piece of paper and construct the inner triangle. Label the vertices α and β and the bottom edge 1. All the other sides and vertices can be labeled using only m, ℓ, α, β, their sums, products, or their compliments. Equations (1) and (2) are obvious by inspection of the top-left triangle. These are the traditional sum of angles formulas for tangent and secant in standard trigonometry.

$$m_{\alpha+\beta} = \frac{m_\alpha + m_\beta}{1 - m_\alpha m_\beta} \tag{1}$$

$$\ell_{\alpha+\beta} = \frac{\ell_\alpha \ell_\beta}{1 - m_\alpha m_\beta} \tag{2}$$

Doubling a slope is a common enough occurrence that it is worth noting the simplification of this formula here:

$$m_{2x} = \frac{2m_x}{1 - m_x^2} \tag{3}$$

Solving for the inverse of Equation (3) yields the half-slope formula:

$$m_{x/2} = \frac{\ell_x - 1}{m_x} = \frac{m_x}{\ell_x + 1} = \sqrt{\frac{\ell_x \mp 1}{\ell_x \pm 1}} = \sqrt{\frac{\sqrt{m_x^2 + 1} \mp 1}{\sqrt{m_x^2 + 1} \pm 1}} \tag{4}$$

The one wrinkle to our number theory is that we do have to allow and manage slopes and lengths of infinity (∞), and so we should call our field the Extended Constructible Numbers, after adjoining that one element.

Definition 1.5 (Extended Constructible Numbers)**.** Let $\mathbb{K}$ denote the field of constructible numbers. The extended constructible numbers, denoted $\mathbb{K}^+$, are defined as:

$$\mathbb{K}^+ = \mathbb{K} \cup \{\infty\}$$

where arithmetic operations involving ∞ are largely undefined:

- For all $x \in \mathbb{K}$, $x + \infty$ and $\infty + x$ are undefined.
- For all $x \in \mathbb{K}$, $x \cdot \infty$ and $\infty \cdot x$ are undefined.
- For all $x \in \mathbb{K}$, x/∞ and ∞/x are undefined.
- For all $x \in \mathbb{K}$, $x/0 = \infty$.
- $\infty \div \infty = 1$

Definition 1.6 (Extended Slopes)**.** The set of constructible slopes, denoted $m(\mathbb{K})^+$, is defined as:

$$m(\mathbb{K})^+ = \{m \in \mathbb{K} \cup \{\infty\} : m \text{ is a slope constructible by compass and straightedge}\}$$

Example 1.7. Let $m_1 = \sqrt{3}$ and $m_2 = \infty$ in $m(\mathbb{K})^+$. Then:

$$\begin{aligned} m_1 + m_2 &= \frac{\sqrt{3} + \infty}{1 - \sqrt{3} \cdot \infty} && \text{Equation (1)} \\ &= \frac{\frac{\sqrt{3}}{\infty} + \frac{\infty}{\infty}}{\frac{1}{\infty} - \sqrt{3}\frac{\infty}{\infty}} && \text{Divide through by } \infty \\ &= \frac{0 + 1}{0 - \sqrt{3}} && \text{Definition 1.5} \\ &= -\sqrt{3}/3 \end{aligned}$$

These correspond to $60° + 90° = 150°$. This addition in $m(\mathbb{K})^+$ preserves the geometric relationship between slopes and angles, even when dealing with infinite slopes.

Itt must be noted that slopes do lack one detail of angles: absolute direction. Traveling along a positive slope will take one from the origin into QI or QIII, just as traveling along a negative slope will carry one from the origin into QII or QIV. Context is required to know which is meant. An imperfect rule of thumb is that negative slopes and lengths denote obtuse angles, while positive ones denote acute angles.

2 Triangle Solutions

2.1 General Identities

All the identities of trigonometry find equally succinct or sometimes even more compact expressions in constructible trigonometry. The Law of Sines is equally clear, as seen in Equation (5).

$$a\frac{\ell_A}{m_A} = b\frac{\ell_B}{m_B} = c\frac{\ell_C}{m_C} \tag{5}$$

The Law of Cosines is also very simple (Equation (6)).

$$c^2 = a^2 + b^2 - 2ab/\ell_C \tag{6}$$

It is much easier to write the inverse, when the included angle is sought, as seen in Equation (7)).

$$\ell_C = \frac{2ab}{a^2 + b^2 - c^2} \tag{7}$$

All the slopes of the vertices of a triangle make a horizontal line (0), therefore we can simplify the sum of slopes formula to Equation (8).

$$m_C = \frac{m_A + m_B}{m_A m_B - 1} = -m_{A+B} \tag{8}$$

The results are essentially the same for lengths. The Sine-Area equation becomes (with Q for area) Equation (9).

