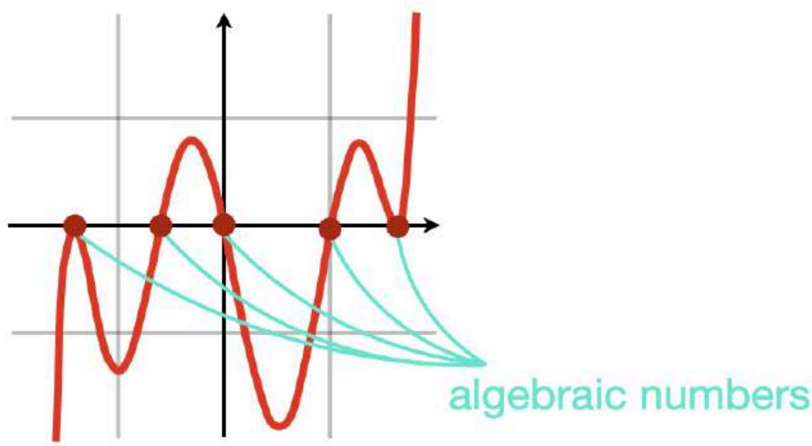
Why are Transcendental Numbers so Interesting?
by DiBeos

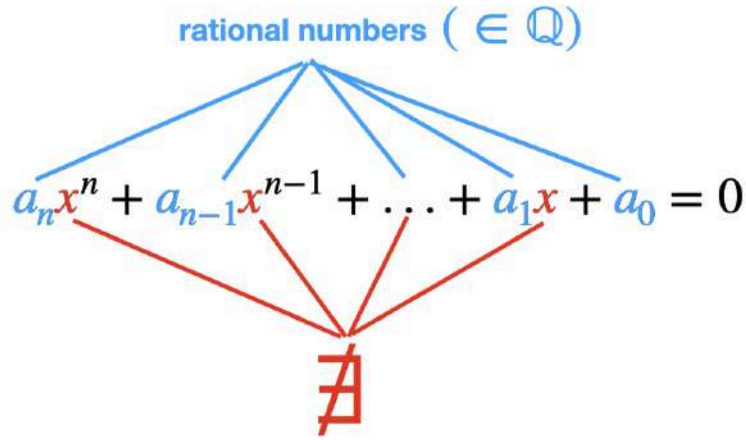
The fundamental difference between *transcendental* and *algebraic* numbers is that a transcendental number is not the root of any nonzero polynomial with rational (or integer) coefficients. In other words, transcendental numbers cannot be expressed as the solution to an algebraic equation like this:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

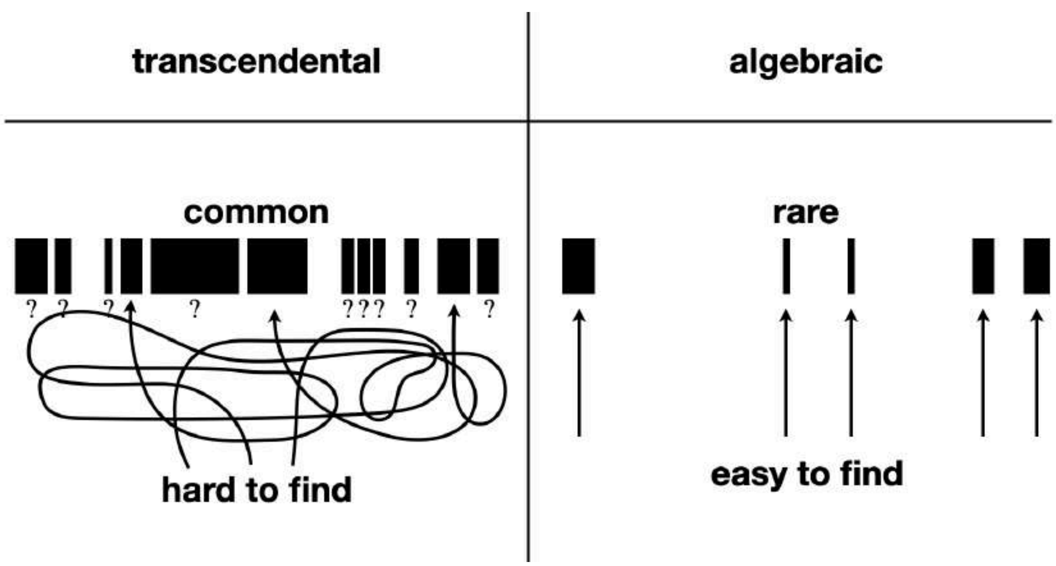
where a_i ($i \in \{1, \ldots, n\}$) are rational numbers and n is a positive integer.



transcendental numbers:



What is weird about them is the fact that almost all real and complex numbers are transcendental, but identifying specific transcendental numbers is very hard. There is a sharp contrast between the "rarity" of algebraic numbers and the huge number of transcendental ones, even though they are so hard to find.



