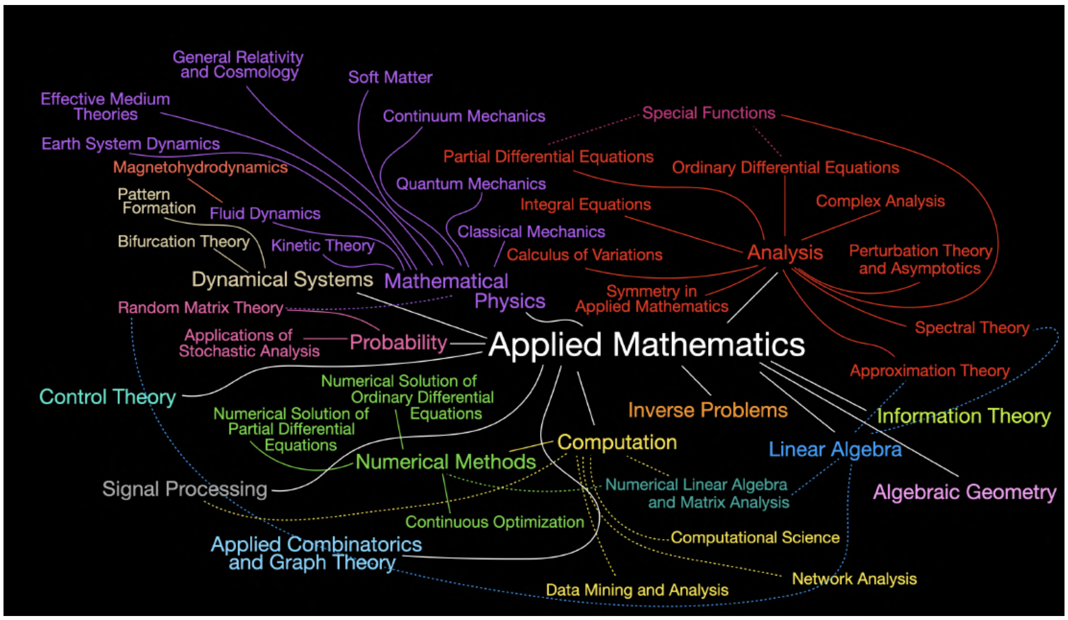
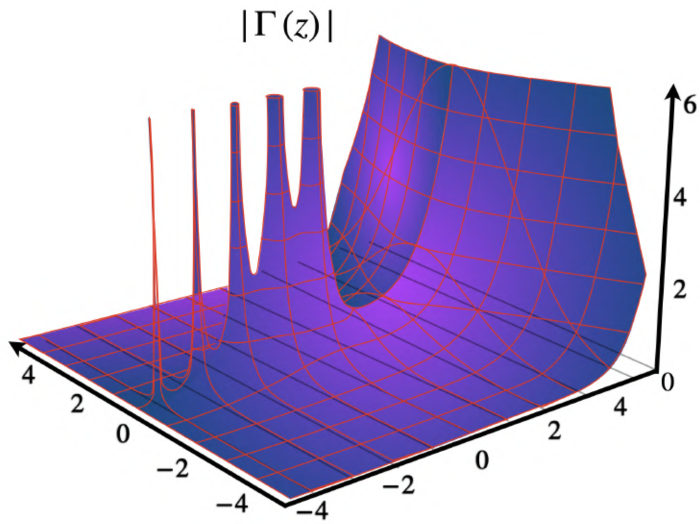
Sub-Branches of Applied Mathematics

by DiBeos

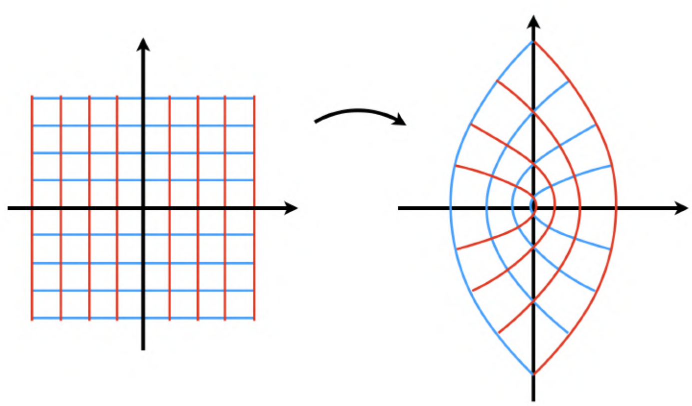


Applied mathematics is built on a surprisingly wide range of foundational areas – some familiar, others less obvious. A complete description of them would probably be impossible, nevertheless let's go through some of the main branches and sub-branches of applied mathematics.

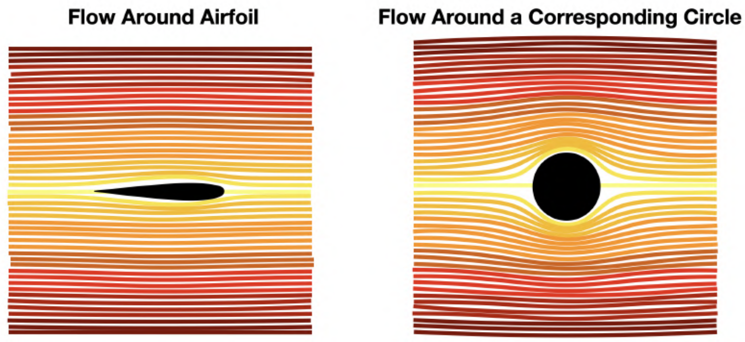
Complex Analysis



Complex analysis studies how complex numbers function. Although it's applied in various fields, an interesting tool is conformal mapping, which transforms shapes into other ones without distorting angles.

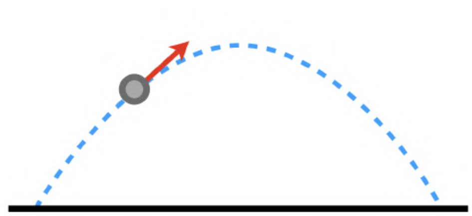


For example, if you're studying heat flow or fluid flow around a strangely shaped object, you can use a conformal map to reshape the domain into a circle, solve the problem there, and then transform the solution back.

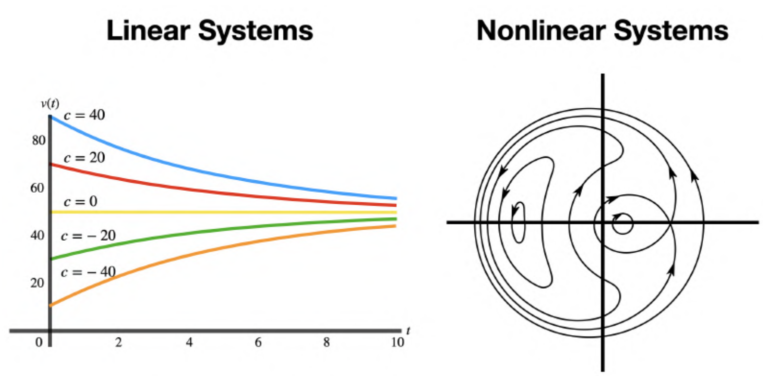


Ordinary Differential Equations

$$\frac{d^2y}{dt^2} = -g$$



Ordinary differential equations (or ODEs) describe how quantities change with respect to a single variable, which is typically time. They are used to study both linear systems, which are solvable analytically, and nonlinear systems, like chaos. ODEs are naturally found when modeling just about anything, from planetary motion to population growth.