$$Q = \frac{1}{2}ab\frac{m_C}{\ell_C} \tag{9}$$

Heron's Formula (Equation (10)) is also indispensable, which utilizes the semi-perimeter $s = \frac{a+b+c}{2}$

$$Q = \sqrt{s(s-a)(s-b)(s-c)} \tag{10}$$

2.2 Classical Cases Reformulated

The formulas already given cover finding all sides and angles of SSS (Side-Side-Side) and SAS (Side-Angle-Side) situations. Not always taught in a trigonometry class, we will also be using this formula for inradius (r):

$$r = \sqrt{\frac{Q}{s}} \tag{11}$$

and this equation for circumradius (R):

$$R = \frac{abc}{4Q} \tag{12}$$

Where constructible trigonometry really starts to shine is in the more obscure cases, where normal trig has to solve for an unknown side or angle, and then use rounded numbers to find more information. The errors compound. For example, consider ASA, where we might know the slopes at A and C, and the included side b. First, we use the Law of Sines:

$$\begin{aligned}
c\frac{\ell_C}{m_C} &= \frac{b\ell_B}{m_B} && \text{Equation (5)} \\
c &= \frac{bm_C}{\ell_C}\ell_B \cdot \frac{1}{m_B} \\
&= \frac{bm_C}{\ell_C} - \ell_{A+C} \cdot \frac{1}{-m_{A+C}} && \text{Equation (8)} \\
&= \frac{bm_C}{\ell_C}\frac{\ell_C\ell_A}{1-m_Am_C} \cdot \frac{1-m_Am_C}{m_A+m_C} && \text{Equation (1)} \\
&= \frac{bm_C\ell_A}{m_A+m_C}
\end{aligned}$$

All such formulas can be derived on the fly, and never require rounding. For example, squaring Equation (9), a little manipulation, and substituting into Equation (6) simplifies to Equation (13). Such equations could, of course, be found in normal trigonometry, but students are not encouraged to do so.

$$\begin{aligned} Q &= \frac{1}{2}ab\frac{m_C}{\ell_C} && \text{Equation (9)} \\ \frac{2Q}{ab} &= \frac{\sqrt{\ell_C^2 - 1}}{\ell_C} \\ \frac{4Q^2}{a^2b^2} &= 1 - \frac{1}{\ell_C^2} \\ \frac{\sqrt{a^2b^2 - 4Q^2}}{ab} &= \frac{1}{\ell_C} \end{aligned}$$

$$c^2 = a^2 + b^2 - 2\sqrt{(ab)^2 - (2Q)^2} \tag{13}$$

2.3 First Ambiguous Case

The first challenging situation is SSA, Side-Side-Angle. This is called the ambiguous case in textbooks, and for good reason. Typically, we would set up the Law of Cosines to find the two c's

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A && \text{Law of Cosines} \\ 0 &= c^2 + (-2b\cos A)c + (b^2 - a^2) && \text{Coefficients of } c \\ c &= \frac{2b\cos A \pm \sqrt{4b^2\cos^2 A - 4(1)(b^2 - a^2)}}{2(1)} && \text{Quadratic Formula} \\ &= b\cos A \pm \sqrt{a^2 - b^2\sin^2 A} \end{aligned}$$

However, stated in terms of slope, we get an exact answer using the constructible equivalent in Equation (14).

$$c_{1,2} = \frac{b \pm \sqrt{(a\ell_A - bm_A)(a\ell_A + bm_A)}}{\ell_A} \tag{14}$$

If we label the radical expression as $D = \sqrt{(a\ell_A - bm_A)(a\ell_A + bm_A)}\ell_A$, then we find it comes up in all the other unknowns of SSA. We can calculate any part of both possible triangles instantly, without going through intermediaries, as seen in Equations (15) - (18).

$$m_{C_{1,2}} = m_A \frac{b \pm D}{bm_A^2 \mp D} \tag{15}$$

$$Q_{1,2} = \frac{m_A b}{2\ell_A^2}(b \pm D) \tag{16}$$

$$m_{B_{1,2}} = \pm \frac{m_A b}{D} \tag{17}$$

$$r_{1,2} = \frac{(m_A b)(b \pm D)}{\ell_A(a+b) + (b \pm D)} \tag{18}$$

Of course, it all depends on whether a reaches down to side c or not. The cases can be evaluated before calculations, by find the minimum length, called h, where $h = b\frac{m_A}{\ell_A}$.

- If A is obtuse, a must be greater than b for any triangle to exist. If it is, then only one triangle is possible.
- If A is acute, then a must be greater than h. If it is not, no triangle is possible. It they are equal, one right triangle is possibe.
- If $h < a < b$, then two triangles are possible.
- If $a > b$, then only one triangle is possible.