Some examples of transcendental numbers are the following:

(1) Euler's number e ;

e

2.71828...

(2) The number π ;

π 3.14159...

(3) Liouville's Constant:

$$L = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = 0.110001000000000000000001\ldots$$

(4) 2^{π} , which is known to be transcendental by the **Gelfond-Schneider theorem** ;

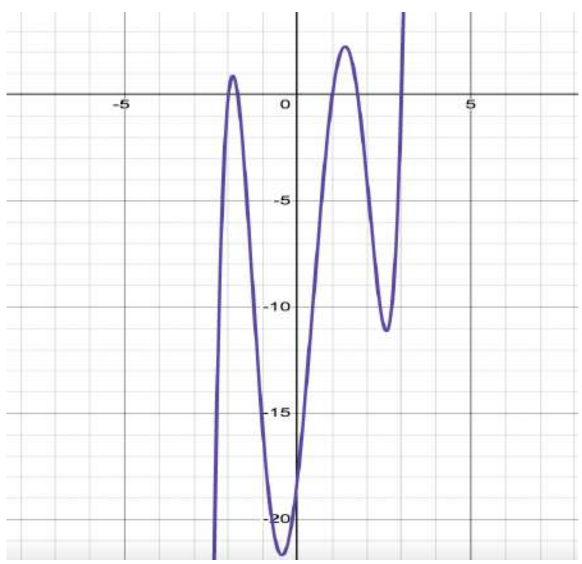
(5) e^{π} , which is called Gelfond's Constant ;

A weird fact is that if you add 2 transcendental numbers together, or multiply them, it doesn't guarantee that you will end with another transcendental number. For example:

(6) $\pi + e$ and $\pi \cdot e$ are suspected to be transcendental, but nobody could prove it so far…

In order to understand transcendental numbers, let's take a step back and talk about the definition of algebraic numbers. Imagine the following polynomial

$$P(x) = x^5 - 2x^4 - 8x^3 + 12x^2 + 15x - 18$$

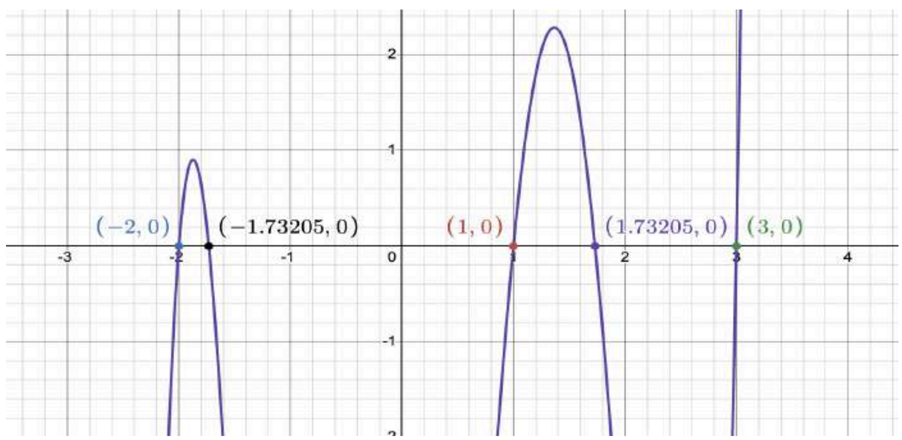


Fortunately, this polynomial can be decomposed in such a way that its roots become obvious.

$$P(x) = (x - 1)(x + 2)(x - 3)\left(x^2 - 3\right)$$

$$roots\ of\ P(x) = 0: \qquad x = 1, -2, 3, +\sqrt{3}, -\sqrt{3}$$

Therefore, by definition, these are algebraic numbers, so they cannot be transcendental.