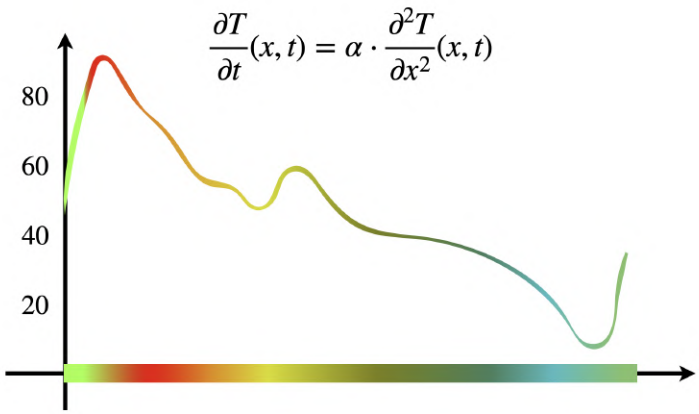


Partial Differential Equations

The **Heat Equation**

$$\frac{\partial T}{\partial t}(x, t) = \alpha \cdot \frac{\partial^2 T}{\partial x^2}(x, t)$$

Partial differential equations (abbreviated as PDEs) involve functions of several variables and their partial derivatives. They're absolutely essential when modeling distributed systems, like heat flow, sound, fluid dynamics, and electromagnetism, showing the local rates of change relate to one another across space and time. In applied math, they can be classified into three types: elliptic, parabolic, and hyperbolic, each with its own behavior and solution techniques.

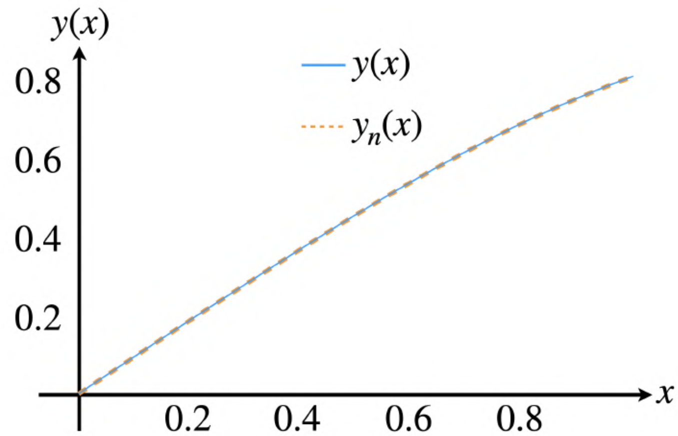


Integral Equations

Volterra Integral Equation of the second kind

$$u(x) = f(x) + \lambda \int_a^x K(x, t)\, u(t)\, dt$$

Integral equations are basically equations where an unknown function appears under an integral sign. The techniques used include classification into Fredholm or Volterra types, as well as exploiting properties of kernels. For example, take the Volterra integral equation of the second kind, where the unknown function appears both inside and outside the integral. It models systems with memory – like how a population's current growth depends not just on today's size, but on how it's been growing over time.



Perturbation Theory and Asymptotics

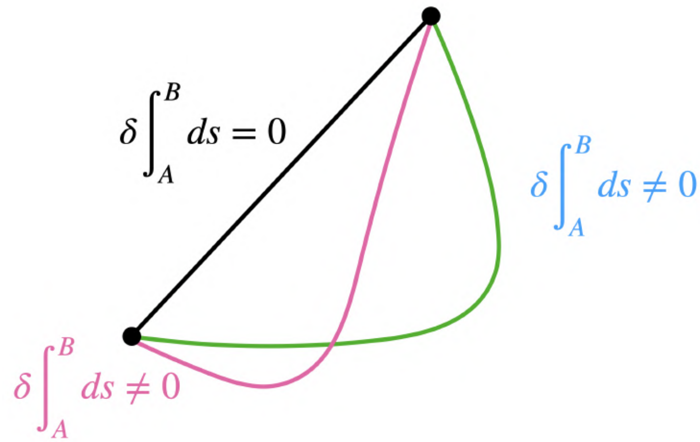
Perturbation theory deals with problems that depend on a small parameter, often denoted as ε, where exact solutions are hard or impossible to find. The idea is to start from a simpler, exactly solvable problem, whose solution is A_0, and then build corrections step by step. Each A_n captures the 'n-th order' correction. If ε is small, higher-order terms get smaller.

We even use *Big $\mathcal{O}$ notation* to show the size of the remaining error:

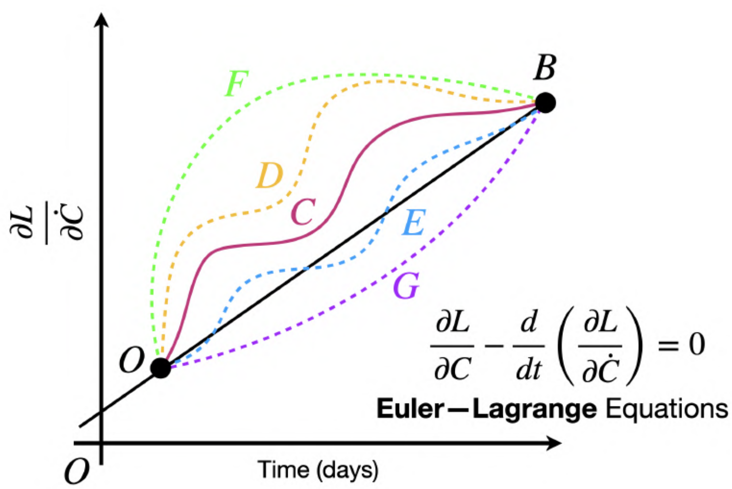
$$A = A_0 + \varepsilon A_1 + \mathcal{O}\left(\varepsilon^2\right)$$

A typical example would be singularly perturbed differential equations, such as boundary layer problems in fluid mechanics.

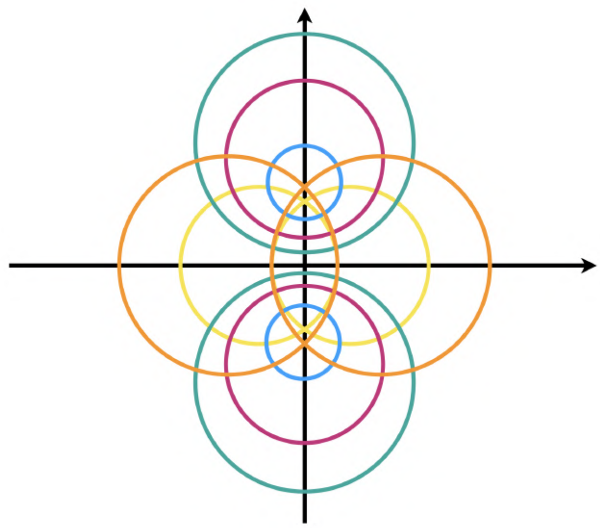
Calculus of Variations



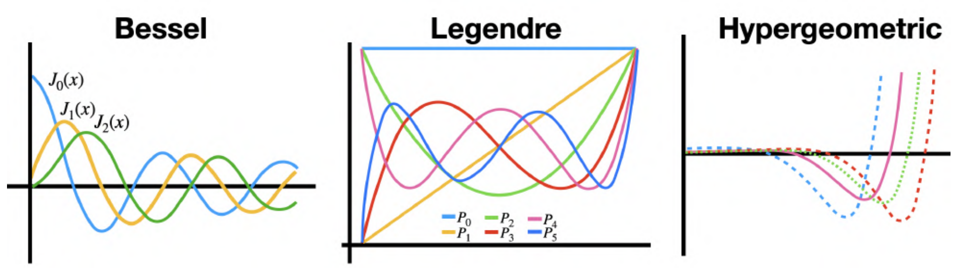
Calculus of variations studies how to find functions that minimize or maximize certain quantities that are typically expressed as integrals. In applied math, this area leads us to Euler–Lagrange equations, which deal with optimal solutions. The famous Euler–Lagrange equations tell you exactly what condition that optimal function must satisfy, for example, they explain why light follows the path of least time, not just the shortest distance.



