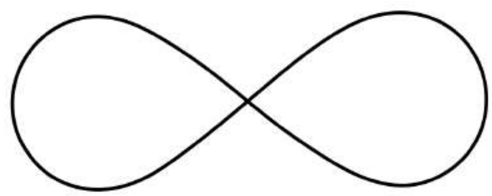
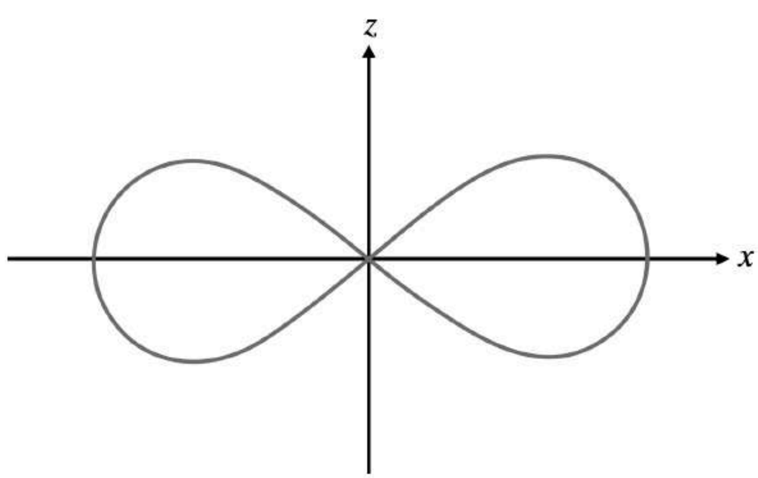
Did I Discover A New Topological Surface?

by Dibeos

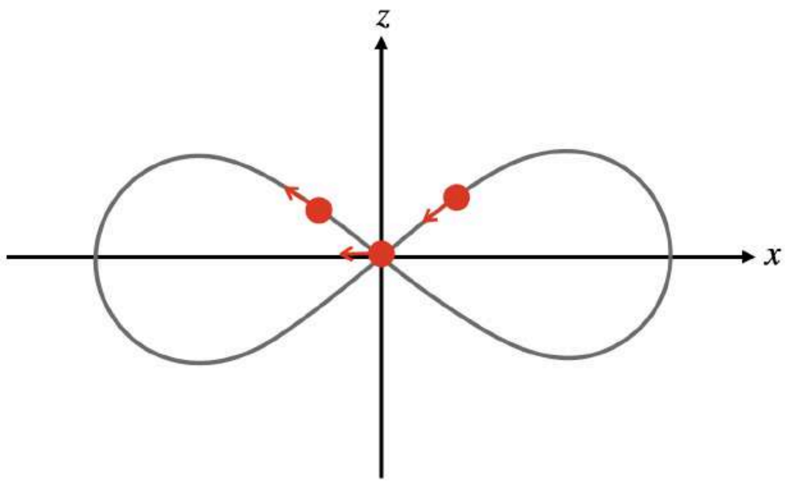
Imagine a *figure-eight*:



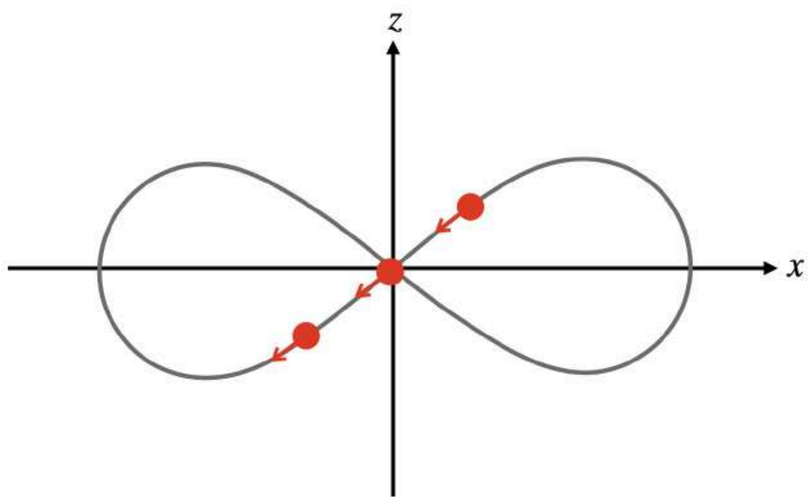
Now, let's place it on a xz cartesian plane in order to systematize its description and build a formula that faithfully describes it. You will see shortly why we chose xz rather than the more standard xy cartesian plane.



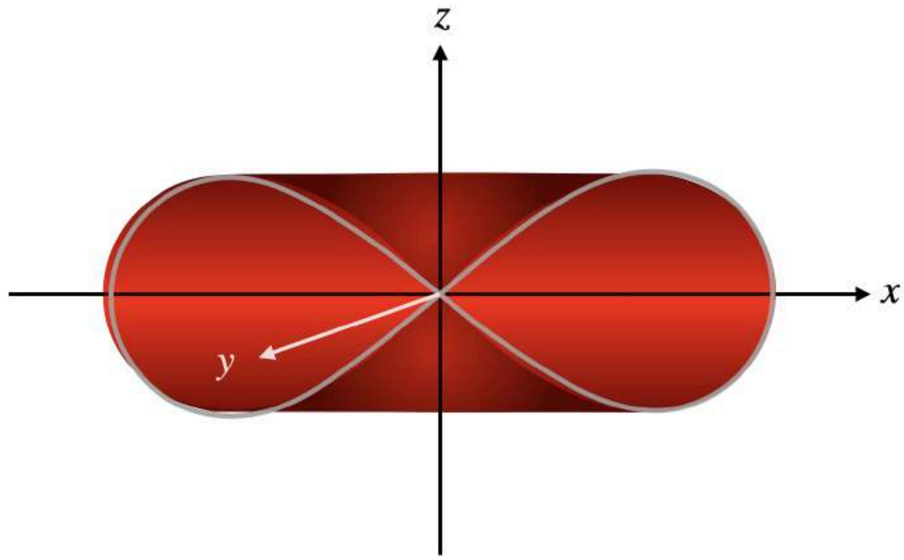
Imagine a point travelling along this figure-eight. It would not cross the intersection this way:



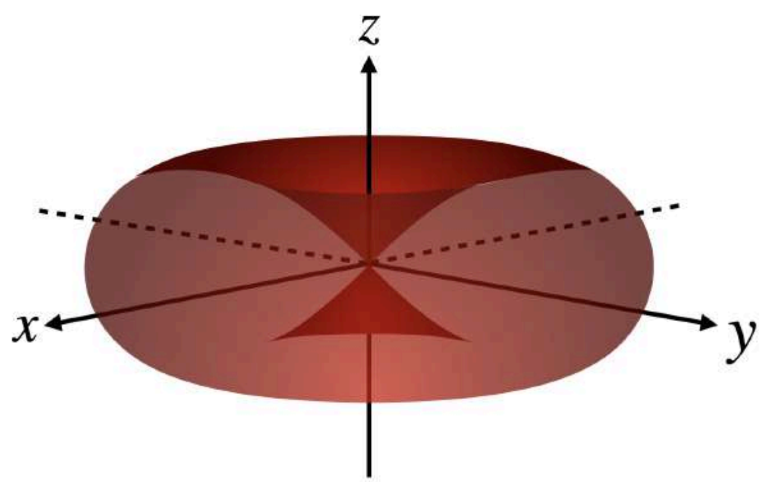
But rather, this way:



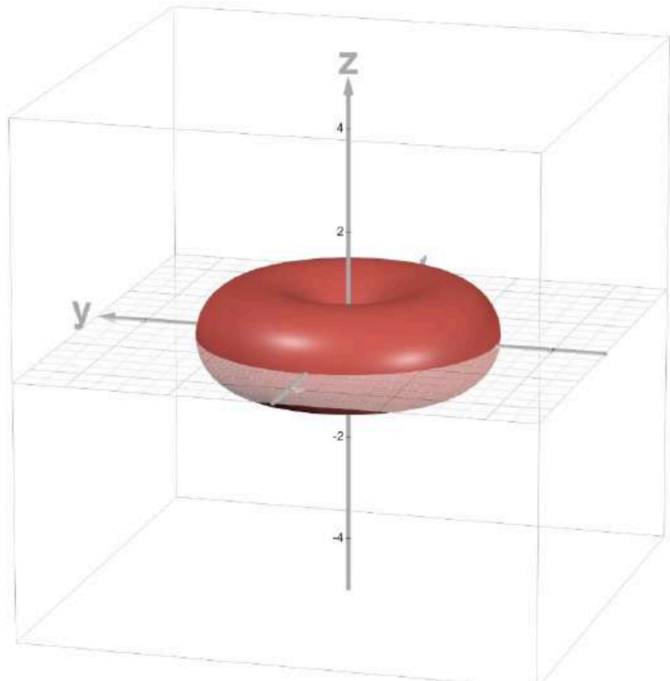
The next step is to create a *solid of revolution* by introducing a third axis (denoted as y) and rotating the figure-eight with respect to the z-axis:



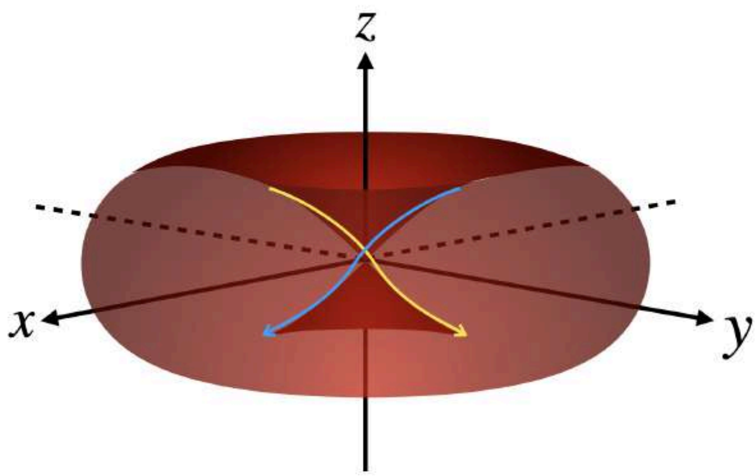
z
x
y



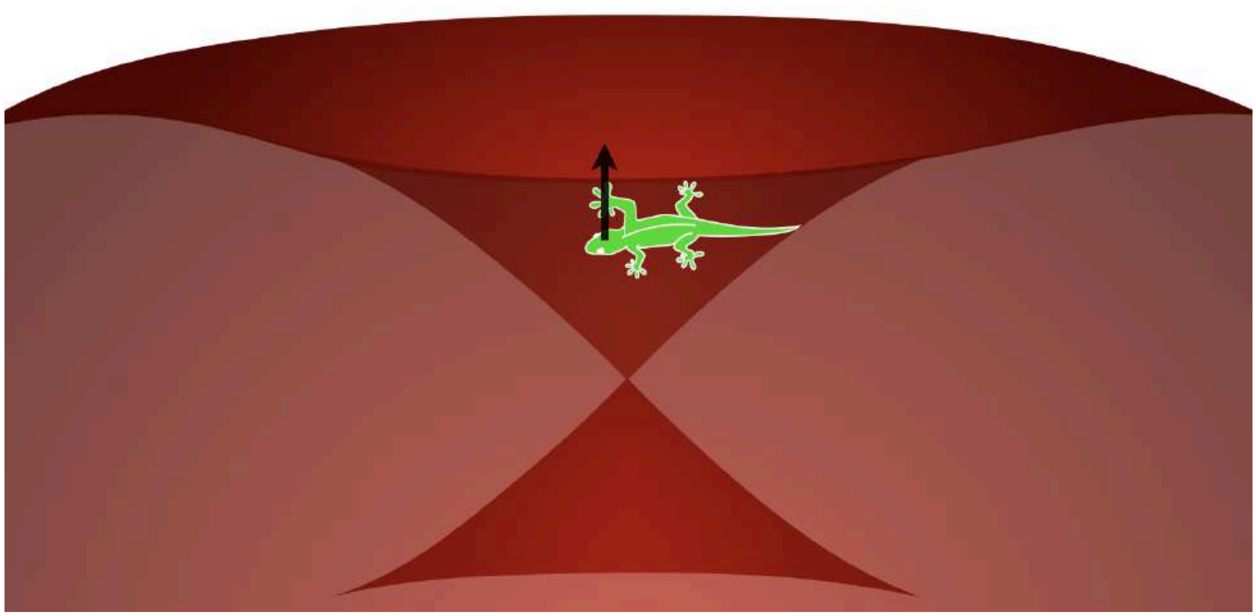
z
x
y



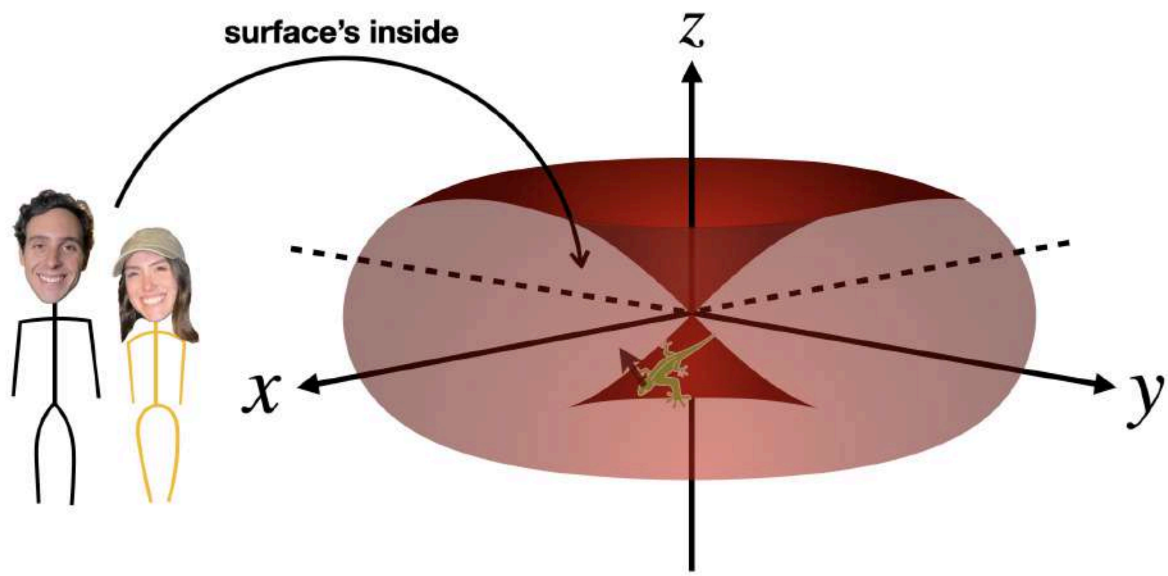
This is a very simple surface (a $2D$ space) embedded in 3 dimensions, but it has some very interesting properties. First of all, it is a *closed* surface (in the sense that it has no *boundaries*). It has a *self-intersection*, which is a unique point, and it is a direct consequence of the way in which this surface was created, out of a figure-eight. This surface probably has a *Möbius strip* in it. But I'm honestly not completely sure about that. It is, however, *non-orientable*. There's no doubt about it. The self-intersection is present just because it is embedded in 3 dimensions. If it were embedded in 4 dimensions, it would not have this self-intersection.



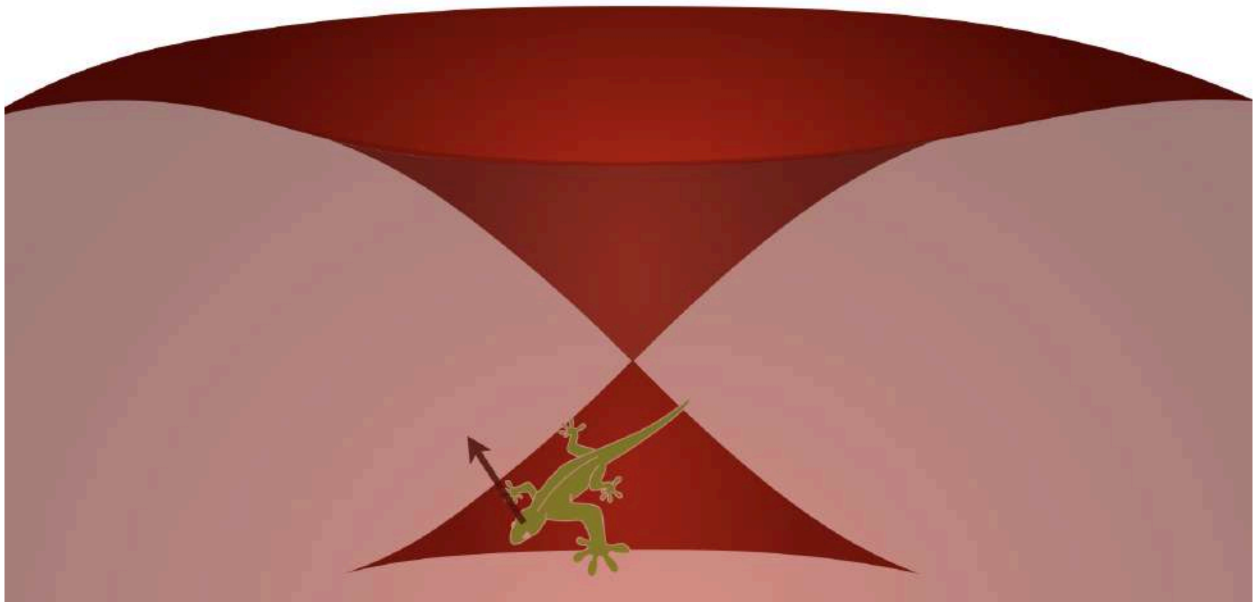
Let’s explore the fascinating way in which this surface is non-orientable: Imagine a 2 -dimensional lizard living in this surface. This lizard is “right handed”.



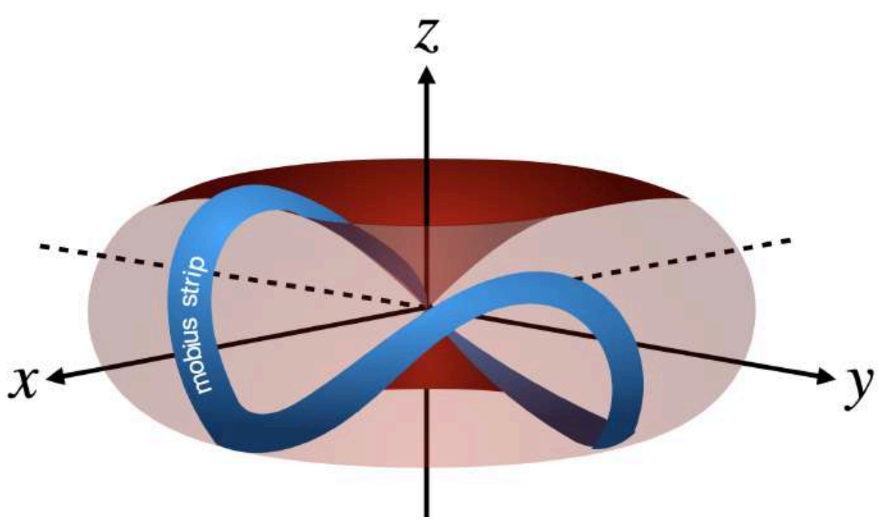
It moves around with no problems along the curvature of the space, but when it decides to cross the central point, a few very weird things happen. First of all, at least for us observing the scene in 3 dimensions, we would see the lizard *shrinking* as it approaches the central point, until it becomes just a point, and then he comes out the *other side* with opposite orientation. This other side is what we, observing the scene in 3 dimensions, would call the “surface’s inside”.