We said "fortunately this polynomial can be decomposed" because usually quintic equations (polynomial equations of degree 5) are really hard to solve. This difficulty is expressed through the **Abel-Ruffini Theorem**, which states that there is no general formula using only radicals (like the quadratic, cubic or quartic formulas) for solving quintic equations and higher-degree polynomials.

quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

cubic formula

$$\frac{\sqrt[3]{-27a^2d + 9abc - 2b^3 + 3a\sqrt{3(27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2)}} + \sqrt[3]{-27a^2d + 9abc - ab^3 - 3a\sqrt{3(27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2)}}}{3\sqrt[3]{2a}}$$

As we said, the numbers $x = 1, -2, 3, +\sqrt{3}, -\sqrt{3}$ are called *algebraic numbers* since they are roots of a polynomial with rational coefficients, namely this polynomial:

$$P(x) = 1x^5 - 2x^4 - 8x^3 + 12x^2 + 15x^1 - 18x^0$$

rational coefficients

So, the general definition of an algebraic number α is a real or complex number that is a root of some non-zero polynomial $(P(x) \neq 0, \forall x)$ with rational coefficients:

$$P(\alpha) = 0$$

A transcendental number is a number that is not algebraic. So, if β is a transcendental number, then, no matter how creative you are, you will never be able to find a non-zero polynomial with rational coefficients, such that $P(\beta) = 0$.

$$P(\beta) \neq 0$$

Btw, we are looking for your research! We'd like to encourage you guys to submit your papers or research to us, so that we can publish it on our site where others can read it and peer review it. It's a free, public way of getting your research out there. So please submit it to us here dibeos.contact@gmail.com .

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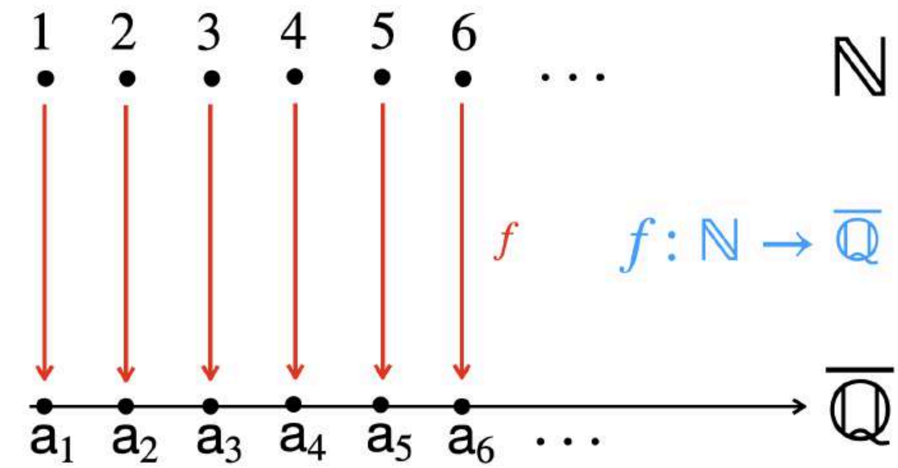
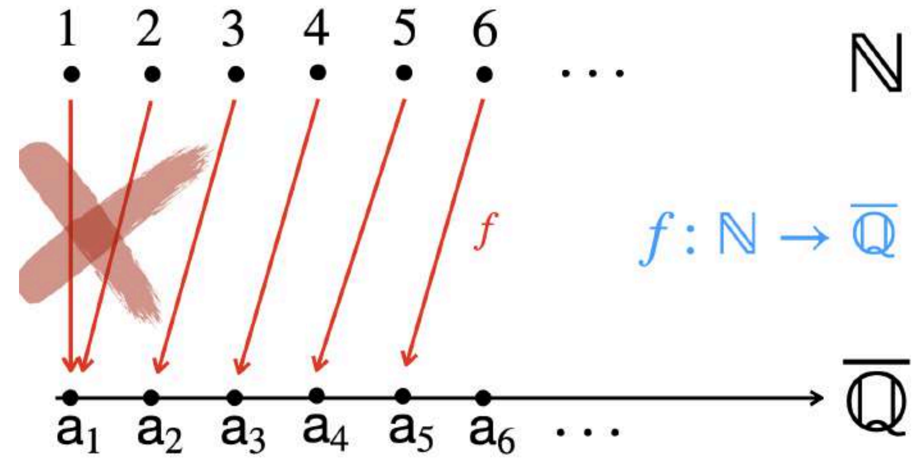
To understand why algebraic numbers are rare, we compare their size to the set of all real numbers:

First of all, the set of algebraic numbers $\overline{\mathbb{Q}}$ is *countable*. What does it mean? It means you can "count them", at least theoretically, even though you would never be able to stop, because they are *not finite*. Rigorously, we say that there is a mapping $f: \mathbb{N} \to \overline{\mathbb{Q}}$ such that this mapping is *one-to-one* (it does not associate two natural numbers to the same algebraic number) and it is also *onto* (it spans all the set of algebraic numbers, i.e. no algebraic number is left out of this mapping). In practical terms, it is like assigning an index (as a natural number) to each algebraic number in this set. It looks like we are counting them, and that's why we say that $\overline{\mathbb{Q}}$ is *countable*.