3 Novel Discoveries

3.1 The ISS Case

Given the $\mathbb{K}^+$ framework, more avenues of exploration are easy, beyond the traditional sides (S) and angles (A). We may posit scenarios involving Area-Side-Side (QSS), Circumradius-Angle-Side (CAS), or Side-Perimeter-Angle (SPA), all of which are determined and have clear formulas for all missing values. The truly surprising result of this investigation is the discovery of a new ambiguous situation: Inradius-Side-Side (ISS).

The incircle is tangent to the sides of a triangle, but without being pinched by a given angle, we might imagine it rolling along the side of segment a. Side b does extend from vertex C — where it met A. The incircle is tangent to them both, but as it comes closer to C, it forces it to become wider, thereby lengthening c. The added criterion that c is tangent to the incircle is very limiting, but two possibilities can still exist.

Example 3.1. Suppose we begin with a 13-14-15 triangle. The semi-perimeter is 21 and the area is 84, so the inradius is 4. Ignore that $c = 15$ and $Q = 84$, and this is now an ISS situation. The construction takes a bit of trial and error, due to the flexible placement of the incircle. Figure 3 shows the two possible triangles, as constructed in Geogebra.

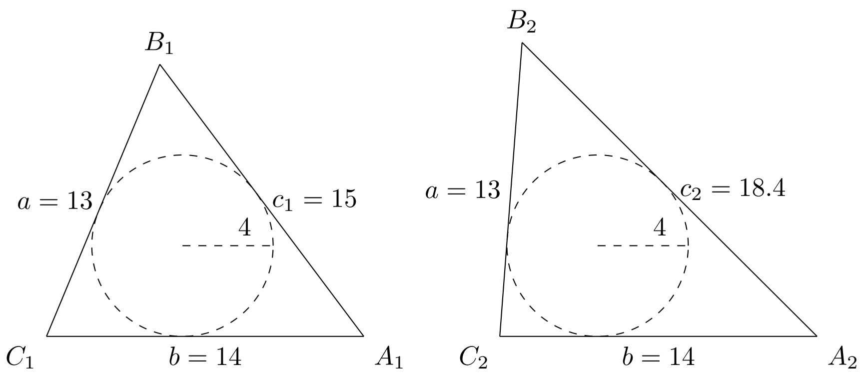

Figure 3: Two triangles with sides of 13 and 14 and an inradius of 4. Notice the different length of the upper-right side in each case.

A formula with all the right ingredients already exists as Equation (11). Unfortunately, it is solved for r, not c. We begin by rewriting without s.

Proof. We start with Equation (11). By using the definition of semi-perimeter, we get

$$r = \sqrt{\frac{\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}{\frac{a+b+c}{2}}}$$

Squaring both sides yields:

$$r^2 = \frac{(-a+b+c)(a-b+c)(a+b-c)}{4(a+b+c)}$$

Distributing all terms:

$$4r^2a + 4r^2b + 4r^2c = -a^3 - b^3 - c^3 + a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2 - 2abc$$

We group by coefficient of c:

$$c^3 + (-a-b)c^2 + (4r^2 - a^2 + 2ab + b^2)c + a^3 + b^3 + (4r^2a + 4r^2b - a^2b - ab^2) = 0$$

Finally, we factor and minimize length:

$$c^3 - (a+b)c^2 + (4r^2 - (a-b)^2))c^2 + (a^3 + b^3 + (a+b)(4r^2 - ab) = 0$$

For constructible inputs, all such polynomials are irreducible over $\mathbb{Q}$ by Eisenstein's criterion with $p = 3$.

□

In the case from Example 3.1, this leads to the cubic, $c^3 - 27c^2 + 63c + 1755$. The solutions are 15, and $6 \pm 3\sqrt{17}$, the positive of which is ≈ 18.3693. The negative result is clearly extraneous and came about as an artifact of our squaring the equation to remove the radical.

Similar expressions exist for the $Q_{1,2}$ (Equation (19)) and $R_{1,2}$ (Equation (20)).

$$Q^3 - 2r(a+b)Q^2 + r^2(r^2 + ab + (a+b)^2)Q - abr^3(a+b) = 0 \tag{19}$$

$$(4r)^3R^3 + 16r^2(r^2 + (a-b)^2 - ab)R^2 - 8abr(2r^2 + (a-b)^2)x + a^2b^2(4r^2 + (a-b)^2) = 0 \tag{20}$$

Most remarkable of all is that one exists for $m_{C_{1,2}}$, as seen in Equation (21).

$$\left(r^2 + ab - (a+b)^2 + \left(\frac{ab}{2r}\right)^2\right)x^3 + (2r(a+b))x^2 + (r^2 + ab - (a+b)^2)x + 2r(a+b) = 0 \tag{21}$$

In this case, two of the slopes correspond to our two cases from above. The third (extraneous) solution does not have the appropriate radius for the incircle, but does have the required r for the *excircle*, tangent to side c.