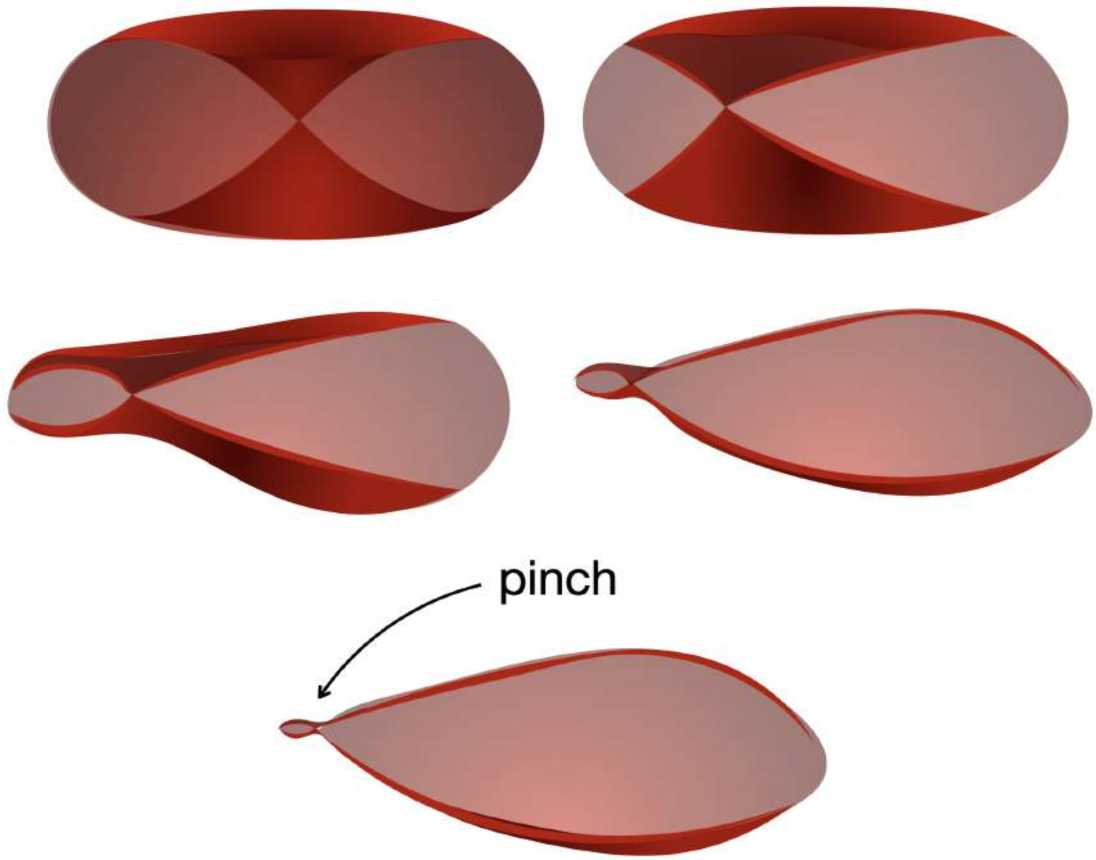
Of course, there is no “inside” or “outside” from the perspective of the lizard, because it is 2-dimensional. For him, the surface is not necessarily embedded in a higher dimensional space. At this point his entire body was flipped on itself and the lizard became “left handed”. He could continue to move around with no problems at all, just as in the beginning, along the curvature of the surface, but this time with its orientation reversed.



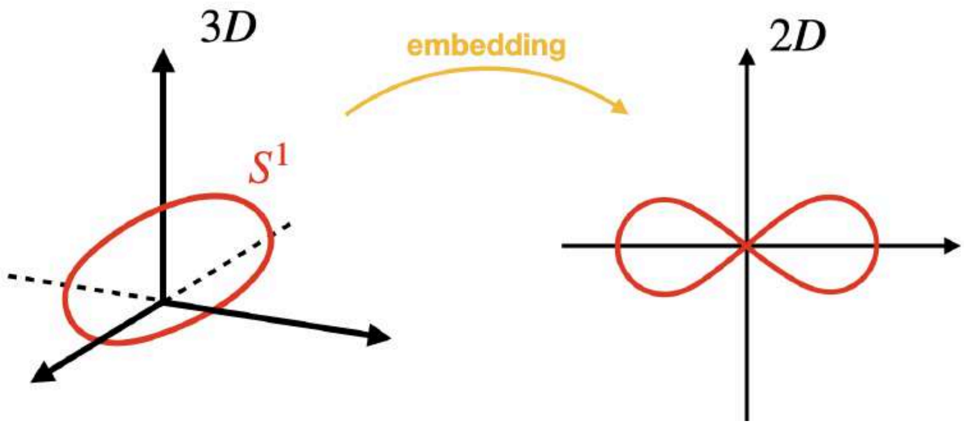
It is really hard to find a Möbius strip in this space, even though it might contain one.



The problem is that, after cutting it and sliding its “walls”, it is very hard to not “pinch” a point. Pinching a point would be illegal if we are trying to perform only homeomorphic (continuous) transformations. It is just a very weird embedding in $3D$.



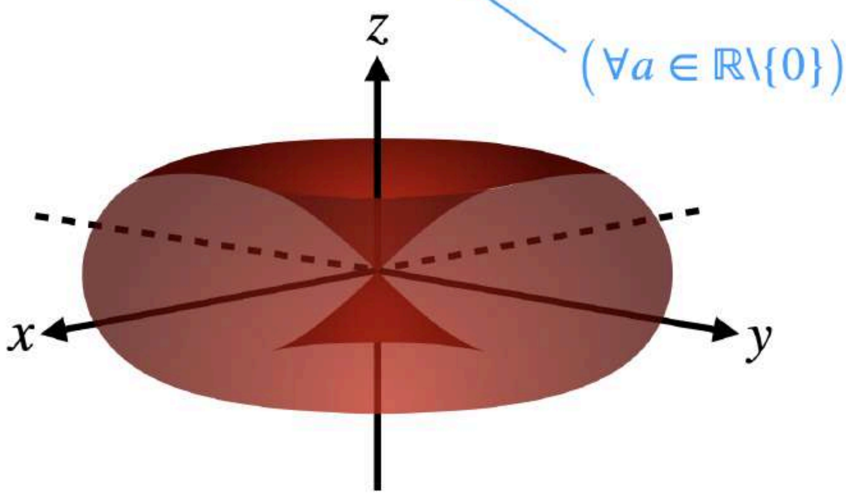
I believe that more insights could be drawn from embedding it in 4 dimensions, rather than in 3 dimensions, for example. The issue here is that we can't draw or imagine 4 dimensions in a tangible way. Just as the figure-eight is a "weird" embedding of a circle S^1 from $3D$ into $2D$, maybe this surface is a sphere S^2 embedded (in a particular way) from $4D$ into $3D$. I really don't know… I'm just guessing here…



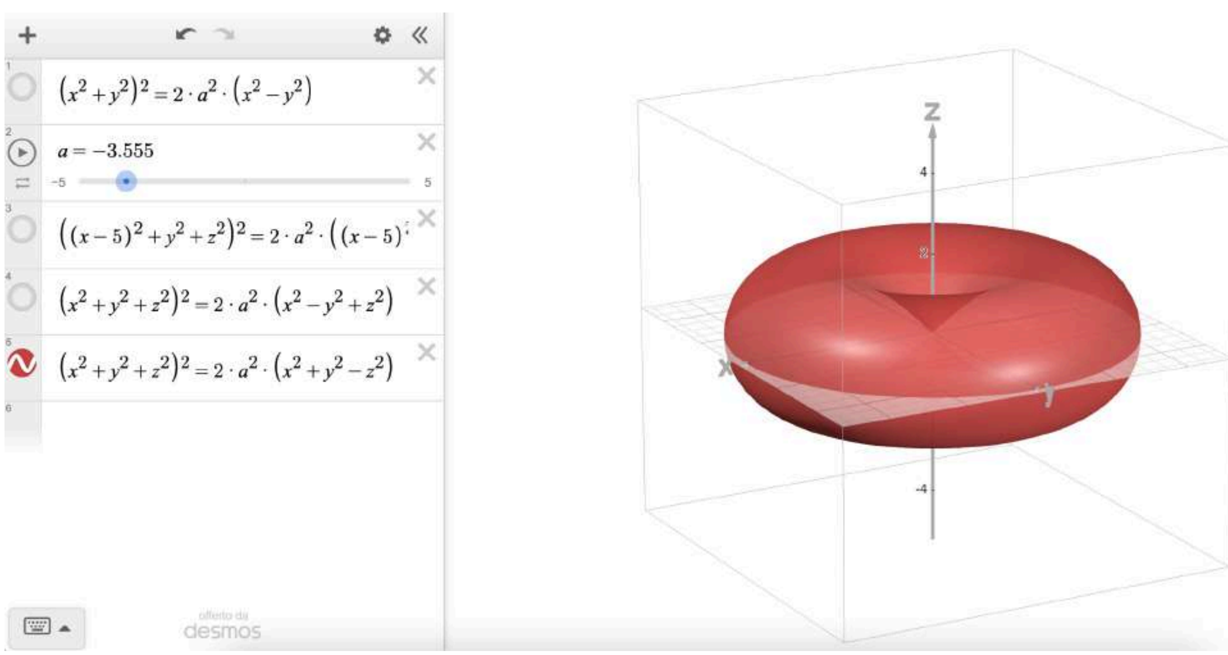
By the way, let's give it a name so that we can easily refer to it: "Podushka Surface". The word "Podushka" means "pillow" in Russian. I came up with this name while I was learning Russian, and living in Ukraine (for my masters degree). It does look like a pillow.

The following is a *quartic formula* that describes the Podushka surface:

$$\left(x^2+y^2+z^2\right)^2=2a^2\left(x^2+y^2-z^2\right)$$



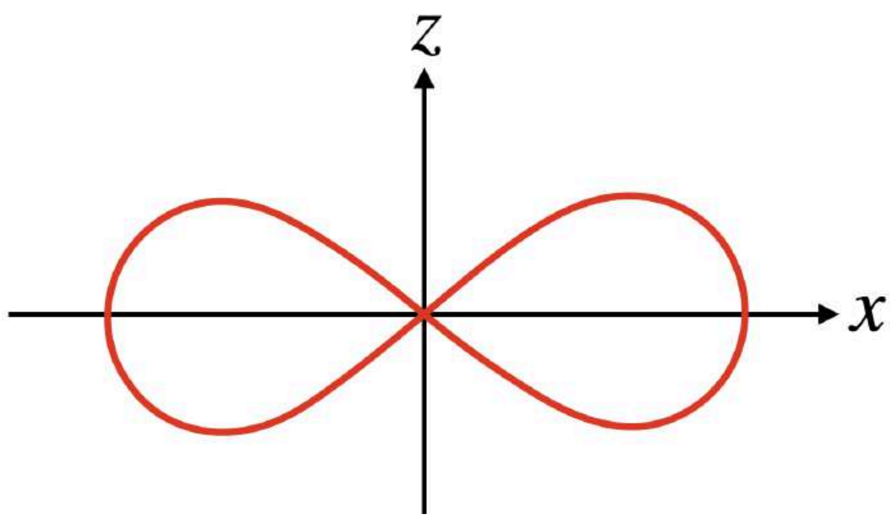
I plotted it in desmos.com/3d (play with the animation in this link), and it was exactly what I expected.