1 2 3 4 5 6 ...
ℕ
f
$f : \mathbb{N} \to \overline{\mathbb{Q}}$
a_1 a_2 a_3 a_4 a_5 a_6 ...
$\overline{\mathbb{Q}}$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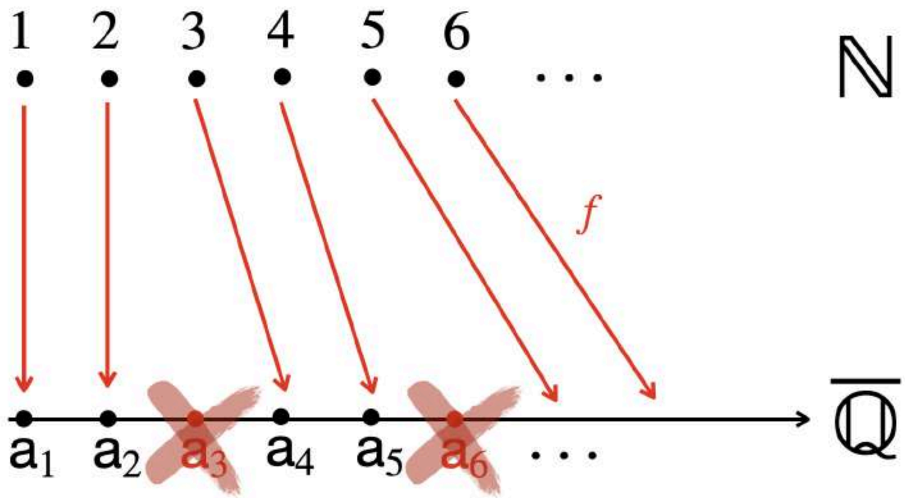
