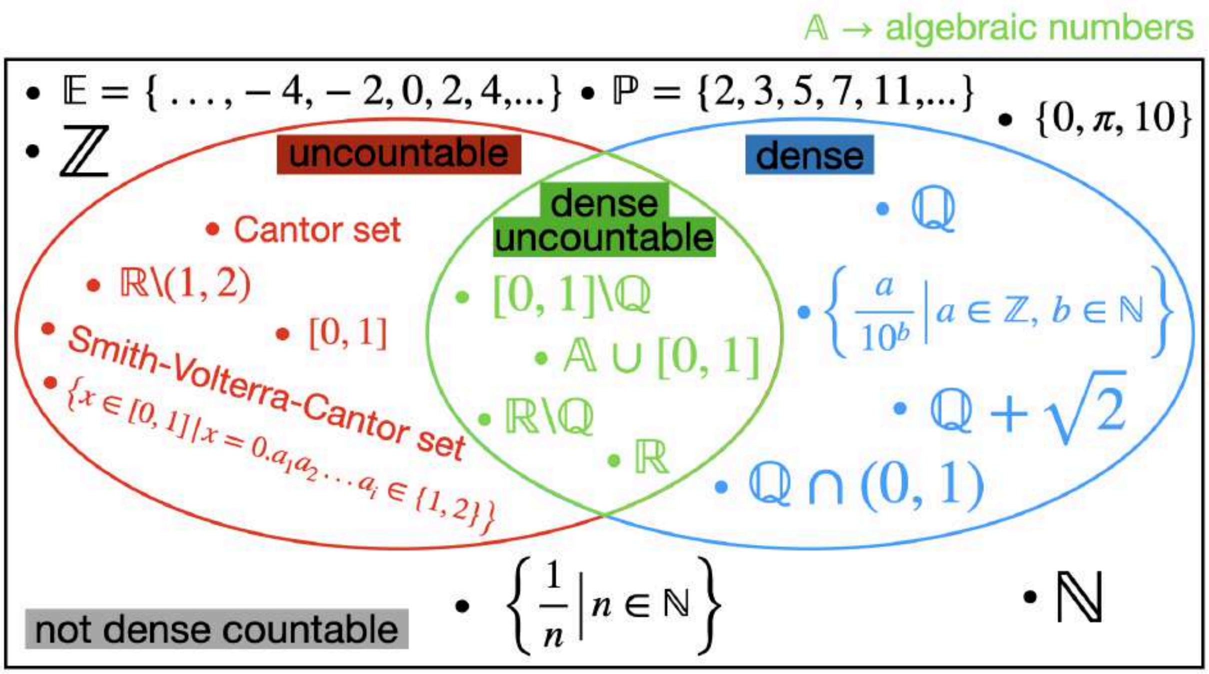
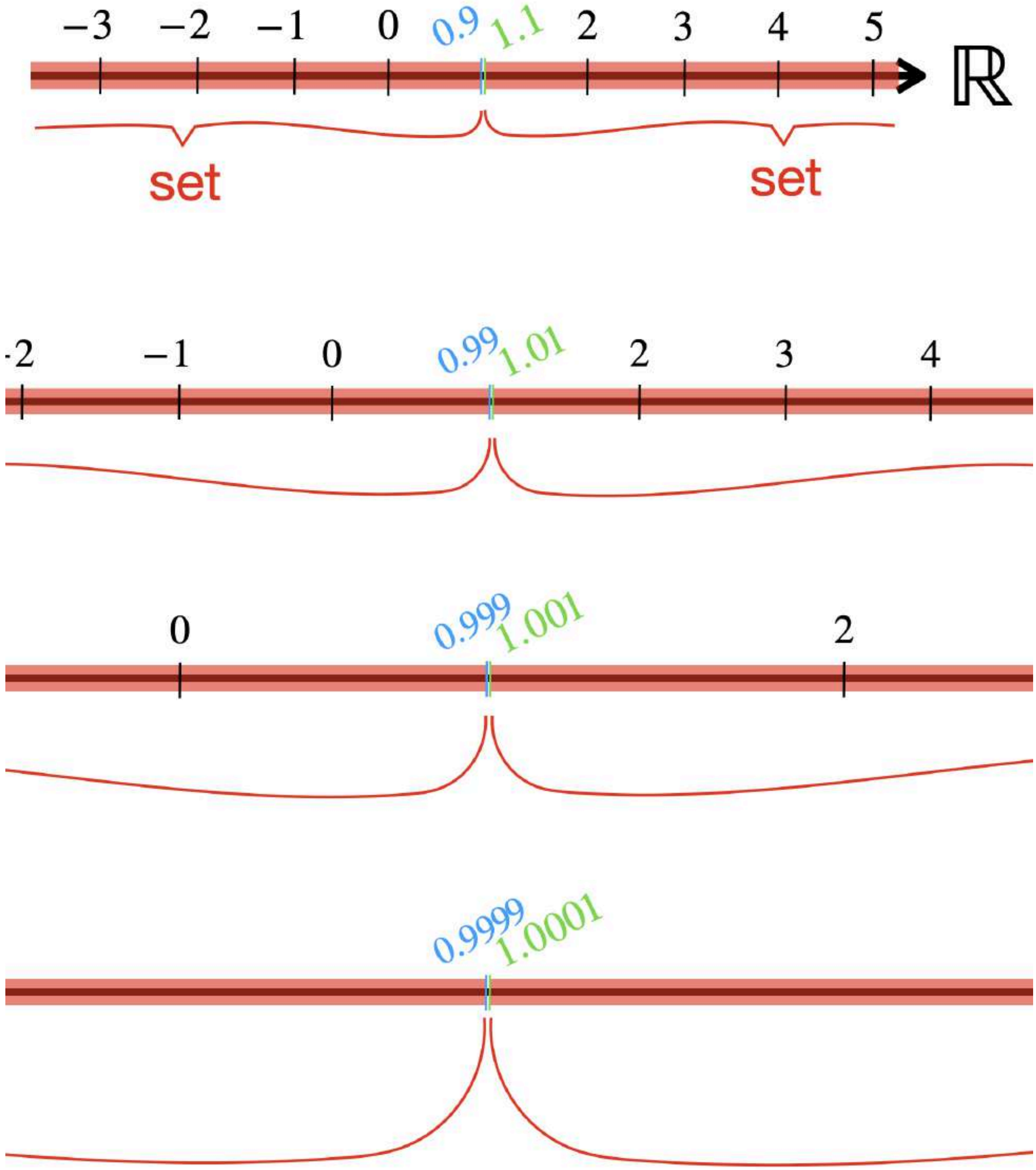
[Dense vs. Uncountable Sets | Are They Really Different?](#)

