On the Non-existence of Exponential Calculus

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November 2023

Abstract

In 2008 Bashirov-Kurpınar-Özyapıcıin [1] generalized the notion of multiplicative calculus introduced by Grossman-Katz [2]. In this paper we explore whether an analogous theory of calculus is possible for the next hyperoperation after multiplication, which is exponentiation.

1 Introduction

During 1967-70 Grossman and Katz [2] explored the notion of geometric/multiplicative calculus by defining the geometric derivative and geometric integral, which is analogous to Newtonian and Riemannian calculus in the multiplicative setup. That is, the basic operation is multiplication instead of addition. They showed that the notion of geometric calculus has a nice relationship with the existing notion of calculus. In 2008, Bashirov et al. [1] drew the attention of the researchers towards the work of Grossman-Katz indicating its usefulness in the different aspects of existing calculus. They also proposed the following generalized philosophical concept of derivatives and integrals to expand upon the work of Grossman-Katz.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bijective function of one variable. Define the † derivative and † integral of the function f by

$$f^{\dagger}(x) = \phi(\phi^{-1} \circ f)'(x) \tag{1}$$

$$\int_a^b f(x)\, d^{\dagger}x = \phi\left(\int_a^b (\phi^{-1} \circ f)(x)\, dx\right) \tag{2}$$

where we assume that the image of f is a subset of image of ϕ. On the basis of these definitions one can develop † calculus. Multiplicative calculus is one such calculi where $\phi(x) = e^x$.

In this scenario, we ask whether an extended theory of calculus is possible for the next hyper-operation after multiplication, which is exponentiation. Since it is well known that exponentiation is non-commutative, it is expected that such theory should not exist. And hence our aim is to give a detailed proof towards the same.

In Riemannian calculus the limit of discrete summation is converted to an integral, this integral will be referred to as the 'regular integral' throughout this paper. Symbolically,

$$\sum_i f(x_i)\, \Delta x_i \;\rightarrow\; \int f(x)\, dx$$

Where (x_i) is a sequence of real numbers and $\Delta x_i = x_{i+1} - x_i$

Similarly for multiplicative calculus, the limit of discrete product is converted to multiplicative integral,

$$\prod_i f(x_i)^{\Delta x_i} \;\rightarrow\; \int^* f(x)^{dx}$$

the * sign on the integral indicates that it is a multiplicative integral.

Where the relation between the multiplicative integral and the regular integral is given in Lemma 1.

We try to extend this notion to exponentiation. Before going into the details, we state a few notations and definitions:-

Tetration, which is repeated exponentiation of a number to itself is denoted by a left superscript to a number.

For e.g. ${}^{3}a = a^{a^{a}}$.

Akin to summation and product operators, we introduce an exponentiation operator denoted by the symbol $\boldsymbol{E}$ which performs repeated exponentiation.

For e.g. $\underset{i=1}{\overset{3}{\boldsymbol{E}}} a_i = {a_1}^{{a_2}^{a_3}}$.

Now along the notion of multiplicative integral we wish to formulate an exponential integral which would be the limit of a discrete exponentiation. Symbolically,

$$\underset{i}{\boldsymbol{E}}\ {}^{\Delta x_i} f(x_i) \rightarrow \widehat{\int}\ {}^{dx} f(x)$$

Here $\frown$ sign on the integral indicates that it is an exponential integral.

2 Preliminaries

2.1 Recursion relation for tetration

$${}^{n+1}a = a^{{}^{n}a}$$

substituting n=0 we get,

$${}^{0}a = 1$$

substituting n= -1 we get,

$$^{-1}a = 0$$

2.2 Domain of tetration

By taking n=-2 in the recursion relation we get

$^{-1}a = a^{^{-2}a}$

From (2.1),

$$a^{^{-2}a} = 0$$

As $a^k = 0$ does not hold for any value of k, ^{-2}a is undefined. Thus for any n less than -2, tetration is not defined.

2.3 An important result in multiplicative calculus

We quote the following result of Grossman-Katz [2] which shows the relationship between multiplicative and regular integral.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly positive continuous function, then the multiplicative integral in terms of regular integral is given by the equation*

$$\int^{*} f(x)^{dx} = e^{\int ln(f(x))\ dx}$$

3 Main Results

We shall show that there does not exist any invertible function which converts exponentiation in terms of multiplication operator, similar to how $ln(x)$ converts multiplication into addition operator. On the contrary we assume there exists a function $zn(x)$ such that

$$zn \circ \underset{i}{\boldsymbol{E}}\ {}^{\Delta x_i} f(x_i) = \prod_i [zn(f(x_i))]\ ^{\Delta x_i}$$

Therefore,

$$\underset{i}{\boldsymbol{E}}\ {}^{\Delta x_i} f(x_i) = zn^{-1} \circ \prod_i [zn(f(x_i))]\ ^{\Delta x_i}$$

Now taking the limit on both sides,

$$\int^{\frown}\ {}^{dx} f(x) = zn^{-1} \circ \int^{*} [zn(f(x))]^{dx}$$

Applying Lemma 1, we get

$$\int^{\frown}\ {}^{dx} f(x) = zn^{-1} \circ e^{\int \ln[zn(f(x))]\ dx} \tag{3}$$

This equation is of the form (2) where $\phi = zn^{-1} \circ exp$. The function $zn(x)$ must satisfy the following properties to obtain the above formulation:

1. $zn(x^y) = zn(x) \cdot zn(y)$
2. $zn({}^y x) = (zn(x))^y$ where $y > -2$.
3. zn takes only positive values

Theorem 3.1. *There does not exist any function that can satisfy the above two properties and has an inverse.*

Proof. Suppose there exists a function zn satisfying the above properties and has an inverse. It is easy to notice that the domain of zn contains all non-negative real numbers since, for any non-negative real y we can write it as $(y^2)^{1/2}$. Note that, $zn(x^y) = zn(x) \cdot zn(y) = zn(y) \cdot zn(x) = zn(y^x)$, and hence for $y = 1$ we have $zn(x) = zn(1)$ for all $x \in \mathrm{dom}(zn)$

Further, since $zn(x^y) = zn(x) \cdot zn(y) = zn(1)$, we see that $zn(1) = zn(1)^2$, and hence either $zn(1) = 1$ or $zn(1) = 0$. As $zn(x)$ is a strictly positive function, the only possibility is $zn(1) = 1$. That is, zn is a constant function. Thus zn^{-1} cannot exist.

□

Corollary 3.2. *The exponential calculus as desired in (3) does not exist*

Proof. Given the nonexistence of the function zn in Theorem 3.1 it follows that the function $\phi := zn^{-1} \circ \exp$ doesn't exist. Therefore, such a formulation of exponential calculus is unattainable. □

4 Conclusion

As all hyper-operations above exponentiation are also non-commutative, by a similar argument we can show that such a formulation of their calculus would not exist either.

5 Acknowledgments

The authors thank Prosenjit Das for his guidance and insightful feedback.

References

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