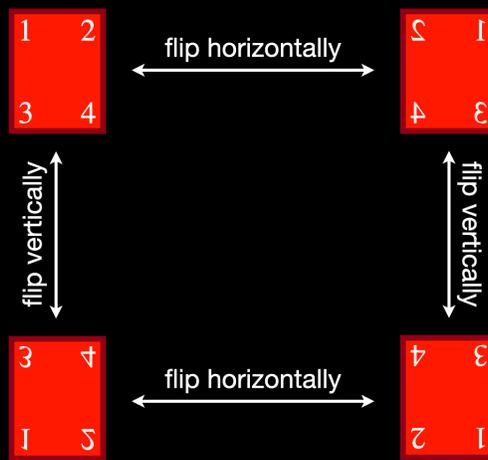# 5 Group Families

These three conditions define the symmetry of an object:

First we identify all the parts of an object that are similar, and give it a number.

Second, we try to find the actions we can perform with the object that can rearrange the numbered parts while always taking up the same amount of space.

And third, map all the possible combinations

The diagram can be transformed.

The arrows become lines. Even though they'll be missing the arrowheads, we'll still call them arrows. The blue represents a horizontal flip and ends at the B node, for blue. The red represents a vertical flip and ends at an R node for red
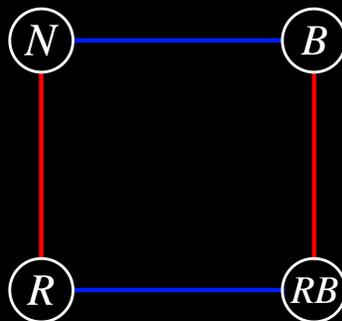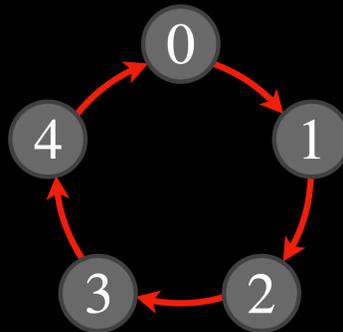
We see that R followed by B is the equivalent of B followed by R. Both of these end at the RB node, and more concisely can be expressed as *RB=BR*. The resulting group is known as a Klein-4 group.

We move on to cyclic groups.

Cyclic groups are commonly named C_n and the index will be the number of elements, or their order, replaces the *n*.

C_5 is represented like this: C_5 = {0,1,2,3,4}

Every rotation (to whichever direction we chose, it can't be both directions) is 1 in this case, so every element of the group is generated by repeatedly adding *1*.

After we reach *4*, we go back to *0*. This is what is known as *modular addition*. *4* would otherwise be known as *n-1*, since *4+1=5*, which completes the cycle by bringing us back to *0*.

If we were to represent that on a Cayley table, you'll see that clearly in cases like 2+3=0, or 4+3=2.

|	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Consider this diagram S_3

S_3
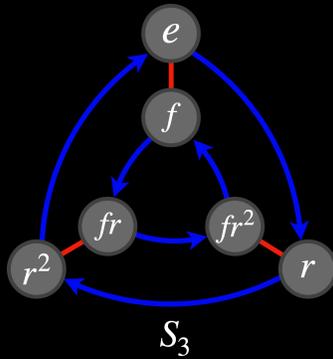
The blue arrows mean rotate, or r. If we start at the identity element e, we see that on the outside, an exact copy of C_3 is traced. The term for that is *orbit* of r. They're usually written together to be treated like a set

We move on to abelian groups. Abelian groups are those in which order of action is irrelevant. Recall our earlier example of V_4. If *R* and *B* are any two actions in an abelian group, then the action *R* followed by the action *B* yields the same result as b followed by *R*, represented by the earlier *RB=BR*

Abelian groups are *commutative*.

This means that, if we follow the blue, then the red arrow in the diagram, it leads us to the same point as following the red, then the blue arrow in the diagram.
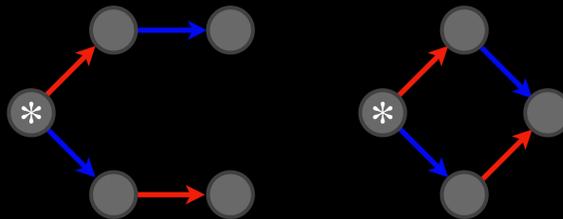

They're also pretty easy to spot on Cayley diagrams, since they pretty much mirror each other. If you were to fold the table in half diagonally, the elements that touch would be identical.
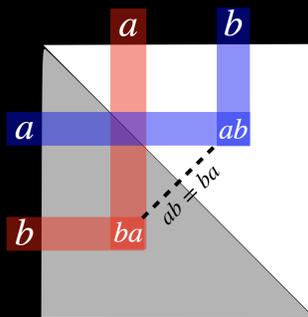
We move to Dihedral groups. Written as D_n

Dihedral groups describe objects which also have bilateral symmetry, meaning that they look the same when reflected.

The outer rings for D_n diagrams are orbits of r, and are copies of cyclic groups C_n. They are rotated clockwise. The inner ring is also a rotation, but counterclockwise. The *f* action connects the inner and outer ring
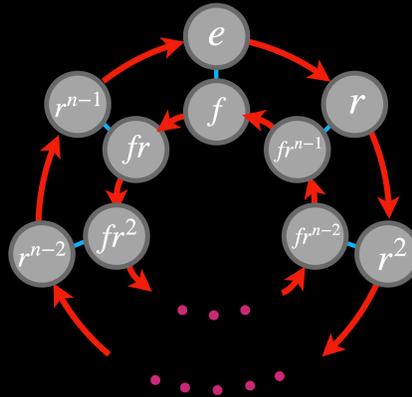
The multiplication table shows that quite clearly, where we can divide it into *4* very distinct quadrants in this example of D_5. We can call them flip and non flip.

Any non-flip times a non-flip is a non-flip
Any non-flip times a flip is a flip
Any flip times a non-flip is a flip
Any flip times a flip is a non-flip

We go on to rearrangements, which are actually called permutations, which lead us to symmetric groups, labeled S_n. We previously showed a symmetric group S_3.

S_n are factorial

$$1! = 1$$
$$2! = 2 \cdot 1 = 2$$
$$3! = 3 \cdot 2 \cdot 1 = 6$$
$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$
$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Despite the fact that a collection of elements forms a group, creating a group of permutations

does not *necessarily* require taking all the permutations of a given size. It is still possible to form a group using just some of the permutations of S_n.

One way to do that is through an alternating group.

original element   the element squared

1 2 3   1 2 3

1 2 3   1 2 3

1 2 3   1 2 3

1 2 3   1 2 3

1 2 3   1 2 3

1 2 3   1 2 3

Out of the *6* possible permutations, we get *3*, so A_3.

Visually, the Cayley diagrams for alternate groups will be the half of the permutation of symmetry groups. Like this A_4 arranged like on a truncated tetrahedron is the half of this S_3 truncated octahedron.

S_4

A_4