by Dibeos

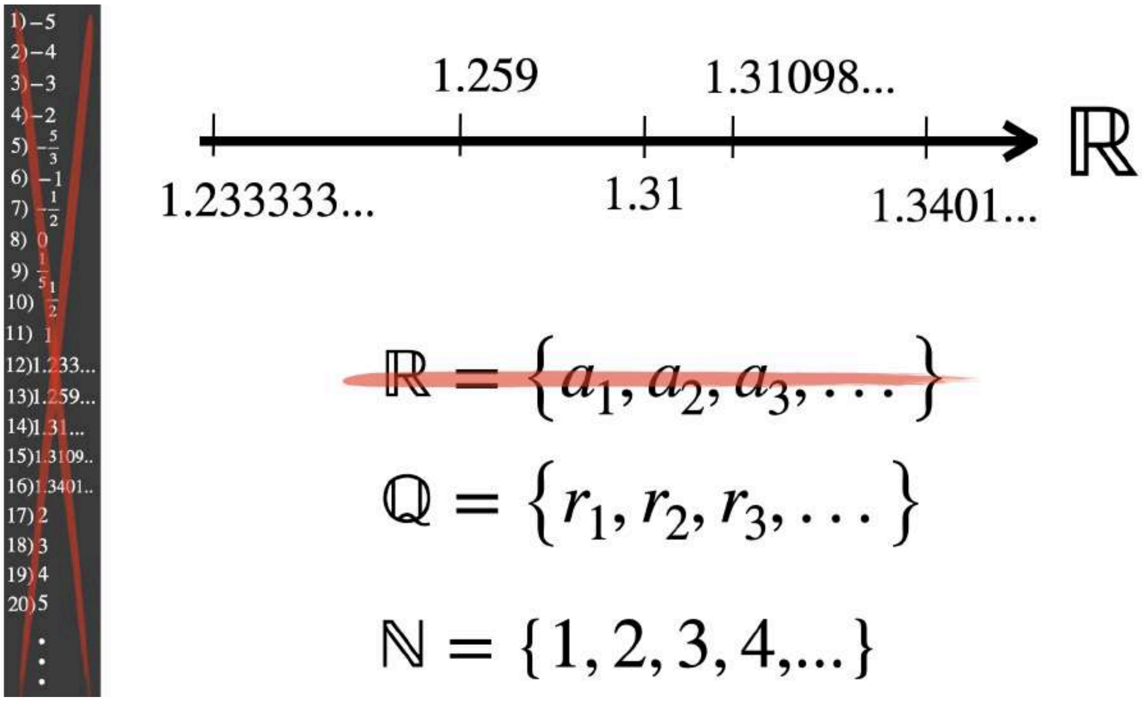


What is the difference between being *dense* and *uncountable*? Above, you can see a Venn diagram that shows how these two concepts are not the same, but can overlap sometimes. A set, in general, can be dense and uncountable, dense and countable, not dense and uncountable or not dense and countable.

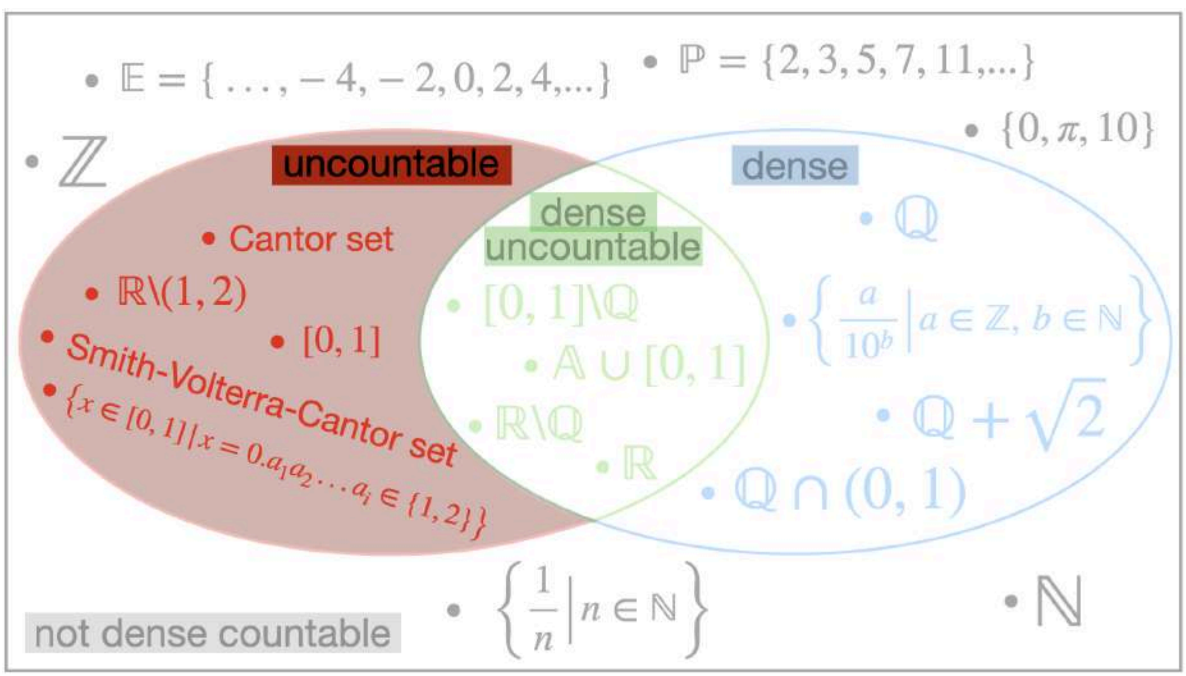
Loosely speaking (just to give you guys some intuition behind these definitions), a set is *dense*, in the real numbers (for example), if its elements are spread out everywhere – you can't find any "gaps", or "holes", in the number line that does not contain at least one of its elements, no matter how small. Now, before some of you start cursing me in your mind, remember: this is just a loose definition, just to give us some grasp of the intuition behind it. We will get to the rigorous definition later on in this video. But, for now, think of dense sets this way: if you zoom in on any two real numbers, no matter how close, you will always find an element of the set in between.



As for uncountable sets, these are sets that are too “big” to list (or enumerate) all of its members, even in an infinite list. So, even if you had, like “forever”, and tried to list them one by one, you'd still miss, not only some, but *infinitely many* of them. There is no way to "line them up" like you can with the natural numbers or rational numbers.

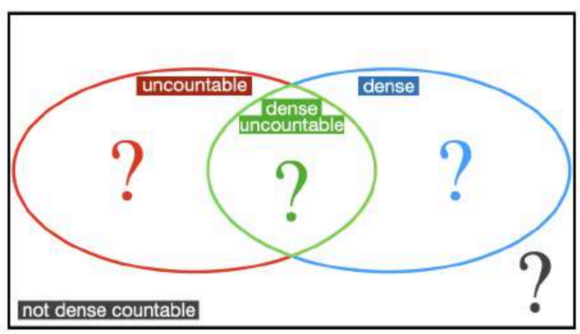


The greatest confusion usually lies in this region (below) of the Venn diagram: **uncountable but not dense**.



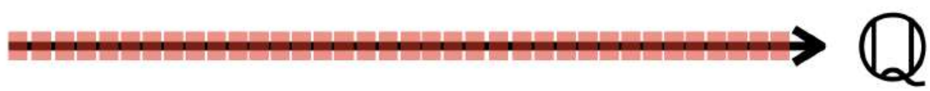
Intuition fails us here because we tend to think that if a set is *“too big to list”*, then its elements must be so tightly packed that there's no room between them — no "gaps." But that mental picture overlaps with a different concept: **density**. In reality, these are two separate ideas. A set can be uncountable without being dense — and vice versa. In fact, a set can live in **any** of the four regions in the diagram.