1 2 3 4 5 6 ...
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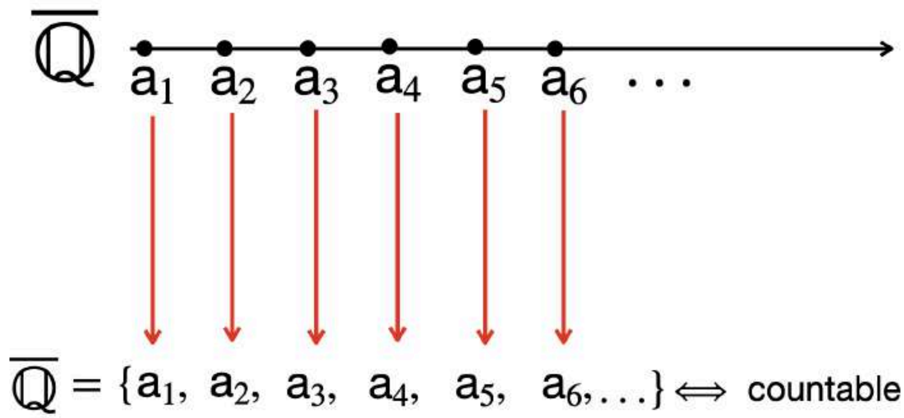
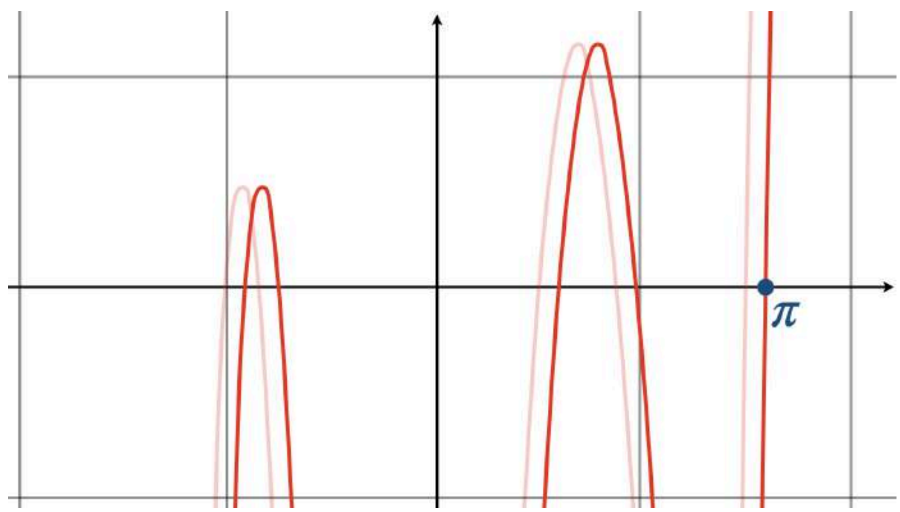
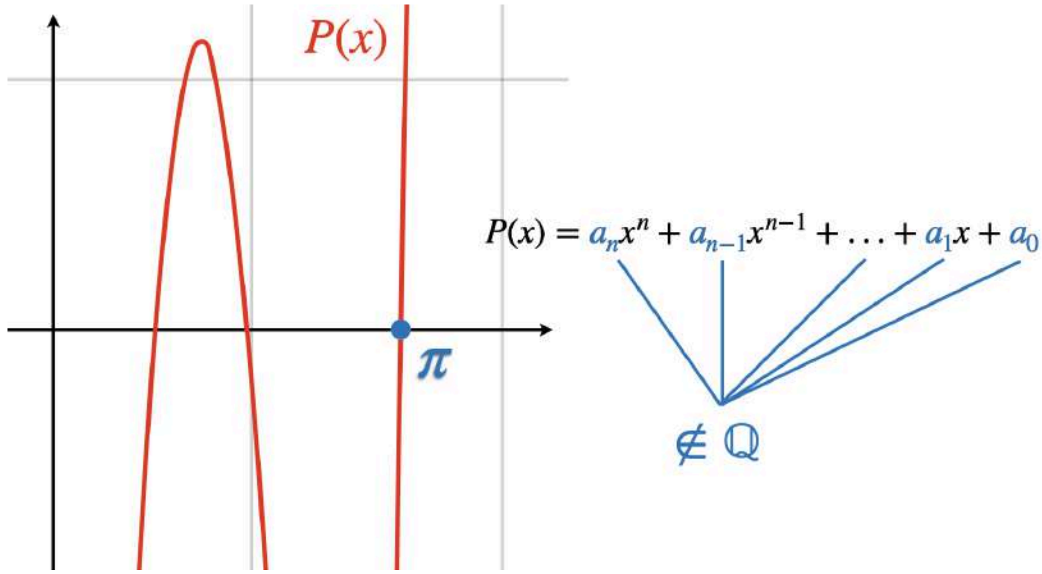