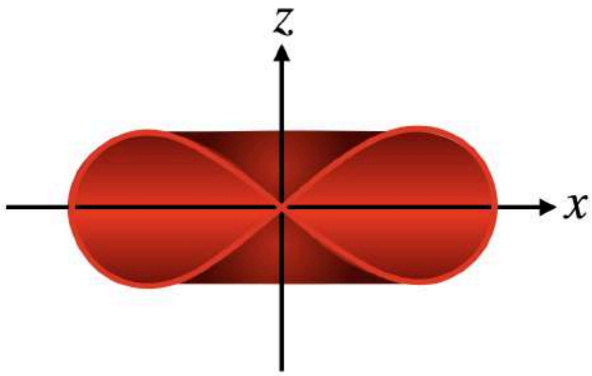
To understand how to come up with this formula, you just have to check (which is not a hard exercise to do) that the equation for the figure-eight shape is the following:

lemniscate of Bernoulli

$$\left(x^2 + z^2\right)^2 = 2a^2\left(x^2 - z^2\right)$$



It is called the *lemniscate of Bernoulli*. The equation for the Podushka surface is just a simple generalization of it in 3 dimensions (since it's a solid of revolution created out of a figure-eight in the xz plane).



$$\left(x^2 + z^2\right)^2 = 2a^2\left(x^2 - z^2\right)$$

$$\left(x^2 + y^2 + z^2\right)^2 = 2a^2\left(x^2 + y^2 - z^2\right)$$

The Podushka surface is, thankfully, described by a quartic equation, not a *quintic or higher*, otherwise it would potentially lack exact algebraic solutions – which is a consequence of the *Abel-Ruffini theorem*. This theorem states that: "*There are no general algebraic solutions (i.e., using radicals) to polynomial equations of degree* 5 *or higher.*"

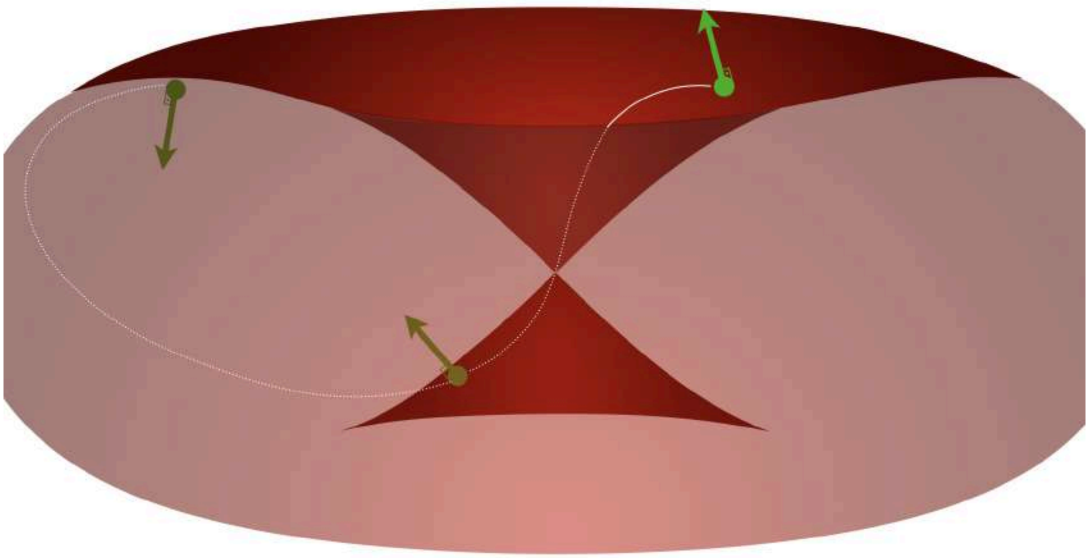
$$\left(x^2 + y^2 + z^2\right)^2 = 2a^2\left(x^2 + y^2 - z^2\right) \implies$$

$$\implies x^4 + x^2y^2 + x^2z^2 + y^4 + x^2y^2 + y^2z^2 + x^2z^2 + y^2z^2 + z^4 = 2a^2x^2 + 2a^2y^2 - 2a^2z^2 \implies$$

$$\implies x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 2a^2x^2 + 2a^2y^2 - 2a^2z^2 = 0$$

quartic

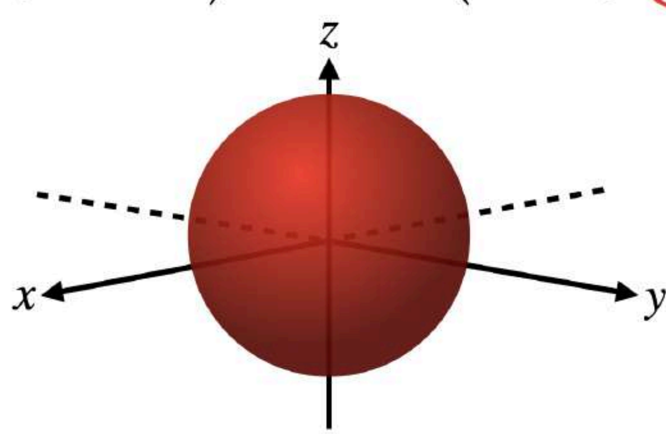
If you look at its plot in the xyz space again, you can notice that the unique self-intersecting point at the origin is the only point responsible for flipping the orientation of any normal vector passing through it.



This shows that it is not possible to define a consistent vector space normal to Podushka at each point.

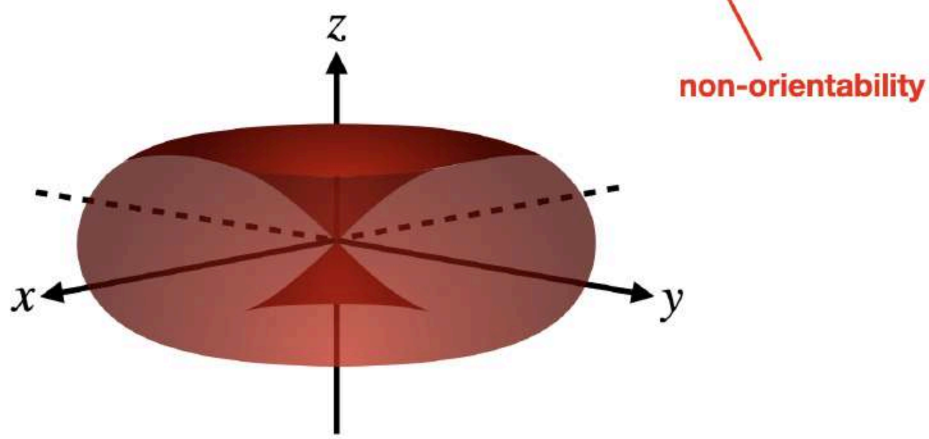
However if we change this quartic formula slightly (changing the sign in front of z^2, in the right hand-side, from $-$ to $+$), we get a sphere:

$$\left(x^2 + y^2 + z^2\right)^2 = 2a^2\left(x^2 + y^2 + z^2\right)$$



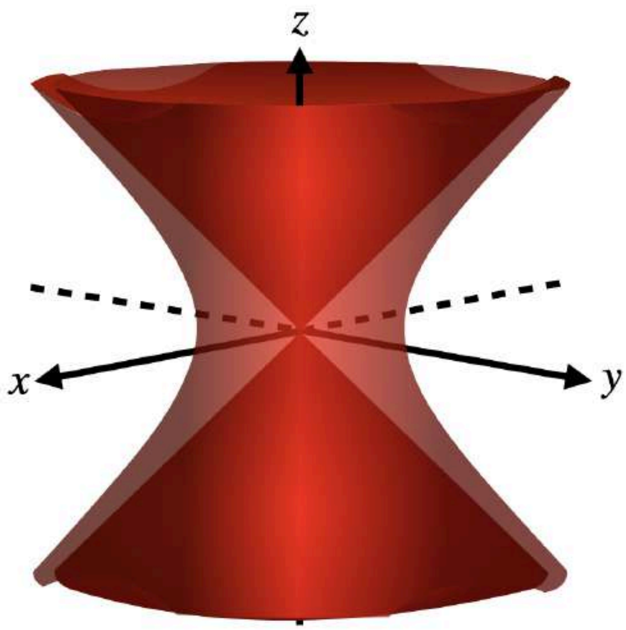
So, this change in sign, from positive to negative, is exactly what creates the *non-orientability* of the surface.

$$\left(x^2 + y^2 + z^2\right)^2 = 2a^2\left(x^2 + y^2 - z^2\right)$$



Another thing that I tried to do, before figuring out the correct quartic equation, was to change signs in front of z^2, on the left hand-side and on the right hand-side.

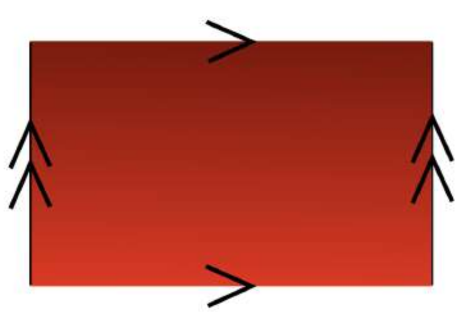
$$\left(x^2 + y^2 - z^2\right)^2 = 2a^2\left(x^2 + y^2 - z^2\right)$$



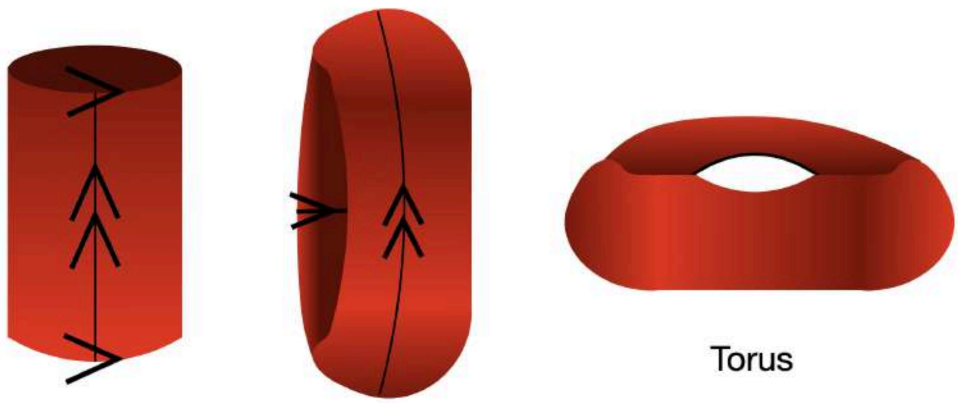
(I'm sharing this just out of curiosity, and to help you guys understand why, and how, this specific formula is the one that effectively describes the Podushka surface).