So, I have some tough questions for you: first, imagine the set $[0, 1] \in \mathbb{R}$. Where would you place it in this diagram? Is it dense and uncountable? Or just dense, or just uncountable?

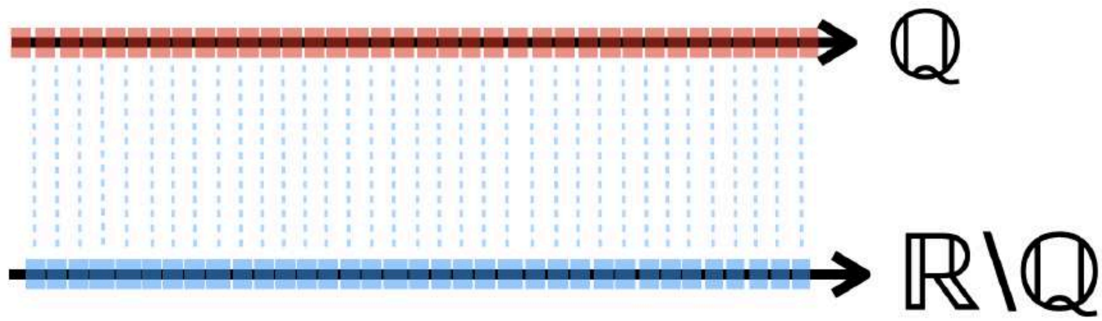


What about the rational numbers ($\mathbb{Q}$)? Is it dense? Is it uncountable?

$$\mathbb{Q} = \left\{ \frac{n}{m} \in \mathbb{R} \,\middle|\, n, m \in \mathbb{Z}, m \neq 0 \right\}$$



What about the irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$) ?



Time to see some formal definitions.

Dense: Let $A \subseteq \mathbb{R}$. We say that A *is dense* in $\mathbb{R}$ if:

$$\forall\, x, y \in \mathbb{R}, \quad x < y \implies \exists\, a \in A : x < a < y$$

Equivalently, we can say the *closure of* A is $\mathbb{R}$:

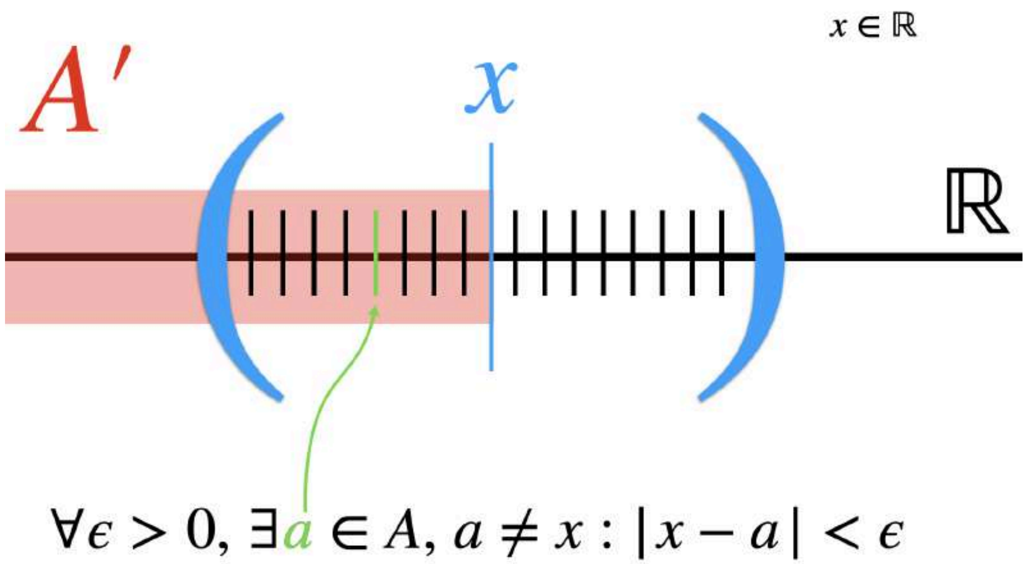
$$\boxed{\overline{A} = \mathbb{R}}$$

The *closure of* A, denoted as $\overline{A}$, is the *smallest closed set* that contains A. Formally, it is defined as:

$$\overline{A} = A \cup A'$$

Here A' is the set of *limit points* (also called *accumulation points*) of A.

A real number x $(\in \mathbb{R})$ is called a *limit point of* A if "every neighborhood around x contains some point of A different from x" – intuitively these are points that are infinitesimally close to A:



So, let's create an algorithm to find out if a set $A \subseteq \mathbb{R}$ is dense or not:

(1) Find the set of limit points of A (denoted as A'), i.e. the set containing points such that every neighborhood around it contains some point of A, other than x itself;

(2) Find the closure of A, which is the union $\overline{A} = A \cup A'$;

(3) If $\overline{A} = \mathbb{R}$, we conclude that A is dense in $\mathbb{R}$, otherwise it is not dense in $\mathbb{R}$.

Remark:

A set is always dense in another set that contains it – or at least in this context. In here, we focus on the simplest case: a set being dense in the real numbers $(\mathbb{R})$ – these are ordered sets. But the definition can be applied more generally – a set A is dense in a set B $(A \subseteq B)$ if the closure of A is equal to B: $\overline{A} = B$.

Let's apply the algorithm to a concrete example:

1. $A = \mathbb{R} \backslash \mathbb{Q}$ (irrational numbers) – is it dense?

(1) Find the set of limit points A'.

By definition, this is

$$\mathbb{R}\backslash\mathbb{Q}' = \left\{x \in \mathbb{R} \,\middle|\, \forall\epsilon > 0, \exists a \in \mathbb{R}\backslash\mathbb{Q},\ a \neq x : |x - a| < \epsilon\right\}$$

I want to prove that $\mathbb{R}\backslash\mathbb{Q}' = \mathbb{R}$. I.e., I want to show that

$$\forall\, x \in \mathbb{R} \quad \wedge \quad \forall\, \epsilon > 0\,,\ \exists\, a \in A\,,\ a \neq x: \ |x - a| < \epsilon$$

Let $x \in \mathbb{R}$ be arbitrary (rational or irrational), and let $\epsilon > 0$. We need to find an irrational $a \in A$, with $a \neq x$, such that $|x - a| < \epsilon$.