Now, this is not intuitive because we could imagine tweaking this polynomial graph to produce different roots in a continuous way:

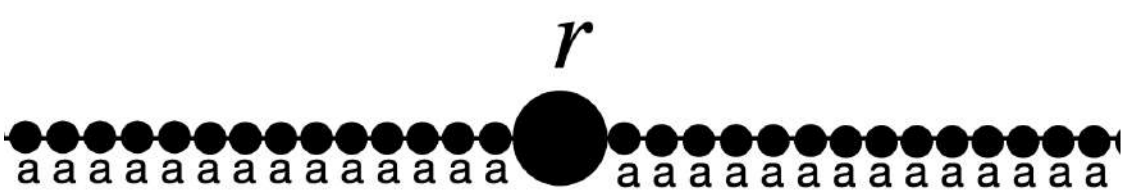


If you find a polynomial such that its graph passes through a transcendental point, then it means your polynomial does not have rational coefficients anymore…

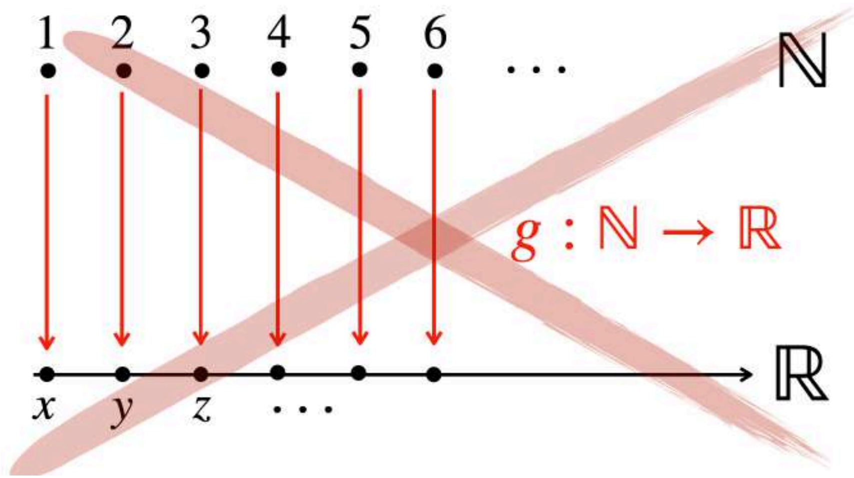


However, with algebraic numbers you can do this continuous transformation such that the polynomial you get at the end still possesses only rational coefficients. And this is so because even though the algebraic numbers are countable, they are also *dense* in $\mathbb{R}$. Intuitively, no matter what real number you choose, you can always find algebraic numbers extremely close to where you are.

Sets can be *dense* and *countable* at the same time. If you guys want an explanation dedicated to how these two “contradictory intuitions” can co-exist, let us know (*dibeos.contact@gmail.com*).



The real numbers, on the other hand, are not only dense, but *uncountable*! Being uncountable is a stronger condition than being just dense. This means that you cannot imagine a function $g: \mathbb{N} \rightarrow \mathbb{R}$ that is *one-to-one* and *onto*.



Well, as we've seen before, the transcendental numbers are just real (or complex) numbers that are not algebraic:

$$\mathbb{R}\backslash\overline{\mathbb{Q}}$$

Another way of expressing it is by saying that

$$\mathbb{R} = \overline{\mathbb{Q}} \cup \mathbb{R}\backslash\overline{\mathbb{Q}}$$

And therefore, if the algebraic numbers are countable, and if we also consider the transcendental numbers to be countable, then the real numbers (which are just their union) must be countable as well. This is a contradiction, and thus we conclude that our hypothesis was wrong. Indeed, the *transcendental numbers are uncountable*. But are they also dense? Yes, even though uncountability does not imply density!

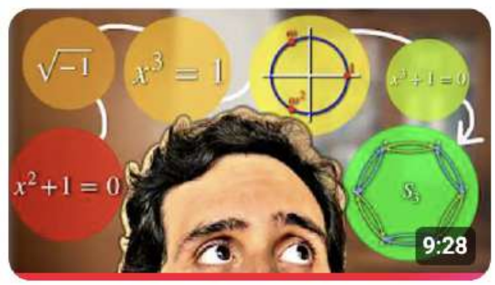
When comparing a set that is dense and countable ($\overline{\mathbb{Q}}$) with a set that is dense and uncountable ($\mathbb{R}\backslash\overline{\mathbb{Q}}$) we can easily see (intuitively) that the latter will be way larger. In fact, that's the intuition behind the statement: "*transcendental numbers are more common than the rare algebraic numbers*".

The best way to see the rarity of algebraic numbers, as well as the difficulty in finding concrete examples of transcendental numbers (even though they compose the vast majority of real numbers), can be best appreciated through the lens of group theory and Galois theory.

As we saw before, an algebraic number is a solution (or root) of a non-zero polynomial with integer (or rational) coefficients. These numbers are associated with **Galois groups** which capture the **symmetries of the roots** of polynomials.

If you want to know more about Galois Groups, check out these video:

How to Get Galois Groups Using Field Extensions

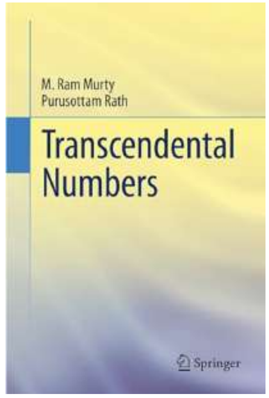


How to Get to Galois Theory Naturally

But, basically, algebraic numbers have a finite and well-defined structure based on its group structure. Transcendental numbers do not have any Galois groups associated with them. They lack structure and symmetry. We can say that it is way easier to find real numbers with no structure (almost “random”) rather than numbers with very clear symmetries and group structure, like the algebraic numbers have.

As a consequence, algebraic numbers are rare, because of the strong requirement of having structure, but “easy” to find (once you have a polynomial). On the other hand, transcendental numbers are very common (almost random, since they have no galois structures), but hard to pin down, since there is no polynomial from which we can start our search.

If you’d like to dive deeper into the fascinating nature of transcendental numbers, you might want to study the following book:



Transcendental Numbers