3.2 Emergence of Cubic Equations

Amidst the bevy of new situations we are able to address using constructible trigonometry — QPS, PSA, SPA, CPA, etc. — that ISS should be a set of cubics is very unexpected. No ancient geometer and no medieval trigonometer was equipped to tackle these problems. It wasn't until the 16th century that Tartaglia, Cardano, and Bombelli invented the necessary toolkit to tackle such problems.

Today, complex numbers are introduced for the first time while solving quadratics. We obtain the complex numbers ($\mathbb{C}$) by adjoining $i = \sqrt{-1}$ to the Reals ($\mathbb{R}$), enabling us to simplify every instance of the quadratic formula. Historically, however, the motivation came from solving cubics which were known to have all real solutions. As part of constructible trigonometry, these cubic equations provide a bridge from geometry and intermediate algebra to calculus and transcendental numbers and functions. For a graphical illustration of these, see a special instance of Marden's Theorem in Figure 4 [9].

This case is much, much more complicated than the SSA situation. Like it, it does typically yield two possible triangles. There is a maximum r value that can fit between the two sides a and b, but it is extremely complicated (another cubic) to calculate. When $a = b$ (isosceles), r is a straight-forward constant, show in Equation (22). A rough approximation for r_{max} can be seen in Equation (23).

Proposition 3.2 (ISS Uniqueness Conditions)**.** *Given sides a, b and inradius r, there exist either 0, 1, or 2 triangles satisfying these constraints. Specifically:*

1. *If $r > r_{max}$ as given by Equation (22) or Equation (23), no triangle exists*
2. *If $r = r_{max}$, exactly one triangle exists*
3. *If $0 < r < r_{max}$, exactly two triangles exist*

$$r_{max,iso} = a\sqrt{\frac{5\sqrt{5}-11}{2}} \approx 0.300283106a \tag{22}$$

$$r_{max} \approx \sqrt{\frac{ab - 0.155(a^2+b^2)}{7.799}} \tag{23}$$

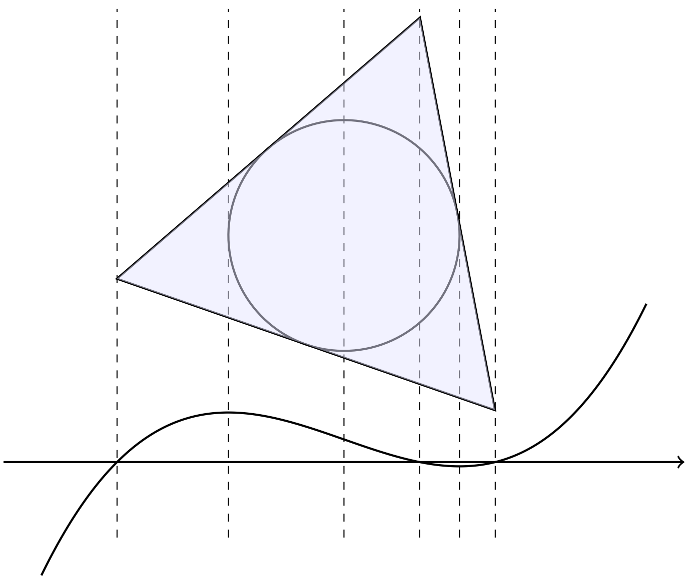

Figure 4: Every cubic equation with three root $\in \mathbb{R}$ has a corresponding equilateral triangle which describes the essential features of it. The incenter's x is above the point of inflection. The ends of a diameter of the incircle are above the critical points. Consider a depressed cubic of the form $x^3 + Px + Q = 0$. The incenter is above 0. The inradius is $\sqrt{\frac{-P}{3}}$. The circumradius is $\sqrt{\frac{-2P}{3}}$. Obviously, the vertices are 120° apart from each other, but where to get the first one? Unfortunately, $3\theta = \cos^{-1}\left(\frac{Q}{2}\sqrt{\frac{27}{P^3}}\right)$. [2] In general, this angle is not constructible. Constructible circumstances have produced an non-constructible — yet visually clear — geometry.

4 Educational Implications

Our students experience a series of quantum leaps in mathematics, from a number theoretical point of view. Basic arithmetic begins with the natural numbers. Next come (positive) fractions. Suddenly, negative numbers enter the scene. Next come radicals, when students must simplify their answers produced from the Pythagorean theorem. Quadratics justify the introduction of complex numbers. Finally, trigonometry and calculus produce new irrational numbers, which are not the result of any finite process.