The irrational number a can be constructed, out of any $x \in \mathbb{R}$ (irrational for the first formula and rational for the second), with the following formulas: $(n \in \mathbb{N})$

$$a := x + \frac{\sqrt{2}}{n} \quad \vee \quad a := x + \frac{1}{n}$$

These formulas effectively construct an irrational number a because of the following two cases:

a) <u>Case 1</u>:

$$x \in \mathbb{Q} \quad \Rightarrow \quad a = x + \frac{\sqrt{2}}{n} \quad \in \ \mathbb{R}\backslash\mathbb{Q}$$

An irrational number $\left(\sqrt{2}\right)$ divided by a natural number (n) always results in an irrational number. And an irrational number $\left(\frac{\sqrt{2}}{n}\right)$ added to a rational number (x) always results in an irrational number $\left(a = x + \frac{\sqrt{2}}{n}\right)$.

b) <u>Case 2</u>:

$$x \in \mathbb{R}\backslash\mathbb{Q} \quad \Rightarrow \quad a = x + \frac{1}{n} \quad \in \ \mathbb{R}\backslash\mathbb{Q}$$

Just as before, an irrational number (x) added to a rational number $\left(\frac{1}{n}\right)$ always results in an irrational number $\left(a = x + \frac{1}{n}\right)$.

In case 1 $(x \in \mathbb{Q})$, we can pick $n \in \mathbb{N}$ large enough such that:

$$\frac{\sqrt{2}}{n} < \epsilon \quad \Rightarrow \quad n > \frac{\sqrt{2}}{\epsilon}$$

Then,

$$|x - a| = \left|x - \left(x + \frac{\sqrt{2}}{n}\right)\right| = \left|x - x - \frac{\sqrt{2}}{n}\right| = \left|-\frac{\sqrt{2}}{n}\right| = \frac{\sqrt{2}}{n} < \epsilon$$

In case 2 $(x \in \mathbb{R}\backslash\mathbb{Q})$, we can pick $n \in \mathbb{N}$ large enough such that:

$$\frac{1}{n} < \epsilon \quad \Rightarrow \quad n > \frac{1}{\epsilon}$$

Then,

$$|x - a| = \left|x - \left(x + \frac{1}{n}\right)\right| = \left|x - x - \frac{1}{n}\right| = \left|-\frac{1}{n}\right| = \frac{1}{n} < \epsilon$$

$\therefore \ \forall\, x \in \mathbb{R} \ \wedge \ \forall\, \epsilon > 0$, we can find an irrational $a \in A$, with $a \neq x$, such that $|x - a| < \epsilon$.

<u>Conclusion</u>: $A' = \mathbb{R}\backslash\mathbb{Q}' = \mathbb{R}$.

(2) Find the closure of A:

$$\overline{A} = A \cup A' = (\mathbb{R}\backslash\mathbb{Q}) \cup \mathbb{R} = \mathbb{R}$$

(3) $\overline{A} = \mathbb{R} \quad \Rightarrow \quad A = \mathbb{R} \backslash \mathbb{Q}$ is **dense** in $\mathbb{R}$.

Next example:

2. $A = [0, 1]$ – dense?

(1) Find the set of limit points A'.

I want to prove that $A' = [0, 1]$ $(= A)$. I.e., I want to prove that

$$\forall\, x \in [0, 1]\,,\ \forall\, \epsilon > 0\,,\ \exists\, a \in A = [0, 1]:\ |x - a| < \epsilon$$

Let $x \in [0, 1]$ be arbitrary, and let $\epsilon > 0$. We need to find a number $a \in A = [0, 1]$, with $a \neq x$, such that $|x - a| < \epsilon$.

The number a can be built, out of any $x \in [0, 1]$, with the following formula:

$$a = x + \frac{\epsilon}{2} \qquad \Rightarrow$$

$$\Rightarrow \quad |x - a| = \left|x - \left(x + \frac{\epsilon}{2}\right)\right| = \left|x - x - \frac{\epsilon}{2}\right| = \left|-\frac{\epsilon}{2}\right| = \frac{\epsilon}{2} < \epsilon \ , \ \forall\, x \in [0, 1]$$

(since $\epsilon > 0$)

$\therefore\ A' = [0, 1]$.

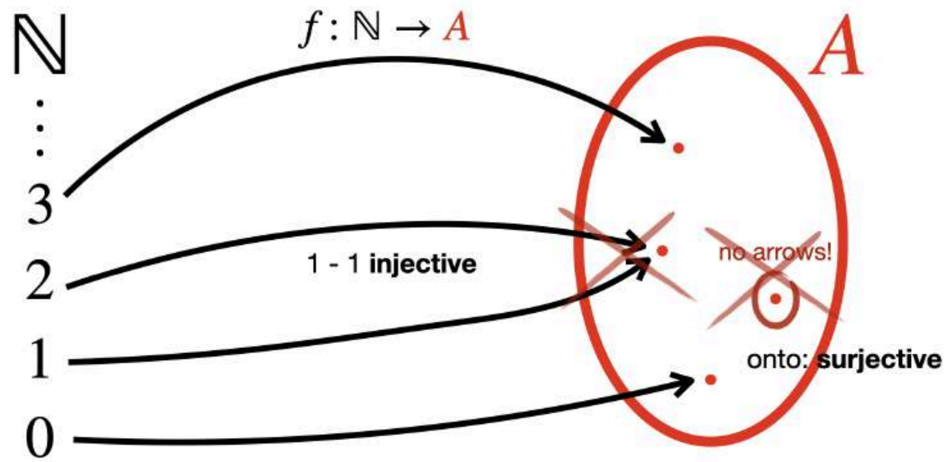
(2) Find the closure of A:

$$\overline{A} = A \cup A' = [0, 1] \cup [0, 1] = [0, 1]$$

(3) $\overline{A} = [0, 1] \neq \mathbb{R} \quad \Rightarrow \quad A = [0, 1]$ is **not dense** in $\mathbb{R}$.

Uncountable: A set A is *uncountable* if there is no bijection (so, a function that is 1-1, or injective, and onto, or surjective, at the same time) between A and $\mathbb{N}$:

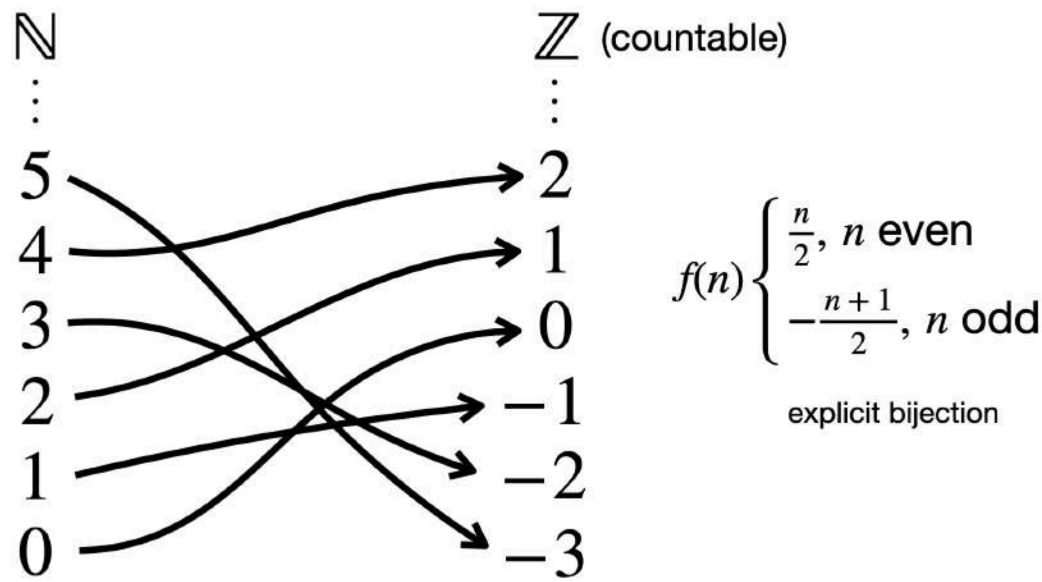
$$f: \mathbb{N} \rightarrow A$$



Equivalently, the elements of A cannot be listed in a sequence $\left(a_1, a_2, a_3, \ldots\right)$ that covers all of them.