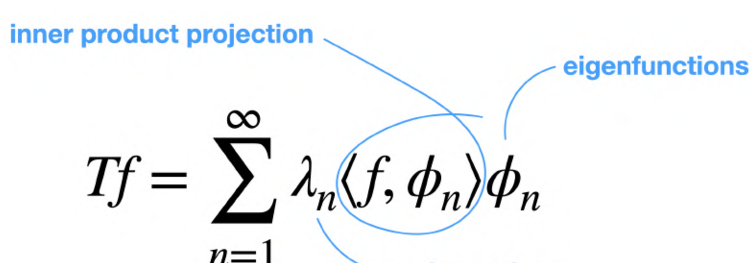
Special Functions



In applied math, special functions are especially important as solutions to differential equations that model physical phenomena, like vibrations, wave propagation, and heat flow. Some examples are Bessel functions, Legendre polynomials, and hypergeometric functions. Bessel functions for example naturally appear when we model the vibration of a drumhead. They capture both the geometry and boundary conditions of the problem.



Spectral Theory



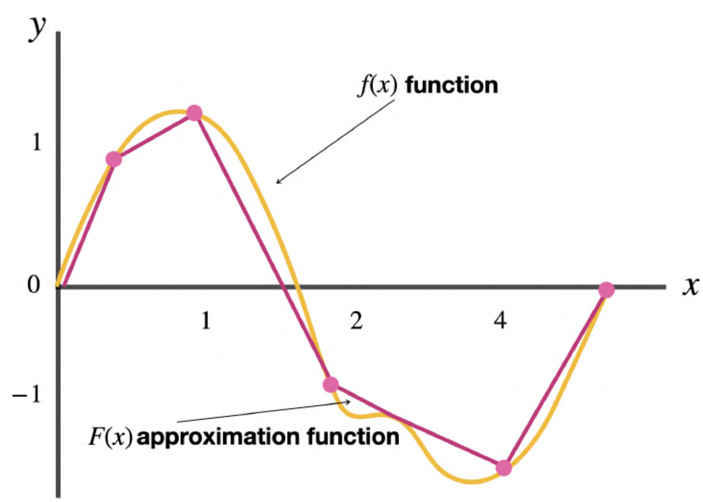
Spectral theory studies how linear operators can be decomposed in terms of their eigenvalues and eigenfunctions. This area generalizes the concept of diagonalizing a matrix into infinite dimensional settings, like as an example, differential and integral operators. It's used in understanding stability, resonance, wave propagation, and signal processing, and many other areas.

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

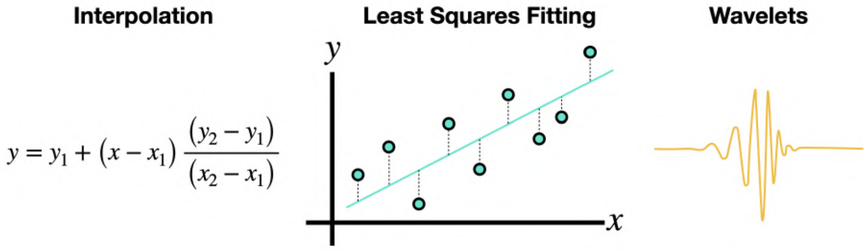
with

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 & 1 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ -1 & 0 & 0 & 1 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

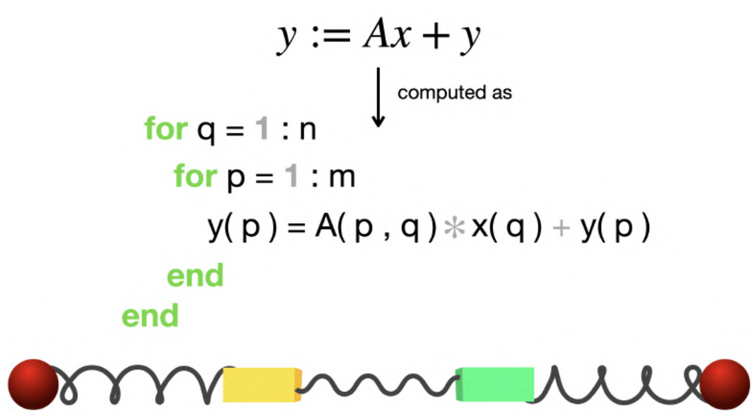
Approximation Theory



This area studies how functions can be approximated by simpler ones, such as polynomials, splines, or rational functions, and how accurately this can be done. It answers questions like, "what's the best way to represent a complex function while having limited data or resources of computing?" Approximation theory uses methods like interpolation, least squares fitting, and wavelets.

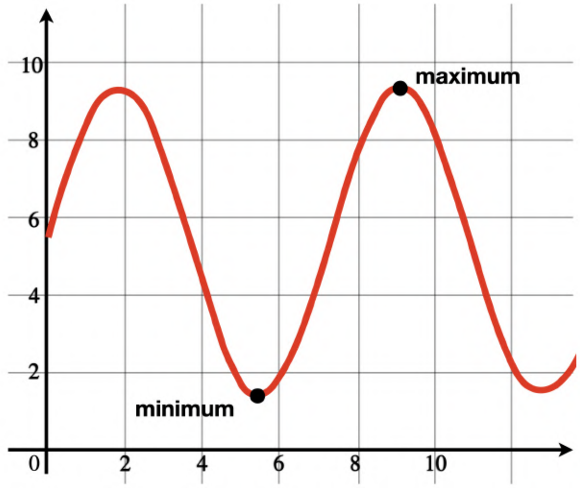


Numerical Linear Algebra and Matrix Analysis

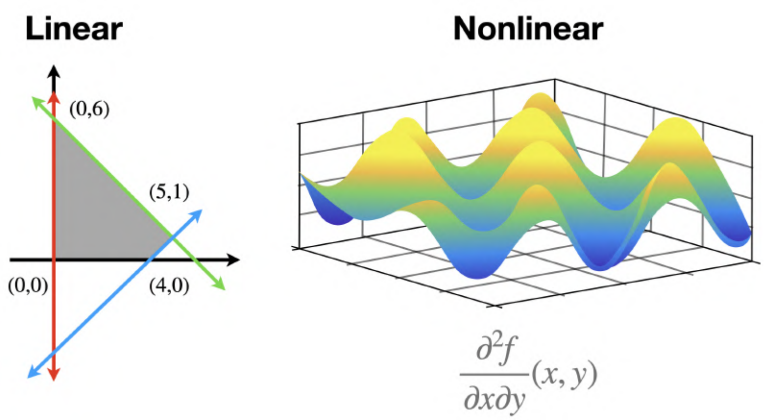


The area keeps its focus on developing and understanding algorithms for solving matrix problems, especially to create computer algorithms. Numerical linear algebra and matrix analysis is the basis for almost all of computational science, from simulating different physical systems, to analyzing data.