4.1 Natural Progression

A clear, logical pathway is afforded to teachers by constructible trigonometry. Our students all learn the Cartesian coordinate plane and the basics of analytic geometry. A great deal of time is spent on lines, and manipulating equations. This means handling slope is already an important skill. Next, geometry provides a great deal of vocabulary and perspective, which is often wasted. SSS congruence and the construction for bisecting an angle are never again exploited and eventually forgotten. Finally, Algebra II teaches even more equation juggling, in the form of quadratic equations, cherry-picked cubics for factoring by grouping lead to overconfidence or despair with respect to real polynomials.

Constructible trigonometry pulls together these loose strings. The basic framework found in this paper should be extended to encompass matrix and vector-based mathematics, like any trigonometry class. Trigonometry is the chance to make analytic geometry robust. The way to do that is to continue the skills of geometry and initially use only constructible numbers [12]. The story of mathematics is often the story of number theory, and we need to lead our students careful in the following progression:

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{K} \to \mathbb{A} \to \mathbb{R} \text{ [8, p. 172]}$$

The importance of "incommensurate" is vital to appreciate different kinds of solutions to various problems. These kinds of discussions are what distinguish mathematics from computation. Seeing that the solutions to other cubics are not constructible means a lot more if you can solve any triangle by hand, yet these are unsolvable. Today, $\mathbb{C}$ is the extension to the field $\mathbb{R}$ by adjoining $i = \sqrt{-1}$, but originally it was just a pathway to extend $\mathbb{K}$ to solve cubics with rational answers. History could've gone differently. Indeed, the field $\mathcal{O}$ – for Origami – has been rigorously defined [1]. Folding paper allows for trisecting an angle and taking cube roots.

Our students need to be part of the discussion about appropriate answers. There is no number that squares to two, if rounding is not allowed. "But I can see and draw the hypotenuse of a unit square!" Even if you grant the infinity of all such numbers, you still cannot double the cube. Grant the infinity of such solutions as those. Grant all such numbers that can be written with arithmetic and any exponents, in any combination. As Emil Artin was fond of pointing out, you still cannot describe the solutions of $x^5 - x - 1 = 0$. Grant the solutions to any such equations, but you still can pinpoint π or e. Our students are like those who look up at the sky, and call everything a star. Some of those twinkling objects are planets like the one you are standing on. Others are galaxies.

5 Conclusion

Most of what ails mathematics education can be solved by following the maxim, that teaching in historical "order happens to be the most convenient for the logical presentation of the subject and is the natural way of examining how the mathematical ideas arose, what the motivations were for investigating these ideas, and how the mathematical creations in turn altered the course of other branches of our culture."[6] Trigonometry could flow naturally from elementary algebra and geometry, if we utilized the extended constructible field. Far from hampering later transcendental studies, this actually motivates them, as the ISS case necessitates understanding cubic equations. This prelude to standard trigonometry would help students distinguish between hand-solvable problems, and approximate, real number mathematics.

Constructible trigonometry is an intermediate state. It serves as a bridge from $y - y_1 = m(x - x_1)$ and orthocenters, to the world of $r = \sin(3\theta)$ and Tangent Half-Angle Substitution. It teaches students to solve $\tan\left(\sin^{-1}\left(\frac{2r}{Q}\right)\right)$ before they know how to write it. It sharpens their algebraic manipulation and their computation by hand skills. What calculus teacher hasn't heard a million times, "The calculus was the easiest part. It's the algebra that's hard"? Don't we all write "exact solutions only", when we wish our students knew which field was appropriate to the problem? Constructible trigonometry makes that discussion logical and necessary, with practical examples.

The impact on surrounding classes and curriculum would be minimal. Pre-Algebra, Algebra I, Algebra II and Calculus would be relatively unaffected. Geometry would have to reconsider some of it's problems involving angles, but that was already true: everybody knows they are padding the hours with "$(2x + 3)^\circ$ and $(3x + 12)^\circ$ are supplementary"! The real change is in precalculus and trigonometry, typically taught over a year. It affords the opportunity to tie together geometry and algebra more than any other class [10, 14]. Teachers should know the rich history of the discoveries and inventions in trigonometry. New problems will be needed with slopes, instead of angles. AI's and CAS's adapt to this new system well, as no floating point arithmetic is needed.

