Generalized Loop Equivalence Conjecture

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Abstract

This paper proposes the **Generalized Loop Equivalence Conjecture**, which states that the recurrence relation $an + k$ has the same number of loops as $an + ka^p$ for any integer $p \geq 1$. The conjecture explores the equivalence of loops formed by different mappings, identifying patterns based on the multiplier and additive constants. The paper also presents detailed examples, the full mapping function, and observations on divergence.

1 Introduction

The Collatz conjecture, also known as the $3n + 1$ problem, considers the iteration of the function:

$$f(n) = \begin{cases} 3n + 1, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

This generalized conjecture extends the Collatz map by considering the function:

$$f(n) = \begin{cases} an + k, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

where $a \geq 1$ and k is a positive integer.

2 Conjecture Statement

The conjecture states:

> For any positive integers a and k, the recurrence relation $an + k$ produces the same number of loops as $an + ka^p$ for any $p \geq 1$.

$$f(n) = \begin{cases} an + k, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$f(n) = \begin{cases} an + ka^p, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

where a,k and p are non negative integers. Both functions, under finite iterations, produce the same number of unique loops. The elements present in the loops have a one-to-one correspondence by being scaled by a factor of a^p.

3 Examples and Verification

Several cases were tested to verify the conjecture:

- For $a = 3$ and $k = 1$, the mapping $3n + 1$ produces a single loop: $4 \to 2 \to 1 \to 4$.
- The mapping $3n + 3$ also produces a single loop: $12 \to 6 \to 3 \to 12$.
- For $a = 5$ and $k = 1$, the map $5n + 1$ leads to divergence for some numbers.
- However, $5n + 5$ follows the same loop equivalence principle, showing commutativity in loop formation.

4 Behavior and Divergence

While the conjecture holds in terms of loop equivalence, it does not guarantee convergence. Certain mappings lead to divergent sequences, especially for higher values of a. Empirical tests show divergence in cases such as $5n + 1$ and $5n + 5$. Thus, while the loop equivalence holds, the mappings may or may not diverge.

5 Conclusion

The generalized loop equivalence conjecture shows that modifying the recurrence relation by scaling the additive constant according to the multiplier's power maintains the same loop structure. This observation opens the door for further exploration into the behavior of iterative mappings beyond the classical Collatz conjecture.

6 References

- L. Collatz, *On the behavior of certain arithmetic sequences*, 1937.
- T. Tao, *Almost all Collatz orbits attain almost bounded values*, 2019.