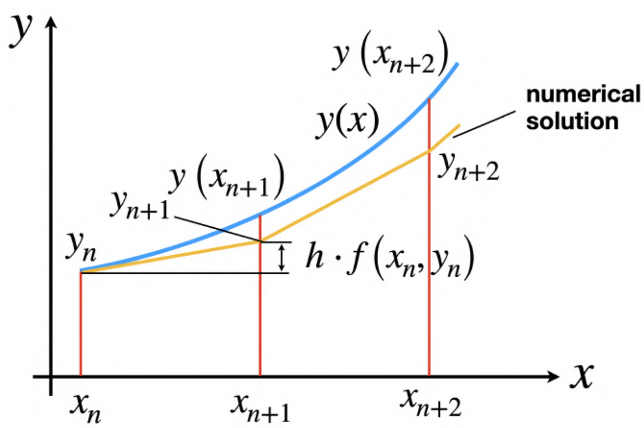
Continuous Optimization



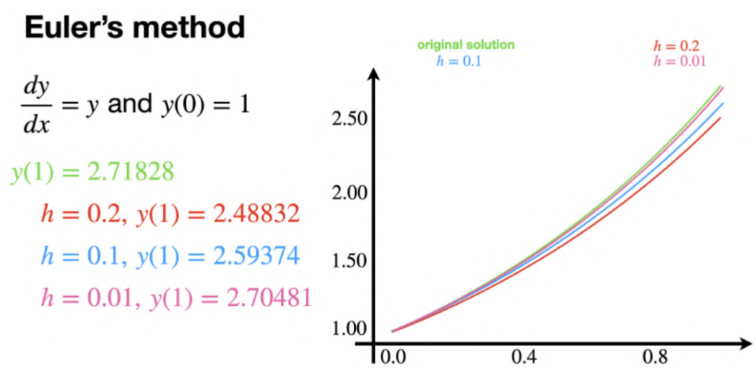
Continuous optimization is concerned with finding minimum or maximum values of a function subject to constraints, where the variables vary continuously. It's important in linear, and nonlinear programming. Linear programming deals with problems with linear objectives and constraints, and nonlinear programming with more general, and often more complex cases.



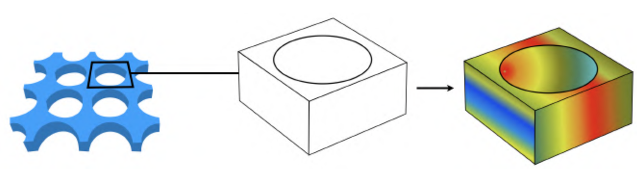
Numerical Solutions of Ordinary Differential Equations



This area studies algorithms for computing approximate solutions to ordinary differential equations (i.e., ODEs), because these often cannot be solved analytically. Take Euler's method, which is often used here to model a step-by-step estimate of how the population evolves over time. Basically speaking, the system gives us reliable results, even when the system behaves chaotically or if it spans multiple time scales.

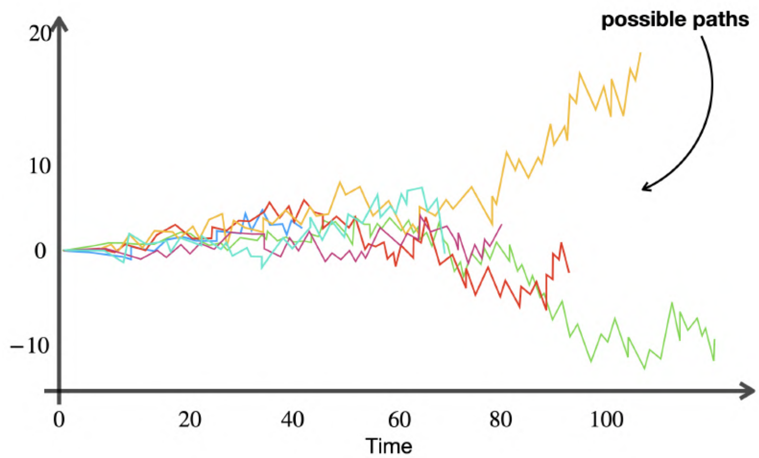


Numerical Solutions of Partial Differential Equations

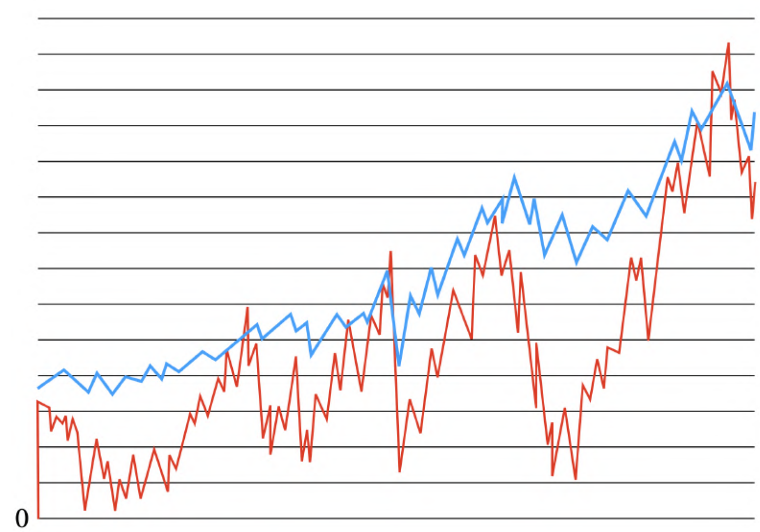


Like the previous area, this one also focuses on computational methods for approximating solutions, but this time to partial differential equations (i.e., PDEs), because they usually cannot be solved exactly. It uses many different types of techniques, but one approach is the finite element method, which breaks the domain into small, flexible pieces and stitches together local solutions. It's especially useful when the geometry is complex, like modeling stress in an irregularly shaped bridge.