As you can see, this surface is the union of a cone and a hyperboloid – which are two classic quartic forms. So, definitely <u>not</u> the Podushka surface.

Let's compare the Podushka surface now to other fundamental topological surfaces. Imagine we have a rectangle with specific directions given to each of its edges:

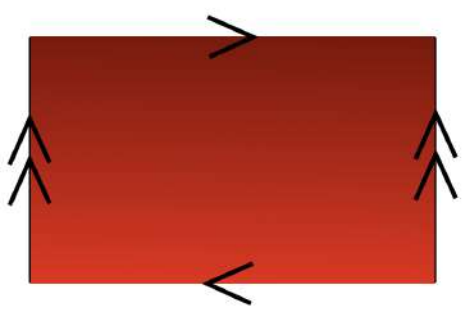


If we embed it in 3 dimensions and glue together its opposite edges following the arrows, we can build a torus.

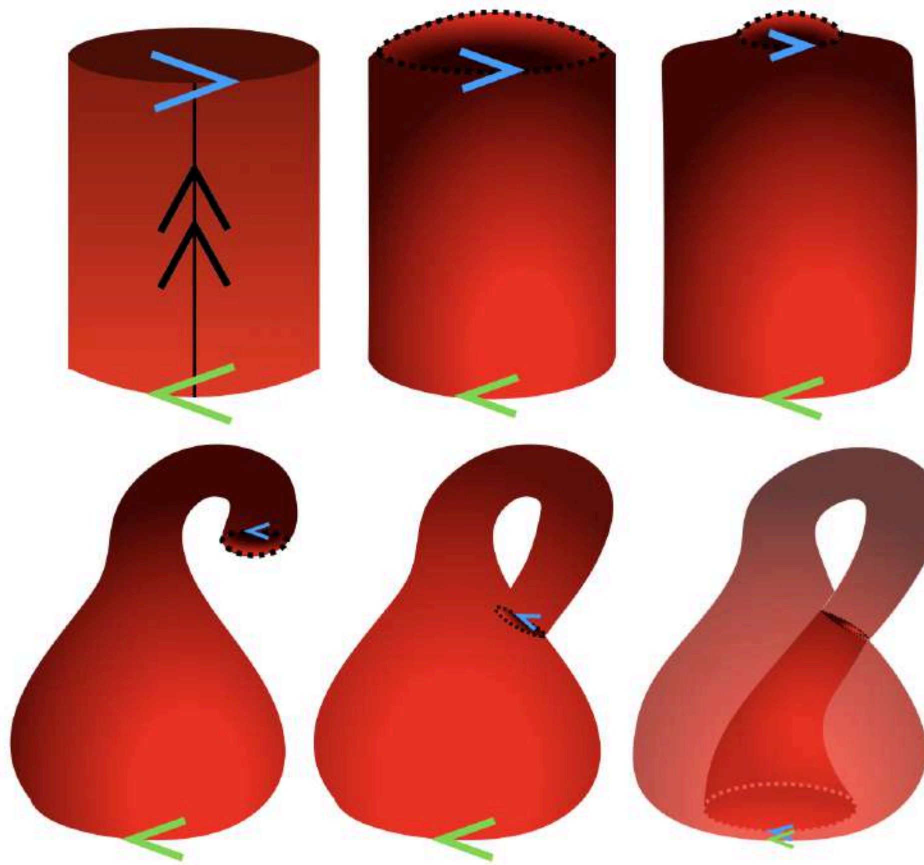


Torus

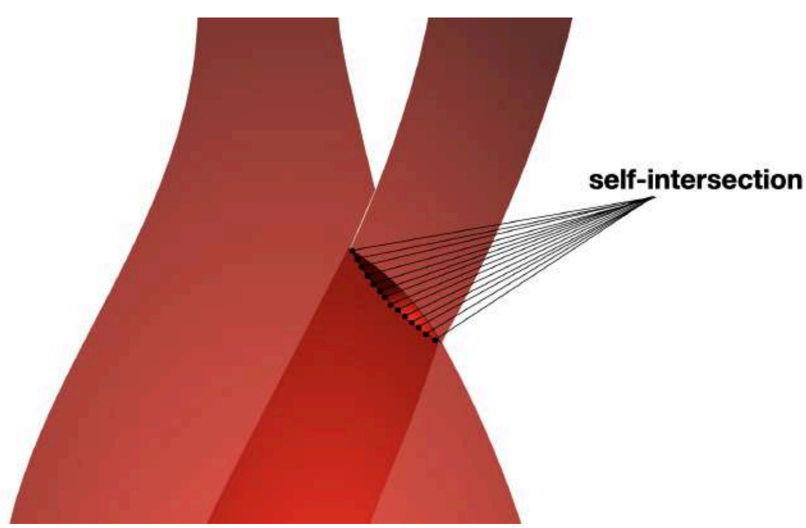
Now, let's start with a rectangle with these orientations for its edges:



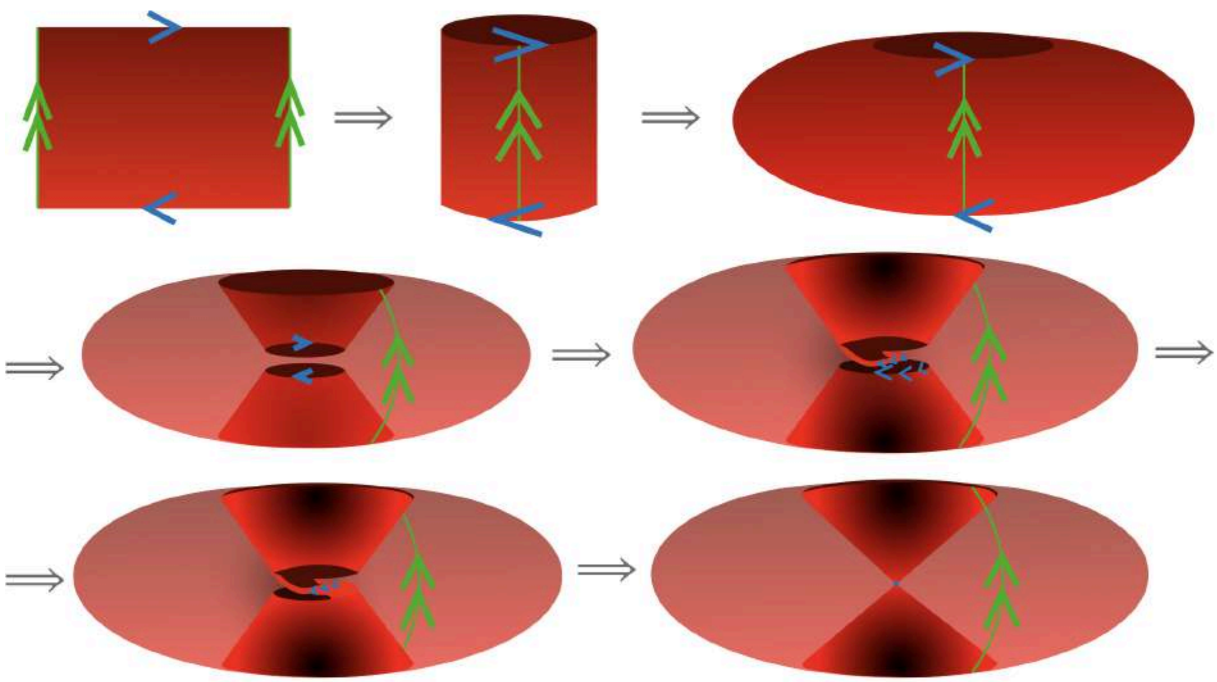
The first two edges (the ones that point in the same direction) are glued together just as before, no surprises here. But then the other two have opposite orientations with respect to each other. The most traditional way of gluing them would be by rotating the circular top and poking it through, forming a self-intersection (which is a circle), and then gluing the two circles with matching directions.



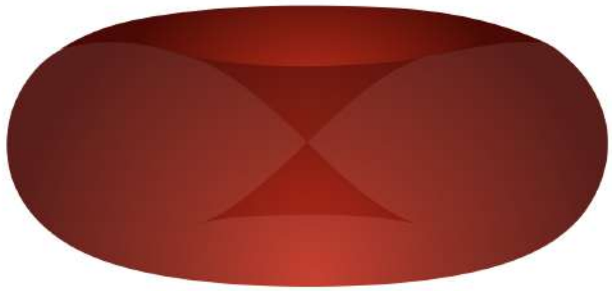
This is just the famous *Klein bottle*. It has infinite points of self-intersection when embedded in $3D$, but no self-intersections at all when embedded in $4D$.