To decide whether a specific set is uncountable or not is a little trickier than finding out if it is dense or not. To conclude that a set A is countable, for example, you need either to find an explicit bijection $f\colon \mathbb{N} \rightarrow A$, or to prove that this set admits such a bijection, even though it might not have an explicitly known one. Therefore, in order to show that a set is uncountable, one often supposes that such a bijection exists, and then finds a logical contradiction out of it. Of course, that's not the only method (by contradiction), but oftenly it's pretty effective.

For instance:



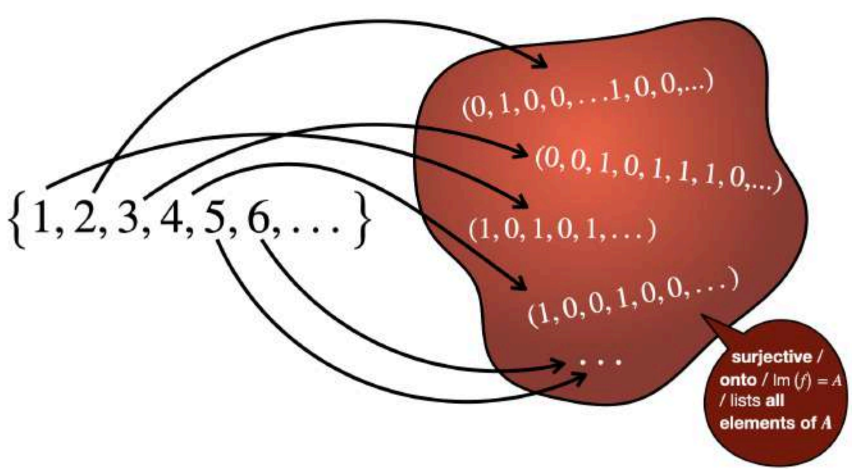
Let's see some examples:

1. $A = \left\{ \left(a_1, a_2, a_3, \ldots\right) \mid a_i \in \{0, 1\}, \forall i \in \mathbb{N} \right\}$ – is it countable?

This is the set of all infinite (binary) sequences of 0s and 1s. For example, some of its elements would be:

$$A = \{ (1, 1, 0, 1, 0, 0, \ldots) ; (1, 1, 1, 1, 0, 0, \ldots) ; (0, 1, 0, 1, 0, 0, \ldots) ; \ldots \}$$

Well, let's suppose that there is a bijection $f\colon \mathbb{N} \to A$ that lists all infinite binary sequences.



We are going to use the famous *Cantor's diagonal method* to construct another infinite binary sequence, denoted as x , that is <u>not</u> an element of A. If $x \notin A$, then the bijection that we supposed that existed in the beginning failed to list this element. In other words, there is no bijection capable of "counting" all elements of A.

$$\left\{\begin{array}{l} f(1) = (a_{11}, a_{12}, a_{13}, a_{14}, \dots) \\ f(2) = (a_{21}, a_{22}, a_{23}, a_{24}, \dots) \\ f(3) = (a_{31}, a_{32}, a_{33}, a_{34}, \dots) \\ f(4) = (a_{41}, a_{42}, a_{43}, a_{44}, \dots) \\ \quad \vdots \qquad\qquad\qquad \vdots \qquad \ddots \quad = (a_{nn}) \;\; n \in \mathbb{N} \end{array}\right\} \not\ni (b_1, b_2, b_3, \dots) \in A$$

$$\Updownarrow \qquad\qquad a_{ij} \in A \quad i,j \in \mathbb{N}$$

$$\nexists\, f : \mathbb{N} \to A$$

The formula that we will use to produce elements x out of the list of sequences of A created by the (supposedly existent) bijection f, is the following:

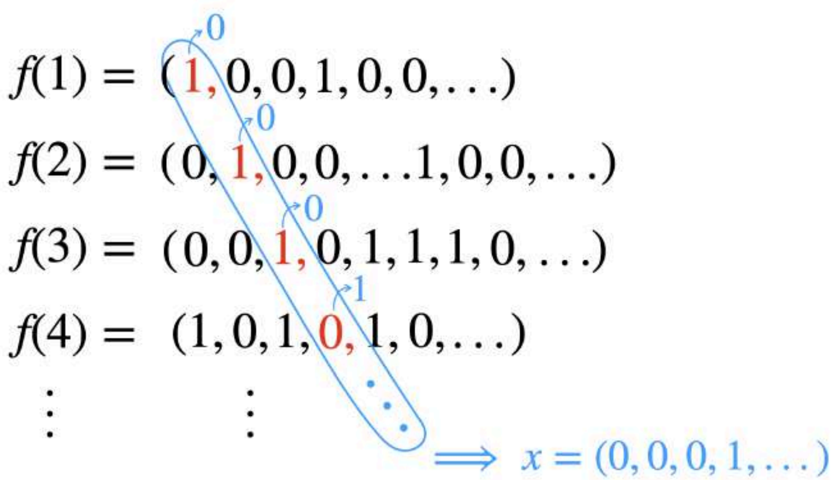
$$\left\{\begin{array}{l} f(1) = (a_{11}, a_{12}, a_{13}, a_{14}, \dots) \\ f(2) = (a_{21}, a_{22}, a_{23}, a_{24}, \dots) \\ f(3) = (a_{31}, a_{32}, a_{33}, a_{34}, \dots) \\ f(4) = (a_{41}, a_{42}, a_{43}, a_{44}, \dots) \\ \quad \vdots \qquad\qquad\qquad \vdots \qquad \ddots \quad = (a_{nn}) \;\; n \in \mathbb{N} \end{array}\right\}$$

$$x := (b_1, b_2, b_3, b_4, \dots) \;:\; b_n \begin{cases} 1, & \text{if } a_{nn} = 0 \\ 0, & \text{if } a_{nn} = 1 \end{cases}$$