Further avenues of exploration are easy to spot. Our r_{max} for the ISS ambiguous case is inexact. All the angles and math from 17-gons — discovered to be constructible in 1796 by Gauss — are avoided like the plague in normal geometry and trigonometry courses. No need for such phobia in constructible trigonometry. We have not explored the transition to sine and cosine trigonometry at all in this work. As mentioned above, vectors and linear algebra are natural progressions. Spherical and hyperbolic trig seem amenable to constructibility, but perhaps not. Nothing has been said about regular or irregular constructible polygons or polyhedra. Indeed, trigonometry touches all of mathematics, as Euler's most famous equation beautifully illustrates (restated using $\mathbb{K}^+$ slopes):

$$e^{i\infty} = \frac{1 + im_\infty}{\ell_\infty} = i$$

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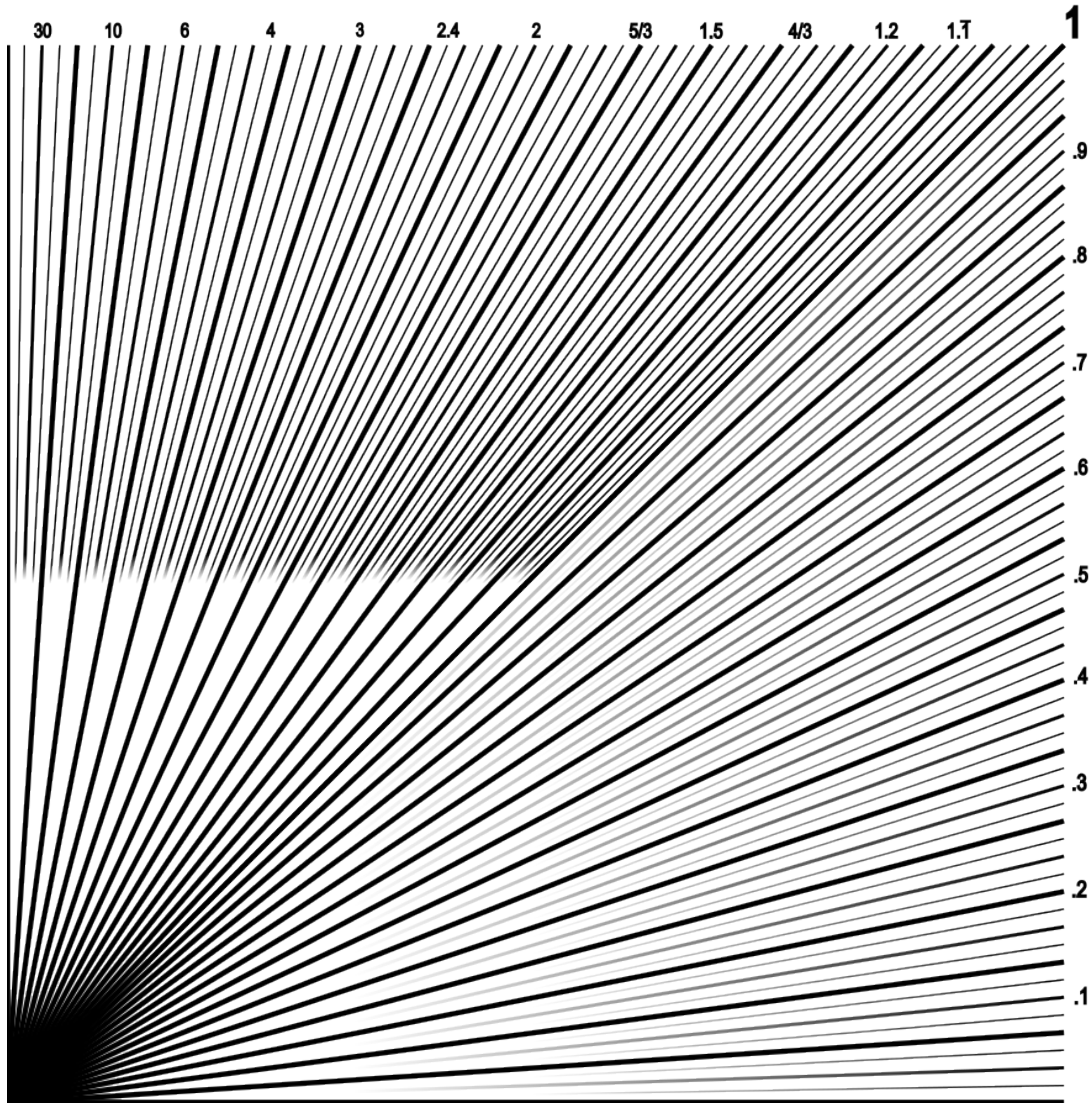
A Slope Protractor

One interesting aside with respect to the adoption of constructible trigonometry is the need for measuring slope of physical objects, just as we use a protractor for angles today. Slopes are inherent more "square" than rotational, so a possible solution is to use a square device. (This could double as a carpenter's square, if made of metal.) Taking a queue from the Babylonians, we could divide it into 60 marks along the each side. Moving up the right, each small tick-mark is $+\frac{1}{60}$. Therefore, every six marks upward is 0.1. A slope of 1 is equivalent to a 45° angle.