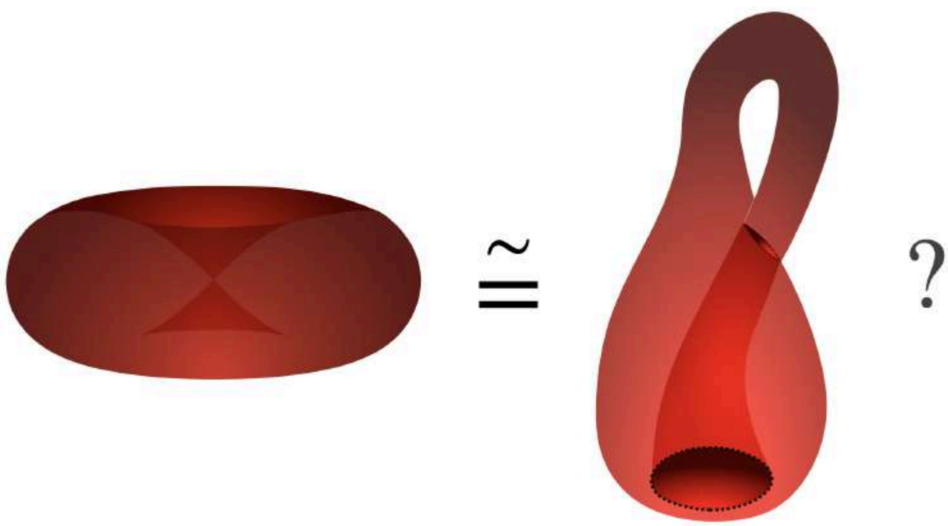
What I propose here is to perform this last step in a slightly different way. We start with the same rectangle of the Klein bottle, glue the first two edges, and then bend the top and the bottom circular edges inside. Finally, glue opposite points in each circle to match the orientations of the edges.



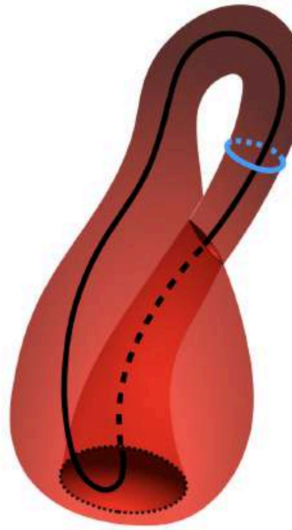
After doing it for all the infinite points on each circle, you will still get a closed surface with self-intersection, just like we got with the Klein bottle. The advantage here, though, is that its embedding in $3D$ has only one (unique) self-intersection, in contrast to the infinite self-intersecting points along the circular region in the Klein bottle.



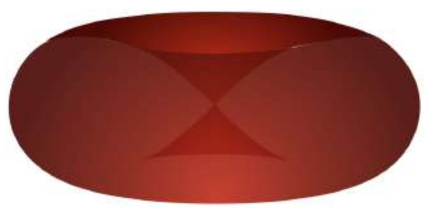
This raises a question: "Is the Podushka space just a variation of the Klein bottle, with a different embedding in $3D$, such that it minimizes the number of self-intersecting points? Or is it an entirely new topological surface on its own?".



In order to understand whether the Podushka space is a variation of the Klein bottle, we need to study its main topological invariants. For example, the Klein bottle has genus 2 (i.e., 2 "non-orientable holes"), but, a priori, the Podushka surface seems to have less than 2 "holes".

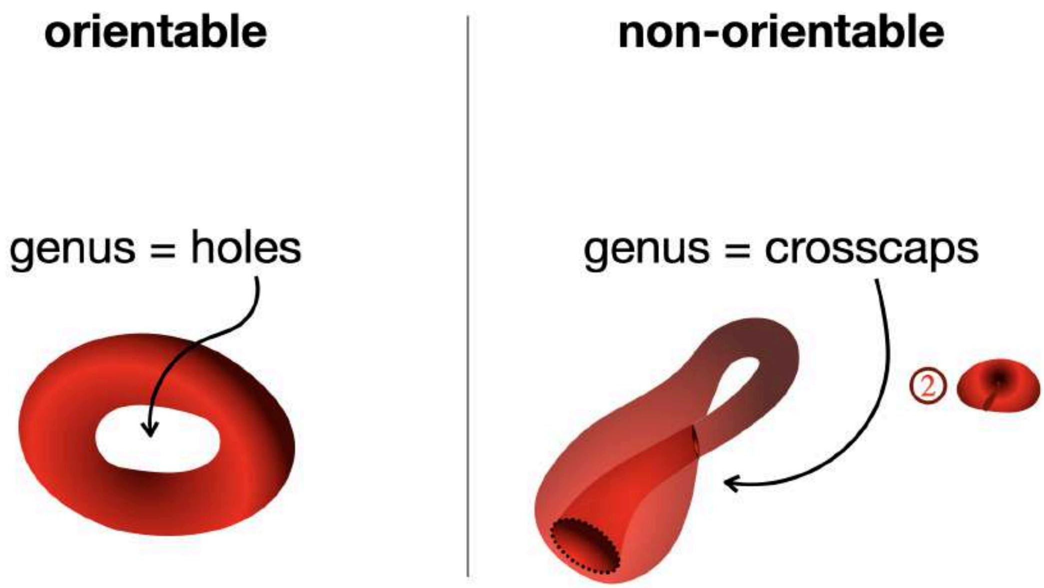


genus 2
(2 non-orientable holes)



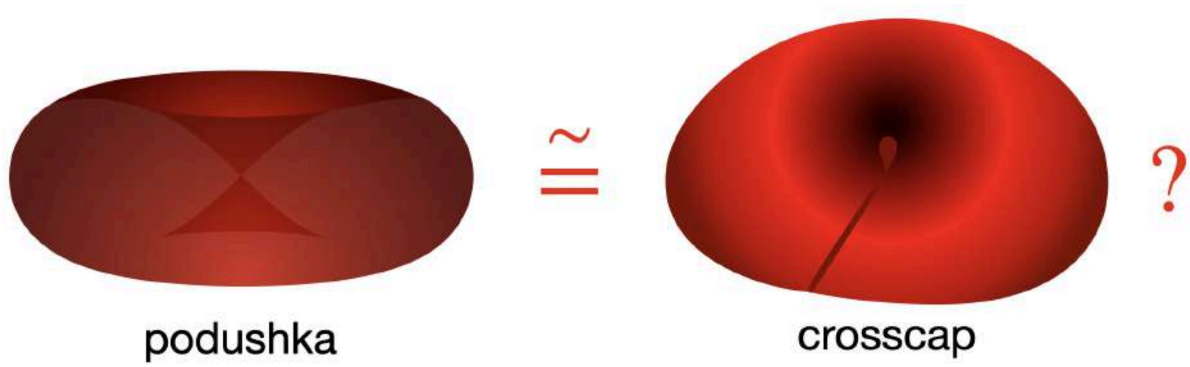
< 2 "holes"?

To be more precise: In orientable surfaces the genus is the number of "holes"; meanwhile in non-orientable surfaces the genus is the sum of the number of crosscaps in it – we will see shortly what exactly a crosscap is.



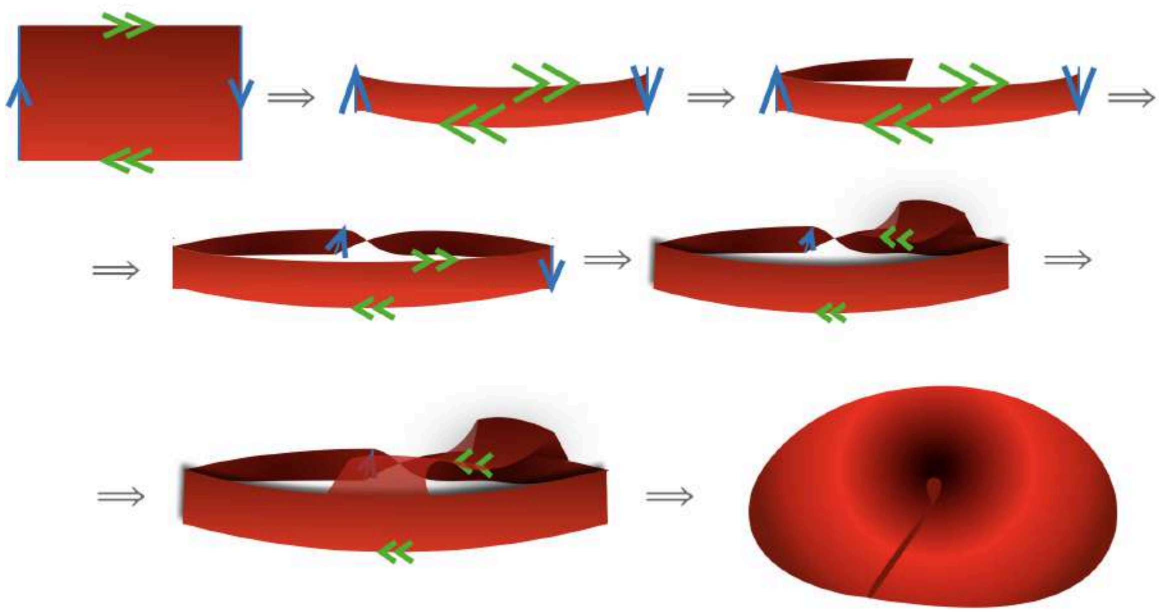
The genus is a topological invariant that is preserved under any kind of embedding. So, if the Podushka surface is merely a variation of the Klein bottle with a different embedding, these two surfaces must have the same genus. But how can we calculate the genus of the Podushka surface? Interesting…

Right after that another question came to mind: “Maybe the Podushka surface is a variation of the crosscap?”. I mean, they look really similar when embedded in $3D$.

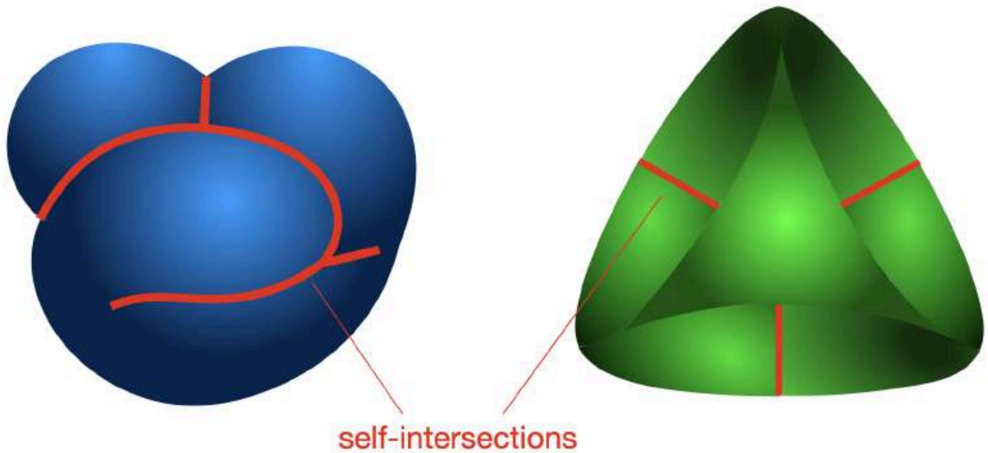


But, what is the *crosscap* afterall and how does it relate to the Podushka surface?

