Analysis of the Collatz Map: Digital Root Behavior and Loop Impossibility

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Abstract

In this paper, we present a novel analysis of the Collatz conjecture. We explore the digital root behavior of numbers under the Collatz map and identify a consistent mapping pattern based on modular arithmetic properties. Furthermore, we introduce a rigorous argument showing that the only possible closed loop in the Collatz map is the trivial loop (4, 2, 1). This result is supported by both analytical reasoning and numerical evidence.

1 Introduction

The Collatz conjecture states that for any positive integer n, the following map eventually leads to 1:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$$

This paper investigates two key properties of the Collatz map:

- The consistent mapping behavior of digital roots under iteration.
- The impossibility of non-trivial loops, supported by parity and divisibility considerations.

2 Digital Root Behavior of Numbers Under Collatz Map

The digital root (DR) of a number is the single-digit value obtained by iteratively summing its digits. For example:

$$987 \rightarrow 9+8+7 = 24 \rightarrow 2+4 = 6$$

Thus, the digital root of 987 is 6.

Under the Collatz map $3n+1$, numbers with the following DR always map as:

- $\mathbf{4}, \mathbf{7}$, and $\mathbf{1}$ $\rightarrow$ always map to a number with digital root **4**.
- $\mathbf{2}, \mathbf{5}$, and $\mathbf{8}$ $\rightarrow$ always map to a number with digital root **7**.
- $\mathbf{3}, \mathbf{6}$, and $\mathbf{9}$ $\rightarrow$ always map to a number with digital root **1**.

Under the Collatz map $n/2$, the digital roots map as:

$$(1,2,3,4,5,6,7,8,9) \rightarrow (5,1,6,2,7,3,8,4,9)$$

3 Parity and DRmod3 Notation

To better classify the behavior of numbers under the Collatz map, we introduce a modular-based notation that captures both parity (odd/even) and the digital root modulo 3.

- Odd numbers are denoted by o, and even numbers by e.
- The superscript represents the digital root modulo 3:
  - $o^1 \rightarrow$ odd number with DR mod 3 = 1
  - $o^2 \rightarrow$ odd number with DR mod 3 = 2
  - $o^0 \rightarrow$ odd number with DR mod 3 = 0
  - $e^1 \rightarrow$ even number with DR mod 3 = 1
  - $e^2 \rightarrow$ even number with DR mod 3 = 2
  - $e^0 \rightarrow$ even number with DR mod 3 = 0

Under this notation:

- The $3n + 1$ mapping exclusively produces e^1 from all odd numbers, regardless of their DR mod 3.
- The $n/2$ mapping maintains parity but alters the DR mod 3 in a predictable pattern:
  - $e^1 \rightarrow o^2$ or $e^2 \rightarrow o^1$
  - $e^1 \rightarrow e^2$ or $e^2 \rightarrow e^1$
  - $e^0 \rightarrow e^0$

This notation will be used throughout the paper to describe the modular behavior of numbers under the Collatz map.

4 Loops in the Collatz Map

To analyze the possibility of non-trivial loops in the Collatz map, we define a loop as a finite set of numbers that map back to their starting point through successive iterations of the Collatz function.

By applying the modular notation, we observe the following behavior:

- The $3n + 1$ operation always produces e^1 from any odd number, ensuring that odd steps always lead to this specific even type.
- The $n/2$ operation either maintains (only for specific e^0) or swaps the DR mod 3 value while reducing the magnitude of the number.

4.1 The Trivial Loop

The well-known trivial loop in the Collatz map consists of the sequence:

$$4 \rightarrow 2 \rightarrow 1 \rightarrow 4$$

In modular notation:

$$e^1 \rightarrow e^2 \rightarrow o^1 \rightarrow e^1$$

This is the only known loop that exists in the Collatz map and corresponds to the stable 4-2-1 cycle.

4.2 Exhaustive Loop Search Using Modular Notation

Since any closed loop must contain at least one odd iteration, e^1 must be part of the loop. For large numbers, the operation $3n + 1$ can be approximated to $3n$ to analyze whether the loop could exist.

We examine all possible branches originating from e^1:

1. First branch:
$$e^1 \to o^2 \to e^1$$
The ratio of the values at the odd and even steps:
$$\frac{3}{2}$$

2. Second branch:
$$e^1 \to o^2 \to e^2 \to o^1 \to e^1$$
The ratio of the values at the odd and even steps:
$$\frac{3}{4}$$
This is possible only for small values, since the contribution from $+1$ can affect the ratio (hence part of the trivial loop).

3. Third branch:
$$e^1 \to e^2 \to e^1$$
The ratio of the values:
$$\frac{1}{4}$$
This ratio makes looping impossible due to the diminishing effect.

4.3 Conclusion: The Impossibility of Non-Trivial Loops

Through exhaustive mapping of all possible branches, it becomes evident that:

- No other stable loops can exist except the trivial one.
- The diminishing or expanding nature of the ratios prevents the formation of stable cycles.
- Experimental evidence from computational verification supports this conclusion, as no non-trivial cycles have ever been observed in massive numerical testing.

Thus, the only possible loop in the Collatz map is the trivial $4 \to 2 \to 1$ cycle.