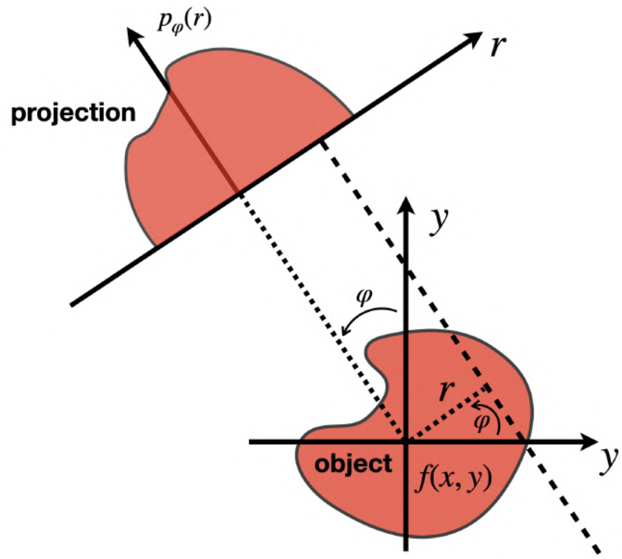
Applications of Stochastic Analysis



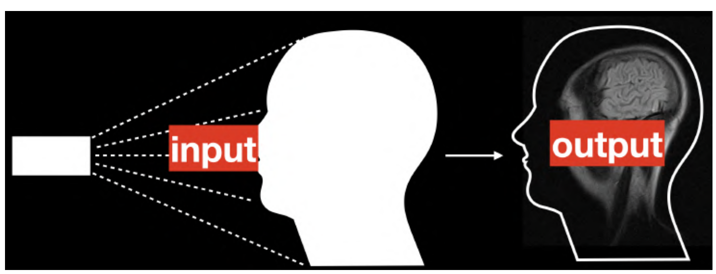
Stochastic Analysis tries to understand probabilistic behavior over time, and does so by developing simulation methods. Basically, it blends the theory of probability with analysis in order to study how randomness shapes the evolution of a system. If you take stock prices, they clearly don't follow neat curves, but wiggle unpredictably. That behavior is captured using stochastic differential equations.



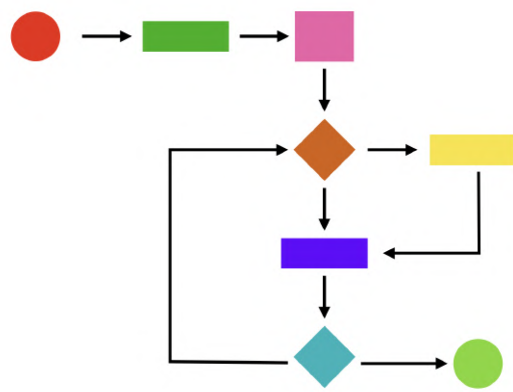
Inverse Problems



Inverse problems involve determining hidden causes from observed effects. For example, reconstructing an image from tomography data, or identifying material properties from surface measurements. Mathematically speaking, inverse problems require inferring unknown inputs to a model from its outputs.

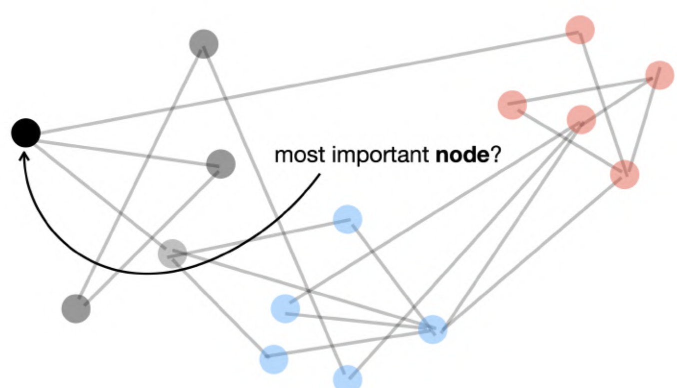


Computational Science

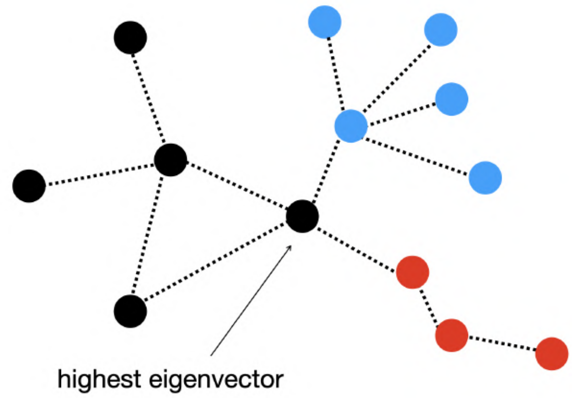


Computational science is the study of using computers to solve mathematical models of complex physical, biological, engineered systems, etc. It brings together applied mathematics, computer science, and other, domain-specific knowledge to study processes that are too difficult or impossible to observe directly or to solve analytically.

Data Mining and Analysis



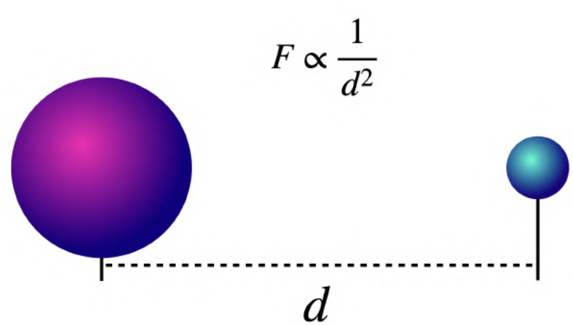
As you might expect from the name, data mining and analysis focus on the extraction of patterns, structure, and useful information from large or complex datasets. This involves statistical techniques, machine learning algorithms, and mathematical modeling in order to interpret, classify, or predict the behavior of the data.



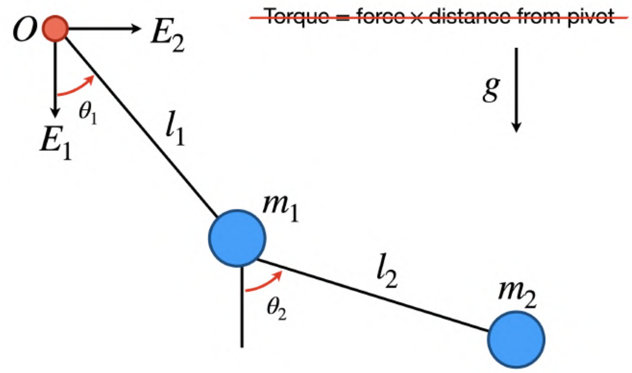
Network Analysis

Network analysis studies systems of interconnected components. It models them as graphs of nodes and edges. One really important idea in network analysis is centrality, which is figuring out which nodes are the most important. Eigenvector centrality, for example, doesn't just count connections, but gives it more weight to being connected to other well-connected nodes. That's how Google's original PageRank algorithm worked: it ranked websites not just by popularity, but by the quality of who linked to these websites.

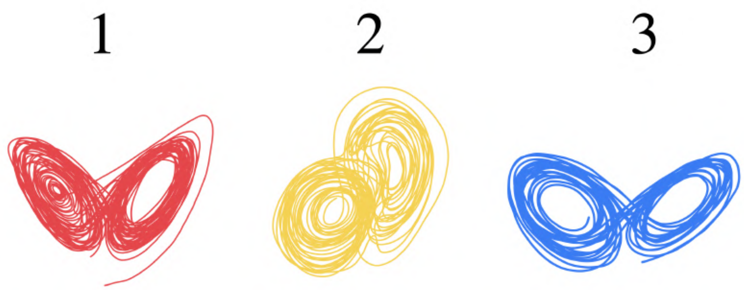
Classical Mechanics



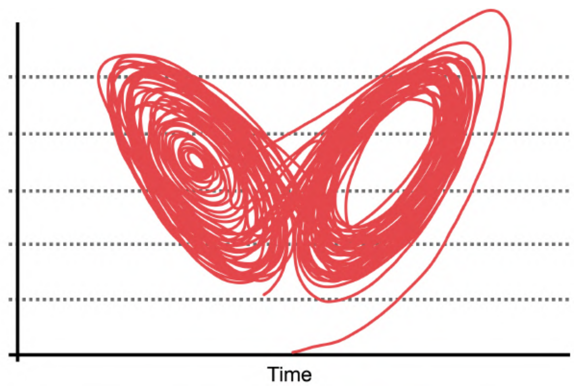
Classical mechanics studies motion and forces in systems that obey Newton's laws. It's a large area of study, and works for a range of topics, from planetary motion to robotics. Though classical mechanics uses many methods, an interesting one is the Lagrangian approach. Basically if you wanted to model a swinging pendulum, you wouldn't need to sum up torques (Torque = force $\times$ distance from pivot (lever arm)), you just write down its energy, and the Lagrangian gives you the equation of motion.