The horizontal tick marks are more difficult. Starting from the 1 corner, we begin subtracting from the *denominator* of $\frac{60}{60}$. The "nice" values are not evenly spaced. Users of this device would need to know fractions. An effort has been made to help always distinguish between the additive first half and the subtractive second half.



B Tangents/Slopes of Constructible Angles

All constructible tangents can be written using nested-radicals [5].

Theorem B.1 (Constructible Slopes). *All slopes of the vertices and exterior angles of regular polygons can be expressed using nested radicals containing only:*

1. *1, $\sqrt{2}$, 3*
2. *The golden ratio pairs $\{\phi, \Phi\}$*

3. The quartet $\{\Xi, \xi, \upsilon, \Upsilon\}$

$\sqrt{1} = \sqrt{2-\sqrt{1}}$. $\sqrt{3} = \sqrt{2+\sqrt{1}}$. Such nested-radicals can be lengthened indefinitely. Similarly, the Golden Ratio and it's conjugate — i.e. $\{\varphi, \Phi\}$ — can be written in this unusual way:

- $\Phi = \sqrt{2-\varphi} = \varphi - 1 = \frac{-1+\sqrt{5}}{2} \approx 0.61803398874989484820458683436563811772030917980576\ldots$
- $\varphi = \sqrt{2+\Phi} = \Phi + 1 = \frac{1+\sqrt{5}}{2} \approx 1.61803398874989484820458683436563811772030917980576\ldots$

Lastly, there are four new interrelated numbers,
$Xi, \xi, \upsilon, \Upsilon$:

- $\Xi = \sqrt{2+\xi} = \frac{-1+\sqrt{5}+\sqrt{30+6\sqrt{5}}}{4} \approx 1.9562952014676112758571334957391990649194756177253\ldots$
- $\xi = \sqrt{2+\Upsilon} = \frac{1+\sqrt{5}+\sqrt{30-6\sqrt{5}}}{4} \approx 1.8270909152852017910042551439706343558816209187549\ldots$
- $\Upsilon = \sqrt{2-\upsilon} = \frac{1-\sqrt{5}+\sqrt{30+6\sqrt{5}}}{4} \approx 1.3382612127177164276525466613735609471991664379195\ldots$
- $\upsilon = \sqrt{2-\Xi} = \frac{-1-5+\sqrt{30-6\sqrt{5}}}{4} \approx 0.2090569265353069427996683096049962381613117389491\ldots$
- $\tan 3^\circ = \tan\frac{\tau}{120} = \tan\frac{\pi}{60} = \frac{\sqrt{2-\sqrt{2+\Xi}}}{\sqrt{2+\sqrt{2+\Xi}}}$
- $\tan 6^\circ = \tan\frac{\tau}{60} = \tan\frac{\pi}{30} = \frac{\sqrt{2-\Xi}}{\sqrt{2+\Xi}} = \sqrt{7-2\sqrt{5}-2\sqrt{15-6\sqrt{5}}}$
- $\tan 7.5^\circ = \tan\frac{\tau}{48} = \tan\frac{\pi}{24} = \frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{\sqrt{2+\sqrt{2+\sqrt{3}}}} = \sqrt{6}-\sqrt{3}+\sqrt{2}-2 = (\sqrt{2}-1)(\sqrt{3}-\sqrt{2})$
- $\tan 9^\circ = \tan\frac{\tau}{40} = \tan\frac{\pi}{20} = \frac{\sqrt{2-\sqrt{2+\varphi}}}{\sqrt{2+\sqrt{2+\varphi}}} = 1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$
- $\tan 11.25^\circ = \tan\frac{\tau}{32} = \tan\frac{\pi}{16} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} = -1-\sqrt{2}+\sqrt{4+2\sqrt{2}}$
- $\tan 12^\circ = \tan\frac{\tau}{30} = \tan\frac{\pi}{15} = \frac{\sqrt{2-\xi}}{\sqrt{2+\xi}} = \sqrt{23-10\sqrt{5}-2\sqrt{3(85-38\sqrt{5})}}$
- $\tan 15^\circ = \tan\frac{\tau}{24} = \tan\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}} = 2-\sqrt{3}$
- $\tan 18^\circ = \tan\frac{\tau}{20} = \tan\frac{\pi}{10} = \frac{\sqrt{2-\varphi}}{\sqrt{2+\varphi}} = \sqrt{1-\frac{2}{\sqrt{5}}} = \frac{\sqrt{25-10\sqrt{5}}}{5}$
- $\tan 21^\circ = \tan\frac{7\tau}{120} = \tan\frac{7\pi}{60} = \frac{\sqrt{2-\sqrt{2+\upsilon}}}{\sqrt{2+\sqrt{2+\upsilon}}}$
- $\tan 22.5^\circ = \tan\frac{\tau}{16} = \tan\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \sqrt{2}-1$
- $\tan 24^\circ = \tan\frac{\tau}{15} = \tan\frac{2\pi}{15} = \frac{\sqrt{2-\Upsilon}}{\sqrt{2+\Upsilon}} = \sqrt{23+10\sqrt{5}-2\sqrt{3(85+38\sqrt{5})}}$
- $\tan 27^\circ = \tan\frac{3\tau}{40} = \tan\frac{3\pi}{20} = \frac{\sqrt{2-\sqrt{2-\Phi}}}{\sqrt{2+\sqrt{2-\Phi}}} = -1+\sqrt{5}-\sqrt{5-2\sqrt{5}}$
- $\tan 30^\circ = \tan\frac{\tau}{12} = \tan\frac{\pi}{6} = \frac{\sqrt{2-\sqrt{1}}}{\sqrt{2+\sqrt{1}}} = 3^{-\frac{1}{2}} = \frac{\sqrt{3}}{3}$