Imagine a rectangle with opposite sides having opposite orientations. Glue the first pair of edges together, and you get a Möbius strip. Now comes the hard part. You need to glue the boundaries adding a twist, i.e. reversing once again their directions. It is really challenging to imagine this process because a line of self-intersecting points is inevitably formed. This is the *real projective plane* $\left(\mathbb{R}P^2\right)$. And its representation here, so the way we decided to embed it in $3D$, is called the *crosscap*.

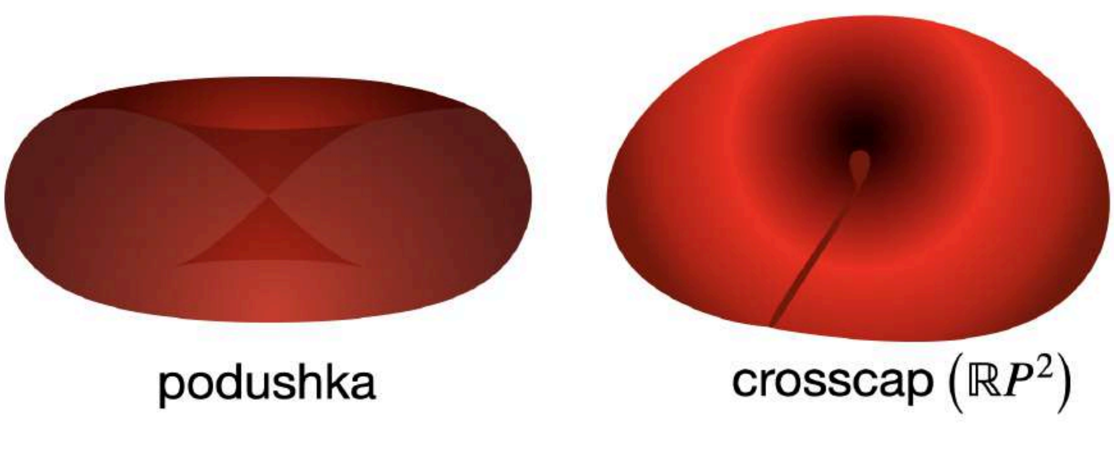


These are other possible embeddings of the real projective plane (they are all topologically equivalent):

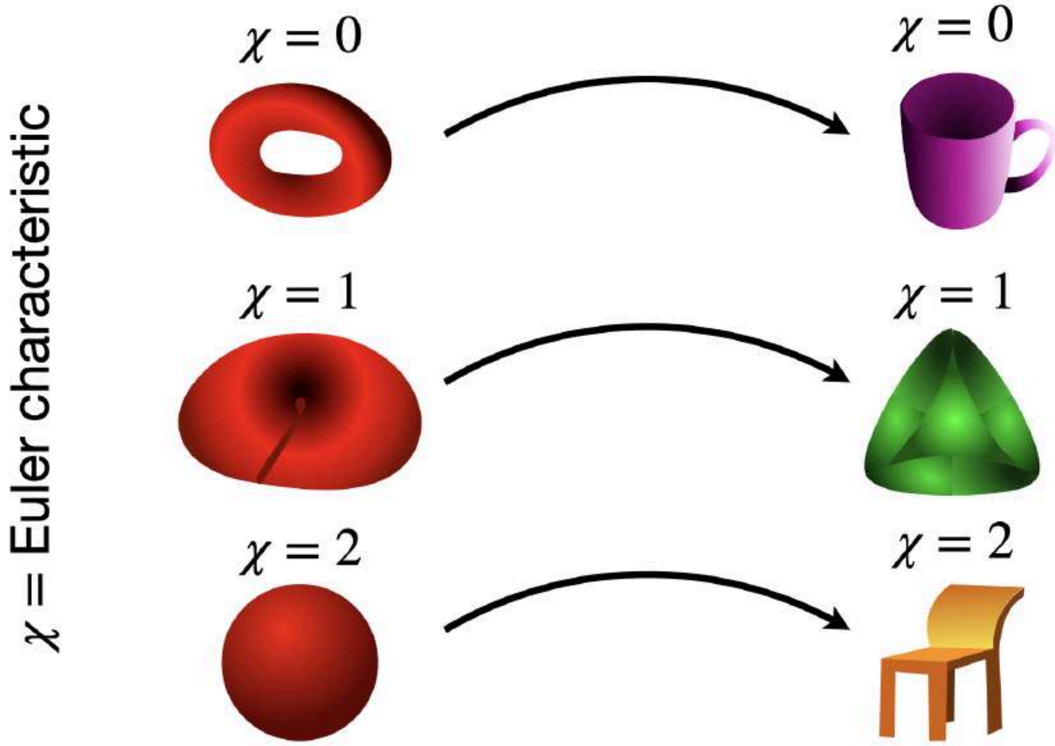


They are, respectively, the Boy's surface and the Roman surface. Notice, though, how it is necessary for all of them to have self-intersections.

As noticed earlier, the crosscap embedding of the real projective plane looks really similar to the Podushka surface. But I don't think they are topologically the same. In order to decide whether they are topologically equivalent or not, we need to add rigor to our discussion. And this can be best achieved, in this context, by talking about some specific topological invariants.



The Euler characteristic is one of the most fundamental topological invariants, and it is a number that is preserved under any kind of embedding or homeomorphic transformation. Basically, if two spaces are topologically equivalent to each other, they must have (among other properties) the same Euler characteristic, denoted with the greek letter χ .

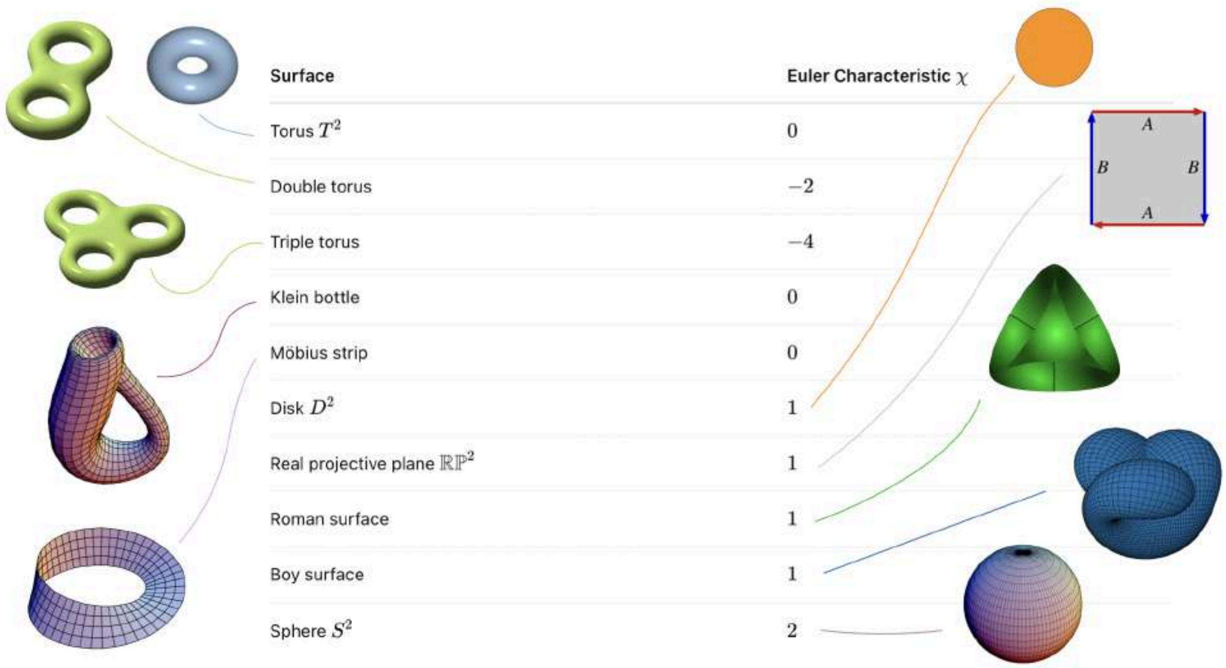


vertices
edges
faces

$$\chi = V - E + F$$

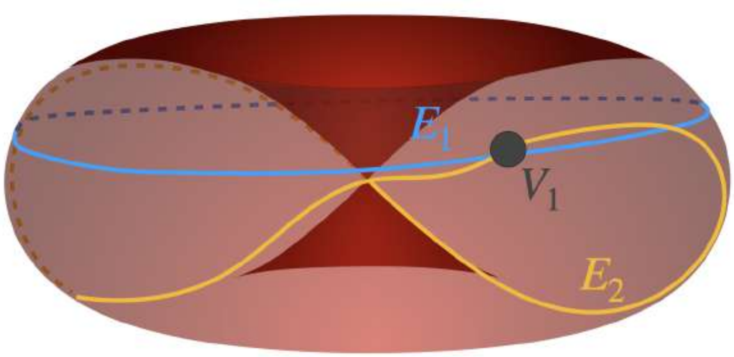
Euler characteristic

The Euler characteristics of many fundamental surfaces are listed below:

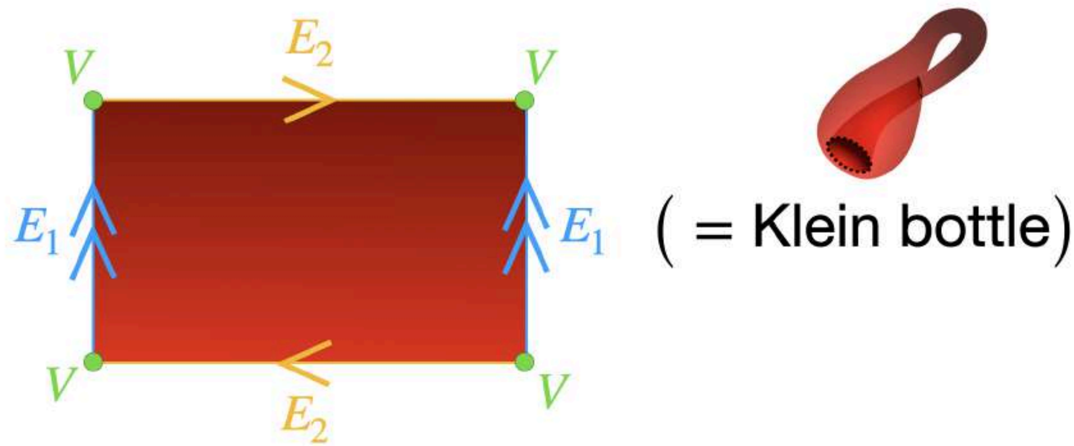


Surface	Euler Characteristic χ
Torus T^2	0
Double torus	-2
Triple torus	-4
Klein bottle	0
Möbius strip	0
Disk D^2	1
Real projective plane $\mathbb{RP}^2$	1
Roman surface	1
Boy surface	1
Sphere S^2	2

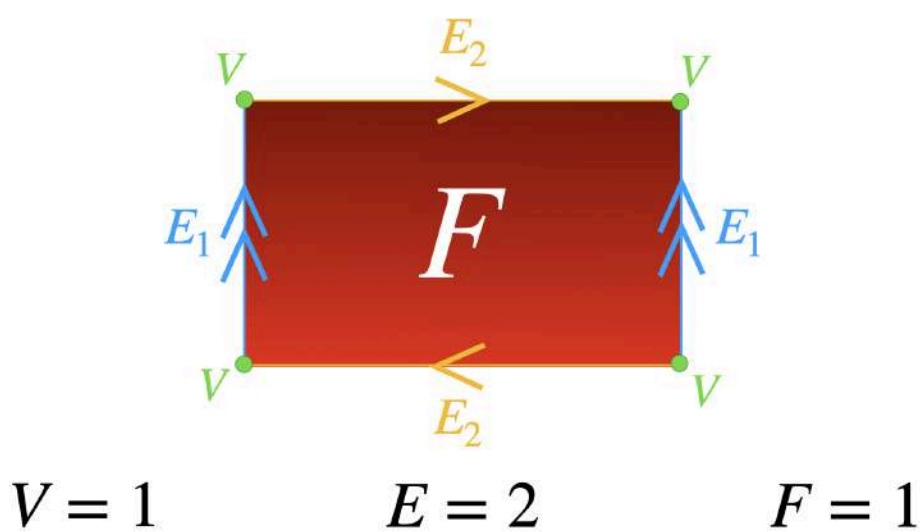
Here we will show just how I calculated the Euler characteristic of the Podushka surface.



Pick a random point in the Podushka surface to be a vertex. Draw two edges that are loops that cannot be continuously deformed into one another, and that cannot be shrunk to a point. These are fundamental loops. If you cut through these edges, you get the polygon representation of the Poshuka surface (which is the same as the one for the Klein bottle, by the way).



Now it is easy to count the number of vertices, edges and faces. There is one vertex, two edges and one face.



$$\chi = V - E + F$$
$$= 1 - 2 + 1 = 0$$

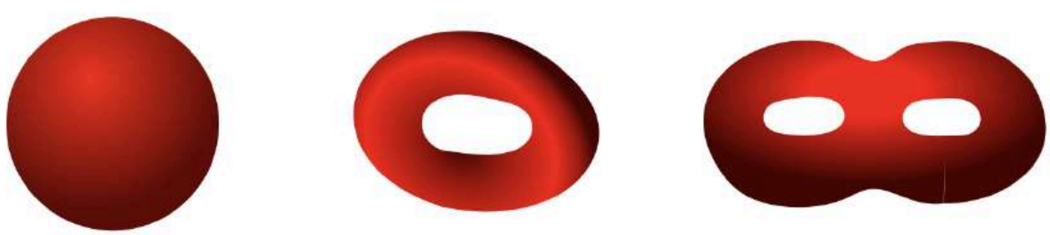
As a consequence its Euler characteristic is *zero*. Which is the same as the Euler characteristic of the Klein bottle! And it is different from the Euler characteristic of the crosscap (or real projective plane), which is one, instead.

Am I missing something?? Let us know via email: dibeos.contact@gmail.com

Now let’s talk about another very important topological invariant. The genus.

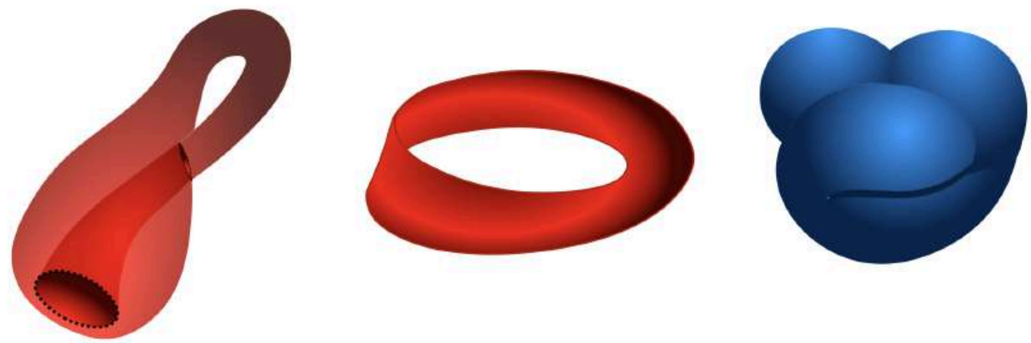
The genus of an orientable surface is calculated as:

$$\text{genus} = g = \frac{2-\chi}{2} \; \text{(number of holes)}$$



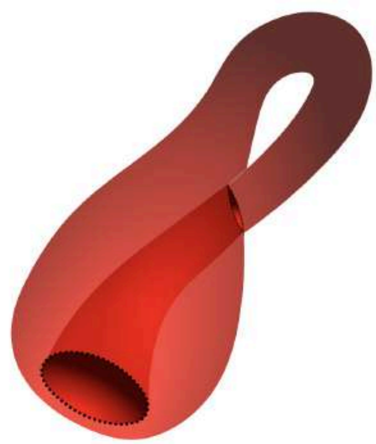
The genus of a non-orientable surface is calculated as:

$$\text{genus} = k = 2 - \chi \left(\text{number of non-orientable holes\textbackslash crosscaps}\right)$$

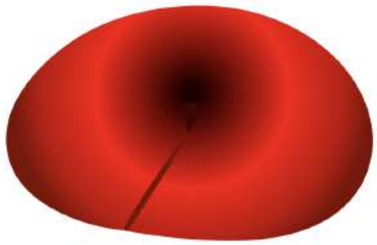


With this in mind, the Klein bottle (which is non-orientable) has genus:

$$k = 2 - 0 = 2$$



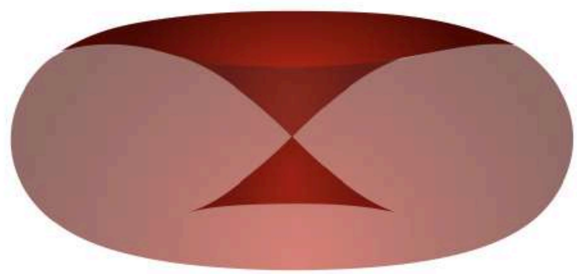
The crosscap, which is topologically equivalent to the real projective plane, and is non-orientable, has genus:



$$\cong \mathbb{R}P^2 \qquad k = 2 - 1 = 1$$

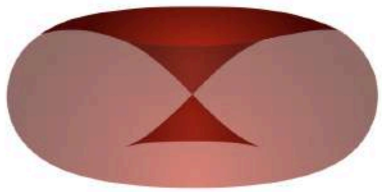
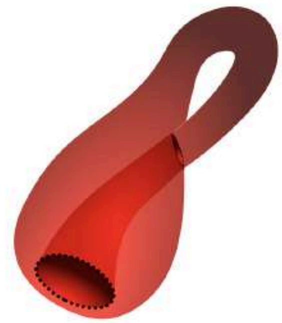
The Podushka surface, which is non-orientable as well (and considering that I calculated its Euler characteristic correctly), has genus:

$$k = 2 - 0 = 2$$



Podushka $(\mathscr{P})$

Therefore, I would guess that the Podushka surface is a variation of the Klein bottle (so, a connected sum of 2 crosscaps), that minimizes the number of self-intersections. It possesses just one self-intersecting point, which is pretty convenient. The Podushka surface can still be considered a *distinct geometric model*, but topologically, I'd guess it to be the same as the Klein bottle. It presents, though, its own kind of embedding.

$\therefore$  $\cong$ 

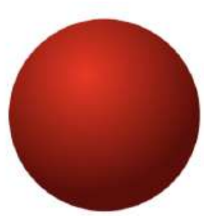
$\mathscr{P}$ $\mathscr{K}$

$$\mathscr{P} \cong \mathbb{R}P^2 \# \mathbb{R}P^2$$

Let me know your thoughts!!! (*dibeos.contact@gmail.com*)

To be very honest, the Podushka surface cannot be a *new* topological surface because of something called “*the classification theorem of closed surfaces*”. Basically, it states that any *connected closed surface* belongs to one of these 3 families:

(1) The sphere (all connected, orientable, closed surfaces with genus 0)



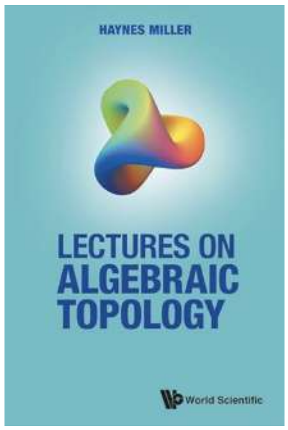
(2) The connected sum of g tori, with $g \geq 1$

(3) The connected sum of k real projective planes, with $k \geq 1$

The Podushka surface definitely belongs to the third family, since this is the only family containing non-orientable surfaces (such as the Klein bottle, the real projective plane, etc).

If you would like to know more about the classification of topological surfaces check out this video:

And if you really enjoy studying *Algebraic Topology* and *Geometric Topology*, you might want to check out these excellent books



Do not forget that we have a website where in one of its sections you can submit your own personal research. Just send it to us via email, more details in the description.

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