O
E2
E1
θ1
l1
m1
l2
θ2
m2
g
Torque = force × distance from pivot

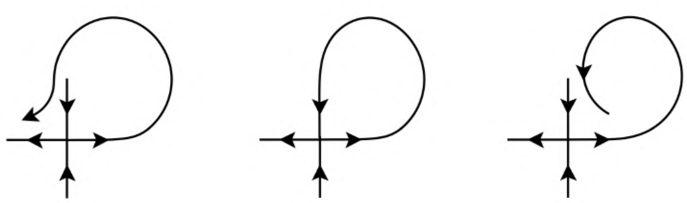
Dynamical Systems



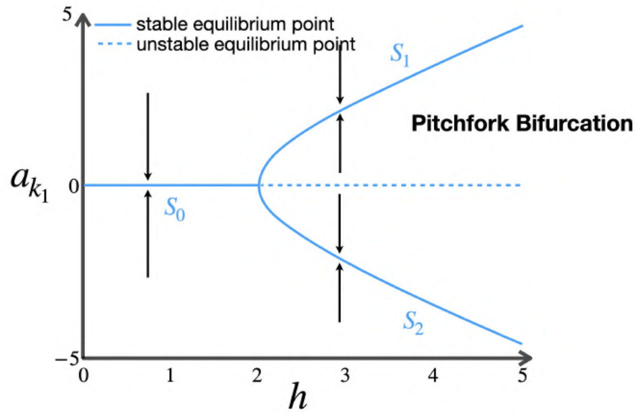
Dynamical systems study how states evolve over time under rules which are deterministic, and delves into many concepts, like stability, periodicity, chaos, and long-term behavior, with tools like phase portraits, bifurcations, and attractors. It especially focuses on behaviour which is constrained by time, especially when explicit solutions are unavailable.



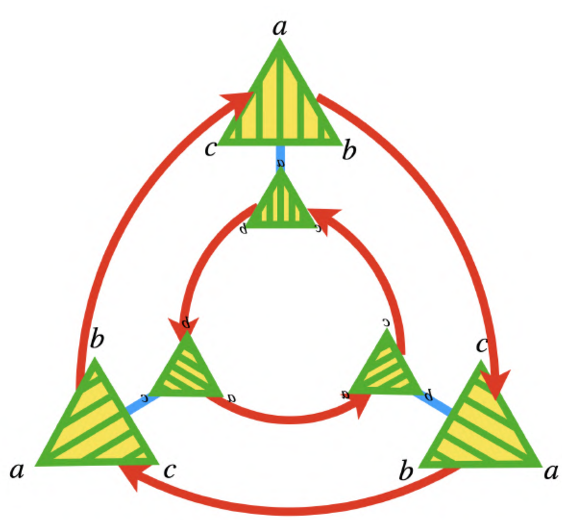
Bifurcation Theory



Bifurcation theory studies how a system's behavior can suddenly change when a parameter crosses a critical value. A bifurcation occurs when small changes in input lead us to sudden changes in structure, like transitioning from stability to oscillation or chaos. A classic example is a straight beam that buckles when compressed. Below a certain force, it stays straight. Go just beyond that, and it suddenly bends left or right – two new stable states emerge. That's called a pitchfork bifurcation.

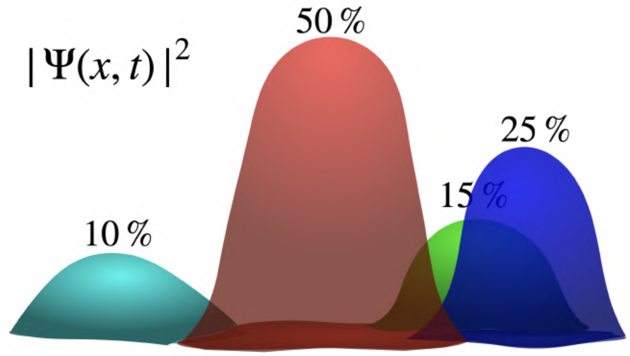


Symmetry in Applied Mathematics

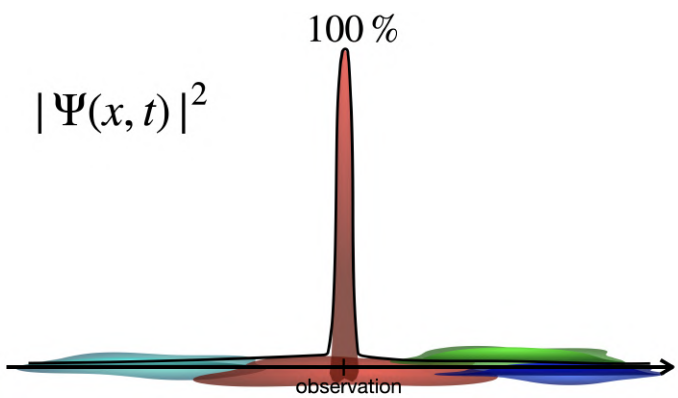


This field looks at how invariance under transformations (in other words, rotations, reflections, or translations) affect mathematical models and their solutions. In physics for example, if a system's laws are time invariant (or don't change over time), then energy is conserved, and this time invariance is a symmetry.

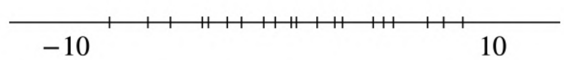
Quantum Mechanics



Quantum mechanics models physical systems at atomic and subatomic scales, where classical mechanics fails us. It replaces deterministic trajectories, as is explained by classical mechanics, with wave functions and operators, led by equations like Schrödinger's equation.

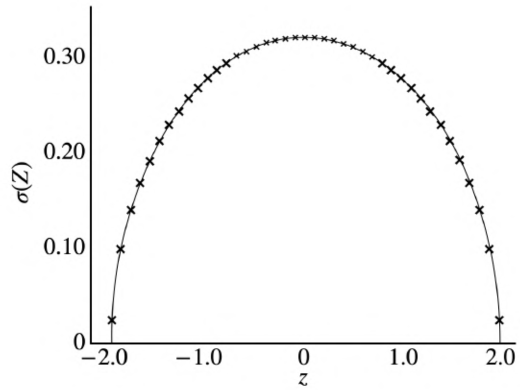


Random Matrix Theory



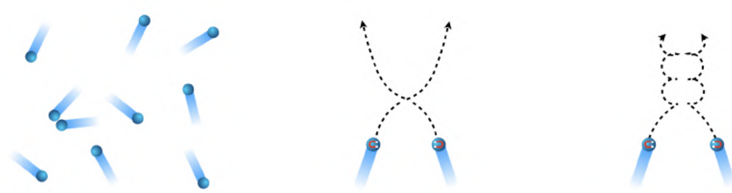
A spectrum of a **randomly chosen complex Hermitian (GUE) matrix** of dimension $N = 20$

The area studies matrices with entries which are chosen at random, and focuses on the statistical properties of their eigenvalues and eigenvectors. It was actually initially developed in nuclear physics, when physicists noticed that energy levels of heavy atomic nuclei did not follow simple patterns. Their statistical behavior actually matched the eigenvalues of large random matrices. So instead of tracking particles individually, the theory models the whole system's uncertainty through randomness.