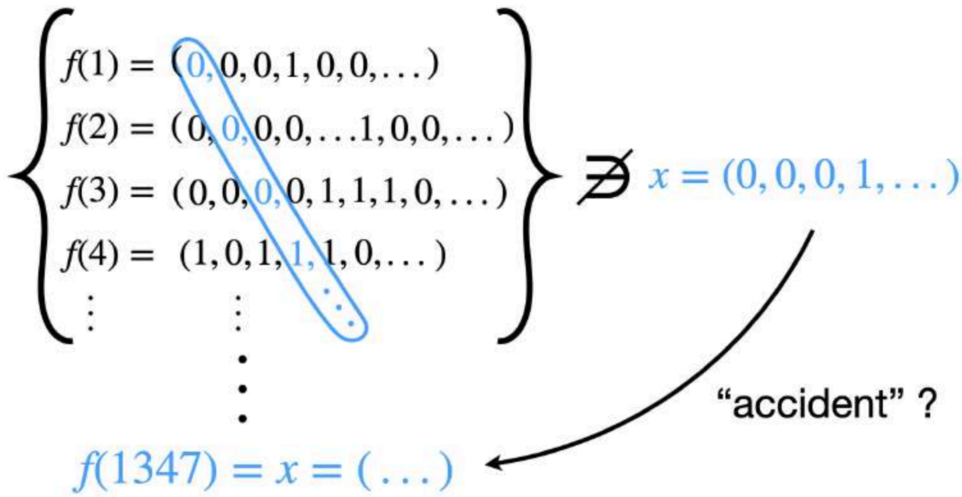
For example, let's suppose that the first four terms of our bijective list are the following:

$$
\begin{aligned}
f(1) &= (\mathbf{1}, 0, 0, 1, 0, 0, \ldots) \\
f(2) &= (0, \mathbf{1}, 0, 0, \ldots 1, 0, 0, \ldots) \\
f(3) &= (0, 0, \mathbf{1}, 0, 1, 1, 1, 0, \ldots) \\
f(4) &= (1, 0, 1, \mathbf{0}, 1, 0, \ldots) \\
&\vdots \qquad\qquad \vdots \qquad\quad \ddots
\end{aligned}
$$

Now, following the formula to construct x , we need to change each diagonal term from 1 to 0, or from 0 to 1. So, the first term in this illustration would be changed from 1 to 0, the second and third as well, but the fourth from 0 to 1, and so on.



The new element formed is $x = (0 , 0 , 0 , 1 , \ldots)$, and it is definitely not on our list. How come? How do we know that the newly constructed sequence x doesn't just coincide with some already existent sequence on the list, by “accident”?



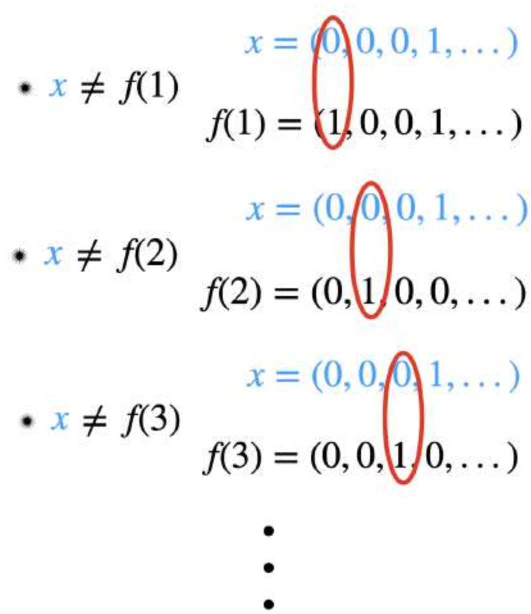
Well, notice the following facts:

✳ x is different from $f(1)$, at least in the 1^{st} digit.

✳ x is different from $f(2)$, at least in the 2^{nd} digit.

✳ x is different from $f(3)$, at least in the 3^{rd} digit.

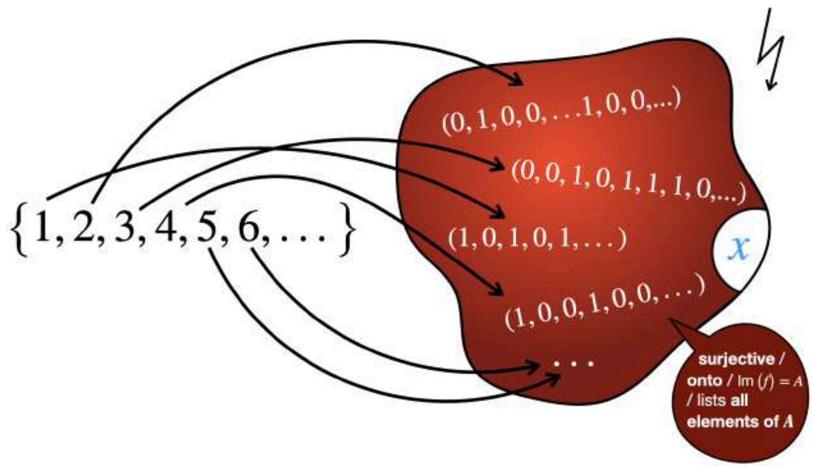
✳ … (and so on…)



So, x cannot be equal to some term $f(n)$ in the list:

$$x \neq f(n) \quad , \quad \forall\, n \in \mathbb{N}$$

This shows that we created a new sequence x , that was not listed in the bijection $f\colon \mathbb{N} \to A$, even though x is clearly an infinite binary sequence, and therefore it qualifies to be an element of A: $x \in A$.



But if f is a bijection, then it is a *surjection*, i.e. it lists ALL elements of A, without leaving any of its members unlisted. So, $x \in A$, but f failed to list it. This is a contradiction, which implies that our hypothesis must be rejected, namely:

$$\nexists \text{ bijection } f\colon \mathbb{N} \to A$$

Therefore, A is uncountable.

2. On the other hand, the set of rational numbers $\mathbb{Q}$ is countable!

We will show that by explicitly constructing a bijection $f\colon \mathbb{N} \to \mathbb{Q}$.

By definition:

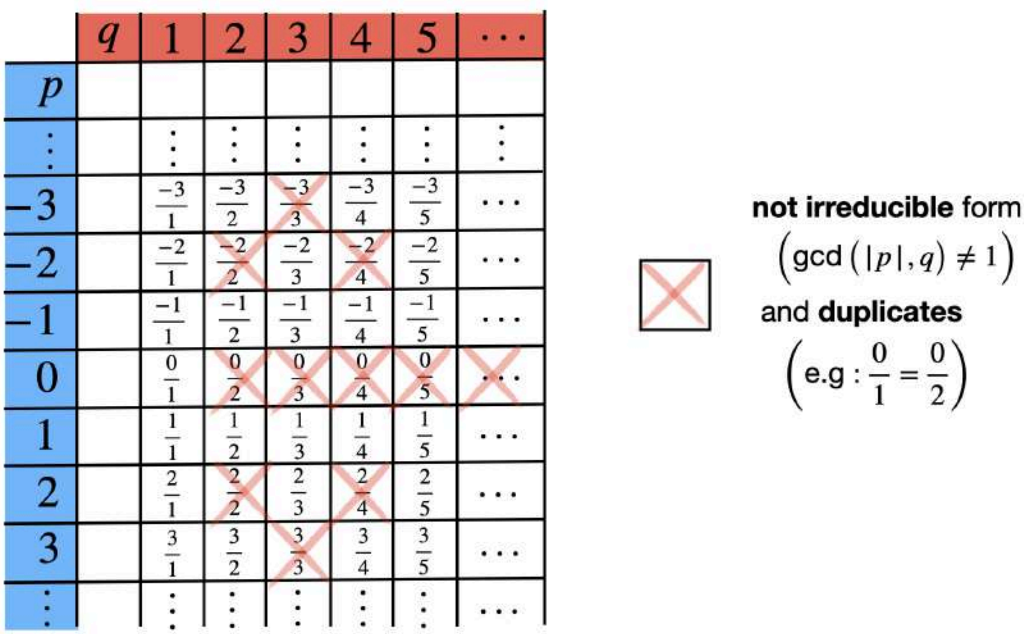
$$\mathbb{Q} = \left\{ \frac{p}{q} \;\; | \;\; p \in \mathbb{Z} \;,\; q \in \mathbb{N} \;,\; \gcd(\,|p|, q\,) = 1 \right\}$$

$p \in \mathbb{Z} \;\rightarrow$ it includes negative and zero numerators.