- $\tan 33° = \tan \frac{11\tau}{120} = \tan \frac{11\pi}{60} = \frac{\sqrt{2-\sqrt{2-\Upsilon}}}{\sqrt{2+\sqrt{2-\Upsilon}}}$
- $\tan 33.75° = \tan \frac{3\tau}{32} = \tan \frac{3\pi}{16} = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}$
- $\tan 36° = \tan \frac{\tau}{10} = \tan \frac{\pi}{5} = \frac{\sqrt{2-\Phi}}{\sqrt{2+\Phi}} = \sqrt{5-2\sqrt{5}}$
- $\tan 37.5° = \tan \frac{5\tau}{24} = \tan \frac{5\pi}{24} = \frac{\sqrt{2-\sqrt{2-\sqrt{3}}}}{\sqrt{2+\sqrt{2-\sqrt{3}}}} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2 = (\sqrt{2}+1)(\sqrt{3}-\sqrt{2})$
- $\tan 39° = \tan \frac{13\tau}{30} = \tan \frac{13\pi}{60} = \frac{\sqrt{2-\sqrt{2-\xi}}}{\sqrt{2+\sqrt{2-\xi}}} = \frac{\sqrt{4-\Phi^2}-1}{\sqrt{3-\Phi}}$
- $\tan 42° = \tan \frac{7\tau}{60} = \tan \frac{7\pi}{30} = \frac{\sqrt{2-\upsilon}}{\sqrt{2+\upsilon}} = \sqrt{7+2\sqrt{5}-2\sqrt{3(5+2\sqrt{5})}}$
- $\tan 45° = \tan \frac{\tau}{8} = \tan \frac{\pi}{4} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

Every angle above these is either the reciprocal of one of these (45 to 90), the negative of one of these (-45 to 0), or both. Any of these angles can be halved by Equation (4).

B.1 Approximations

Many existing trigonometry problems can be adapted into a constructible framework by close approximation.

θ	$\tan\theta$	θ	$\tan\theta$	θ	$\tan\theta$
1°	$\frac{1}{57}$	16°	$\frac{2}{7}$	31°	$\frac{3}{5}$
2°	$\frac{3}{86}$	17°	$\frac{26}{85}$	32°	$\frac{5}{8}$
3°	$\frac{1}{19}$	18°	$\sqrt{1-\frac{2}{\sqrt{5}}}$	33°	$\frac{37}{57}$
4°	0.07	19°	$\frac{21}{61}$	34°	$\frac{29}{43}$
5°	$\frac{7}{80}$	20°	$\frac{4}{11}$	35°	0.7
6°	$\frac{2}{19}$	21°	$\frac{5}{13}$	36°	$\sqrt{5-2\sqrt{5}}$
7°	$\frac{7}{57}$	22°	$\frac{40}{99}$	37°	$\frac{52}{69}$
8°	$\frac{9}{64}$	23°	$\frac{31}{73}$	38°	$\frac{25}{32}$
9°	$\frac{13}{82}$	24°	$\frac{4}{9}$	39°	$\frac{17}{21}$
10°	$\frac{3}{17}$	25°	$\frac{7}{15}$	40°	$\frac{47}{56}$
11°	$\frac{7}{36}$	26°	$\frac{20}{41}$	41°	$\frac{20}{23}$
12°	$\frac{1}{\sqrt{12}}$	27°	0.57	42°	0.9
13°	$\frac{3}{13}$	28°	$\frac{8}{15}$	43°	$\frac{55}{59}$
14°	$\frac{1}{4}$	29°	$\frac{5}{9}$	44°	$\frac{28}{29}$
15°	$2-\sqrt{3}$	30°	$\frac{1}{\sqrt{3}}$	45°	1

Table 1: Approximations and Exact Tangent Values