The spectral measure (**eigenvalue density**) of a **GUE matrix** of dimension N = 500 000, with the eigenvalues divided by crosses.

Kinetic Theory

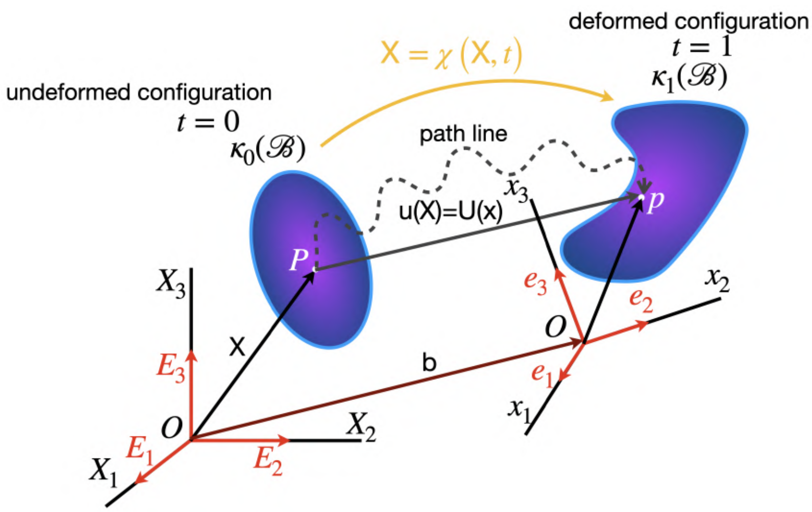


Kinetic theory describes the statistical behavior of systems that have a large number of particles, like gases or plasmas. But it can even be applied in the study of traffic flows, for example. Classically, it links microscopic particle motion to macroscopic quantities like pressure and temperature through equations like the Boltzmann equation.

Boltzmann equation

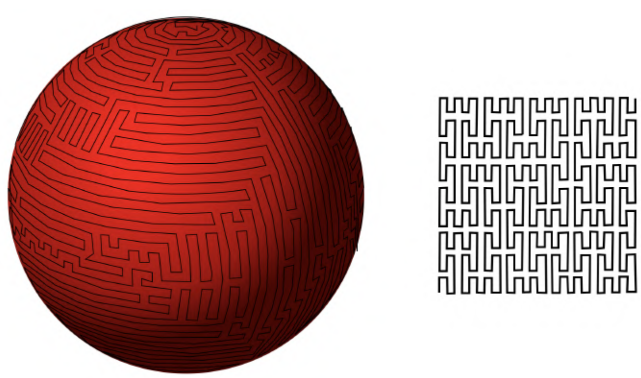
$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{a} \cdot \nabla_{\mathbf{v}} f = Q(f,f)$$

Continuum Mechanics



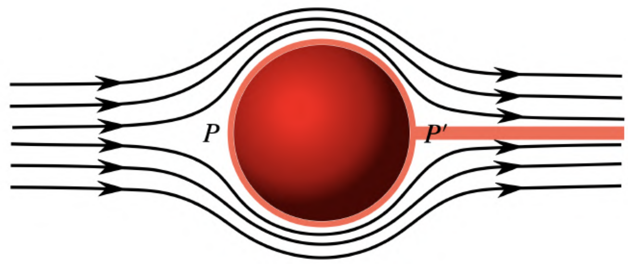
Continuum mechanics treats materials as a continuous media, and ignores their atomic structure. It describes deformation, flow, stress, etc., and includes both solids and fluids. It includes another sub-branch called mechanics of solids, which studies how solid materials deform and respond to forces, their elasticity, plasticity, fracture, and wave propagation.

Pattern Formation



Pattern formation studies, you guessed it, pattern formation. More rigorously speaking, it looks at how complex, ordered structures spontaneously emerge in systems which are spatially extended. It examines mechanisms behind stripes, spirals, spots, and waves seen in nature and experiments, which often arise from instabilities in nonlinear PDEs.

Fluid Dynamics



Fluid dynamics is the study of how liquids and gases move under the influence of forces. Navier–Stokes equations are at the heart of the area, which describe how velocity and pressure evolve in a moving fluid, and are based on Newton's laws. Fluid dynamics models weather, ocean currents, blood flow, aerodynamics, and much more.

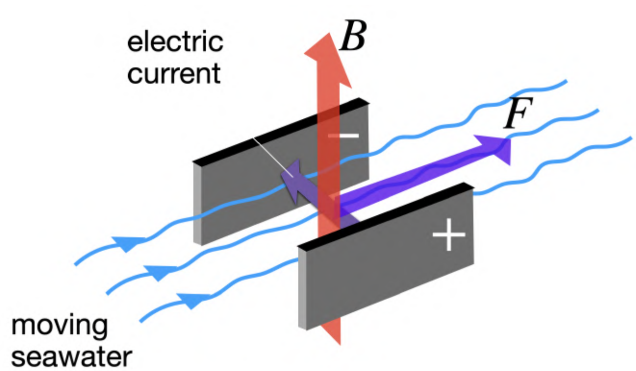
Navier–Stokes equation for an incompressible fluid

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

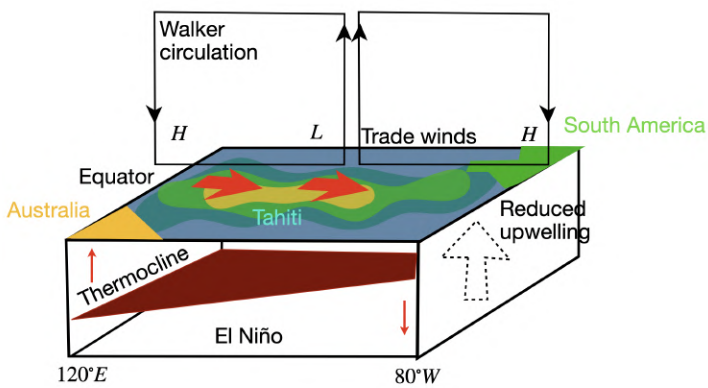
$$\nabla \cdot \mathbf{u} = 0$$

Magnetohydrodynamics

Magnetohydrodynamics (MHD) is a subfield of fluid dynamics, which studies the behavior of electrically conducting fluids, like plasmas, liquid metals, and saltwater, when interacting with magnetic fields. It merges fluid dynamics with Maxwell's equations to model the coupled motion of fluid and electromagnetic fields.