$q \in \mathbb{N} \;\rightarrow$ the denominator is <u>not</u> zero and is always positive. This ensures that there are no duplicates: $\left(\frac{-3}{4} = \frac{3}{-4}\right)$.

$\gcd(\,|p|, q\,) = 1 \;\rightarrow\; |p|$ ignores the sign of p, and $\gcd(\,|p|, q\,) = 1$ ensures irreducible form (no duplicates): $\left(\frac{1}{2} = \frac{2}{4} = \frac{6}{12}\right)$.

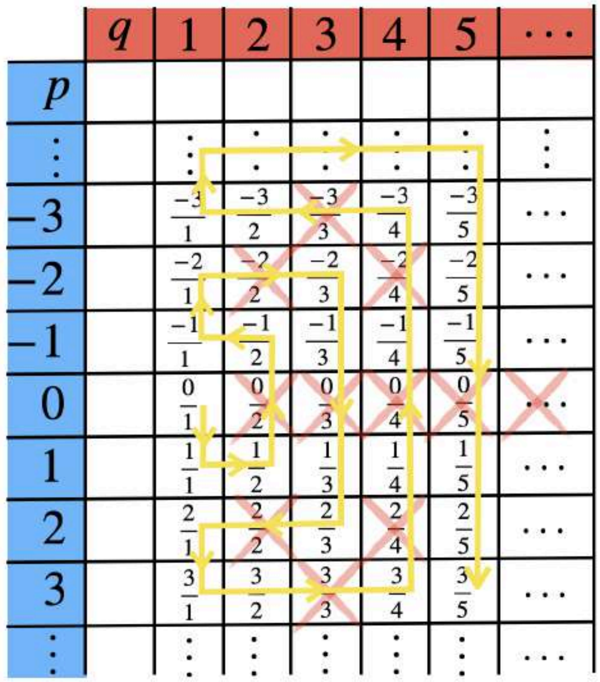
Let's organize all these rational numbers in a table:



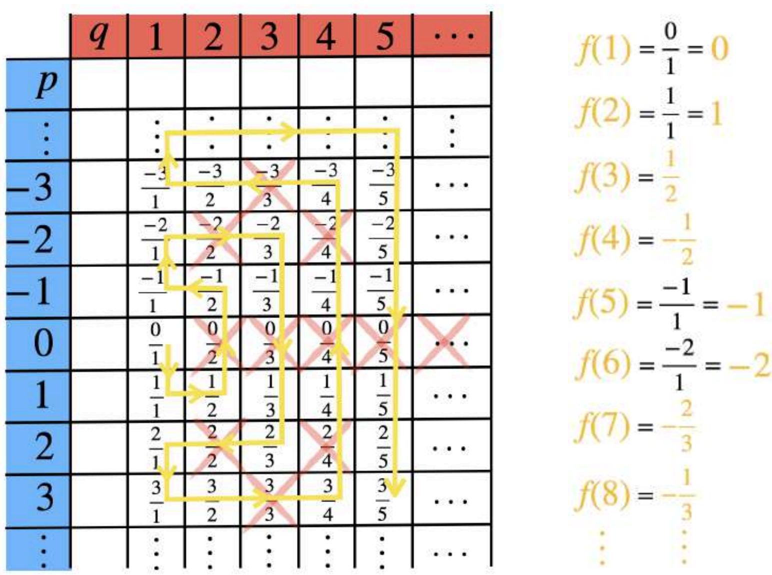
p \ q	1	2	3	4	5	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮
−3	$\frac{-3}{1}$	$\frac{-3}{2}$	~~$\frac{-3}{3}$~~	$\frac{-3}{4}$	$\frac{-3}{5}$	...
−2	$\frac{-2}{1}$	~~$\frac{-2}{2}$~~	$\frac{-2}{3}$	~~$\frac{-2}{4}$~~	$\frac{-2}{5}$	...
−1	$\frac{-1}{1}$	$\frac{-1}{2}$	$\frac{-1}{3}$	$\frac{-1}{4}$	$\frac{-1}{5}$	...
0	$\frac{0}{1}$	~~$\frac{0}{2}$~~	~~$\frac{0}{3}$~~	~~$\frac{0}{4}$~~	~~$\frac{0}{5}$~~	~~...~~
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	...
2	$\frac{2}{1}$	~~$\frac{2}{2}$~~	$\frac{2}{3}$	~~$\frac{2}{4}$~~	$\frac{2}{5}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	~~$\frac{3}{3}$~~	$\frac{3}{4}$	$\frac{3}{5}$	...
⋮	⋮	⋮	⋮	⋮	⋮	...

Notice how cells were crossed out since they are either not in irreducible form (i.e. $\gcd(\,|p|, q\,) \neq 1$) or they are simply duplicates (e.g.: $\frac{0}{1} = \frac{0}{2}$).

Now, all we have to do, in order to associate each natural number to a unique (non duplicate) element of this table and make sure that no element is left out from this "counting", is to create a path that passes only once through each number (injectiveness) and that covers all valid cells in the table (subjectiveness). Such a path could look like this:

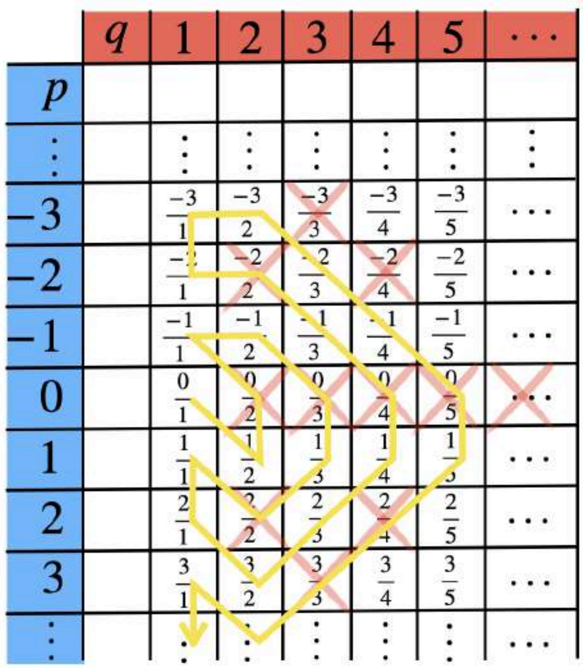


As a list, this would look like this:



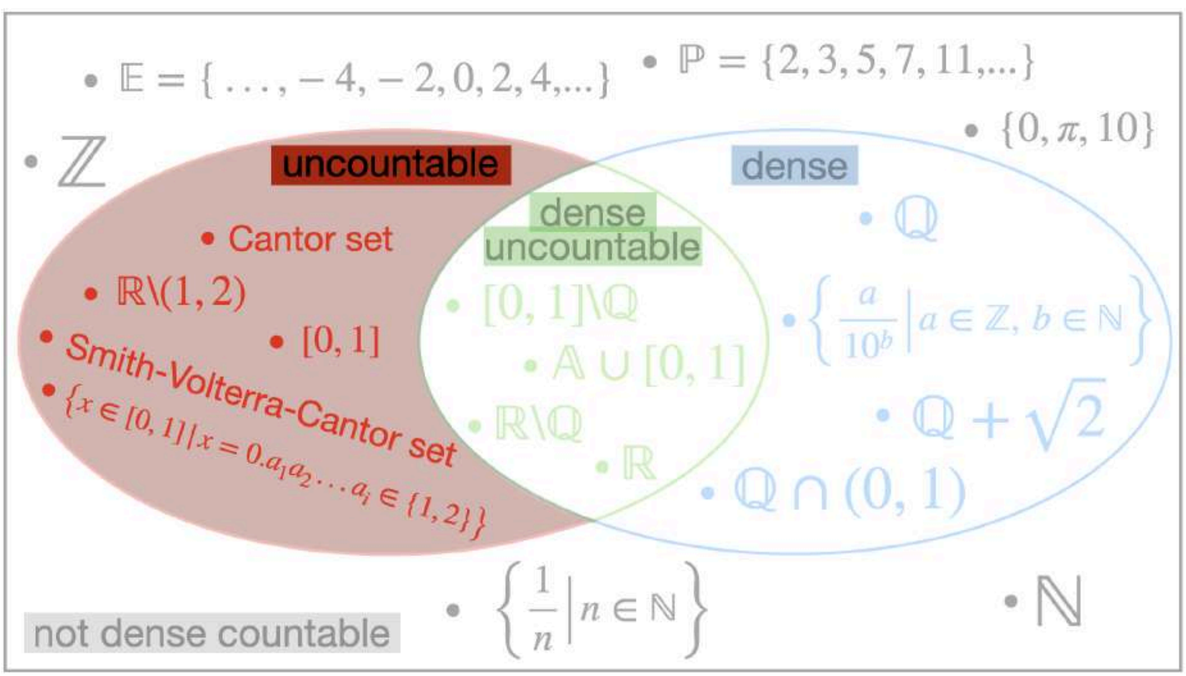
Therefore, $\mathbb{Q}$ is countable!

Remark: Of course, this path is not unique. You could also create some sort of "diagonal zig-zag".



Can you think of others? Let us know! send us an [email](#) with your ideas and personal research to publish on our website [dibeos.net](#).

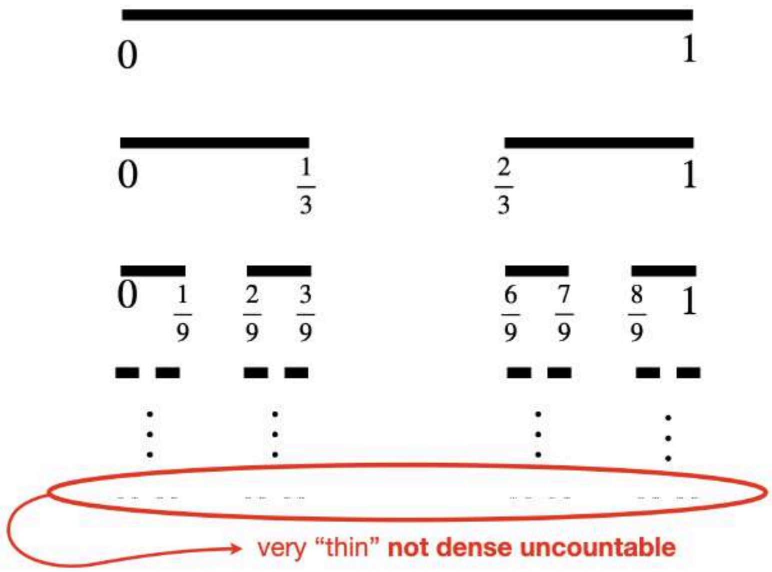
In conclusion, *uncountability* and *density* are two distinct (and independent) possible properties of a set. The greatest confusions usually arise in the following region – where sets are *uncountable*, but *not dense*:



These are some quick examples of this kind of set:

1. Cantor set:

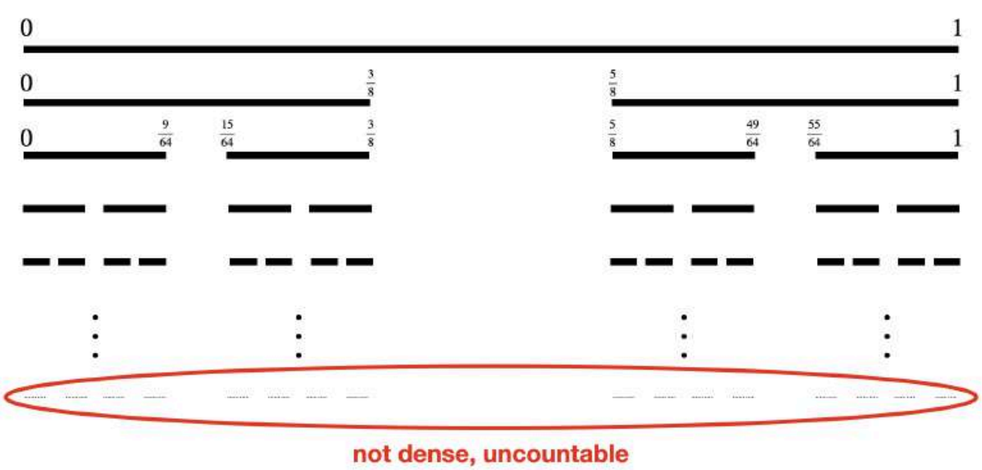
Start with the interval $[0, 1]$ and repeatedly remove the open middle third from each remaining segment:



What's left is a very "thin" set. It is *not dense*, but (surprisingly) *uncountable*!

2. Smith-Volterra-Cantor set:

This is a variation of the Cantor set in which we remove smaller and smaller chunks at each step of the way:



What's left is *not dense*, but *uncountable*.

Remark: "Notice that the concept of 'countability' has nothing to do with whether the members of the set are "sprinkled" into it so trickly that a member of the set can always be found between any two elements." – This is an adaptation of an answer given to a question in math.stackexchange related to the difference between 'dense and uncountable sets'. Click this link to see it.

If you found this document useful let us know. If you found typos and things to improve, let us know as well. Your feedback is very important to us – we're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com