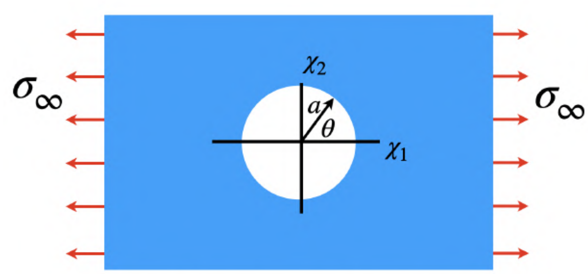


Earth System Dynamics



This area models and studies how components of the Earth system interact, like the atmosphere, oceans, land, ice, and the biosphere, over a wide range of spatial and temporal scales.

Effective Medium Theories



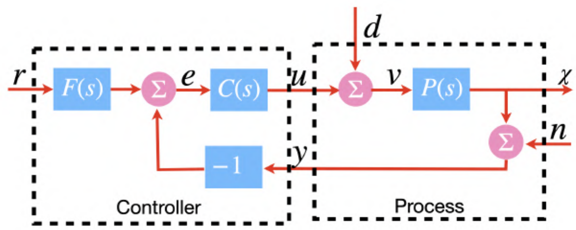
Tension of an elastic plate with a hole

Effective medium theories look at the macroscopic behavior of materials that are heterogeneous, like porous materials, by homogenizing (in normal words, averaging) their microscopic structures. Basically speaking, the goal of effective medium theories is to replace a complex and microstructured material with a simpler, averaged one that behaves the same way at large scales.

Soft Matter

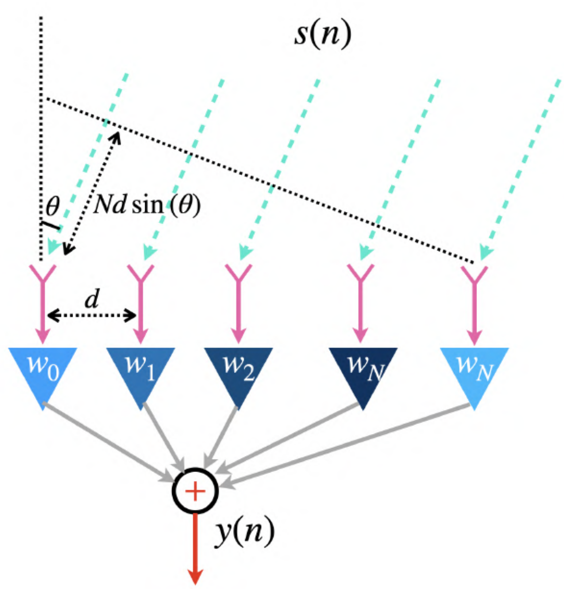
Soft matter refers to materials which are easily deformed by external forces, like liquids, polymers, foams, gels, biological tissues, etc. These soft things show complex behavior, like viscoelasticity, self-organization, or phase transitions.

Control Theory



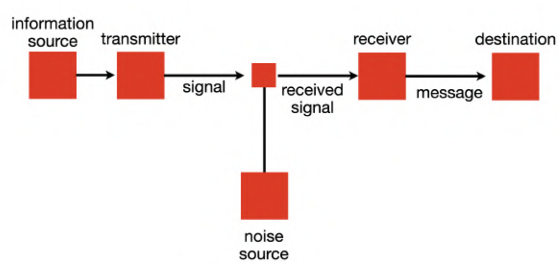
Control theory is interested in the influence of the behavior of dynamical systems through inputs, and aims to achieve desired outcomes, like stability or optimization for example.

Signal Processing



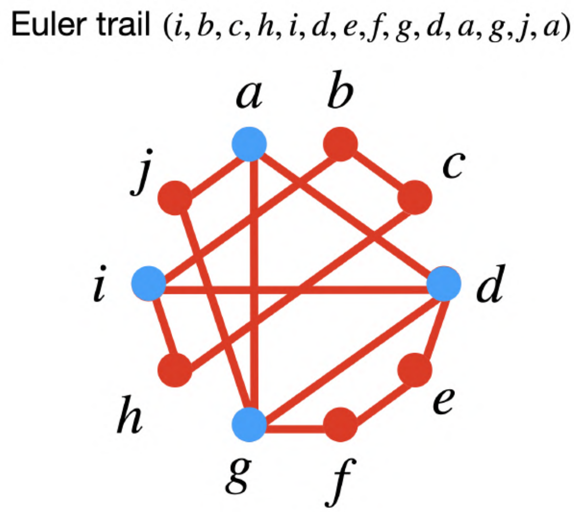
The name says it all, signal processing analyzes, transforms, and interprets signals, which are functions that represent time or space dependent data, like audio, images, or sensors. It might sound straightforward, but it's harder than it sounds. In order to have constructed this image of the black hole, astronomers had to collect and organize noisy, incomplete data from different detectors, combine it over a period of almost 2 years, and turn it into this picture.

Information Theory

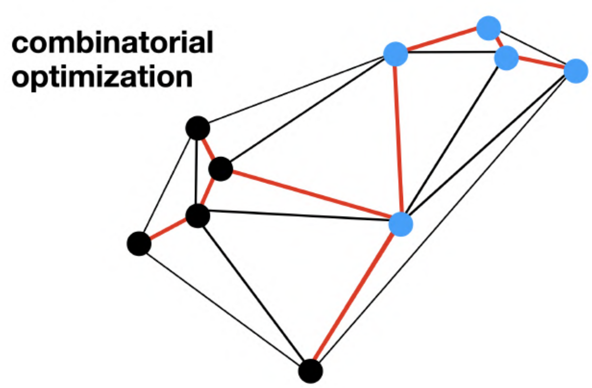


Information theory studies the quantification, storage, and transmission of information. Basically, everything about how information works and its transmission.

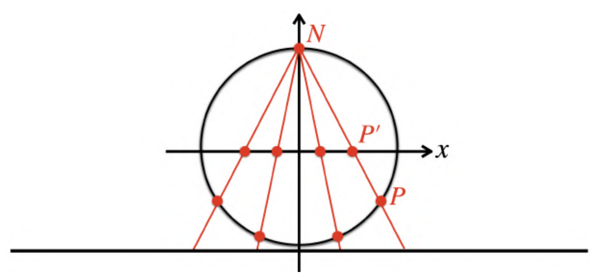
Applied Combinatorics and Graph Theory



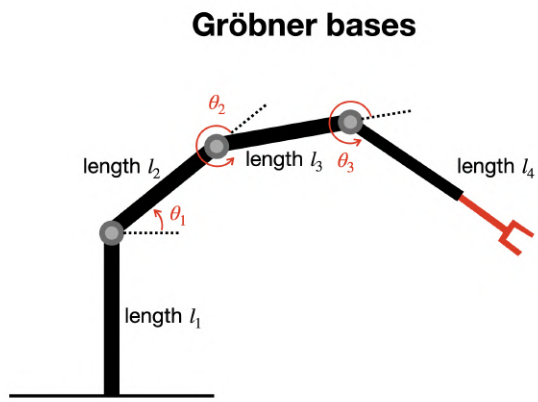
Applied combinatorics and graph theory study discrete structures and relationships, like arrangements, selections, and networks. It includes a branch called Combinatorial Optimization, which looks for the best solution from a finite set of discrete possibilities. It focuses on problems like shortest paths, maximum flows, and optimal scheduling.



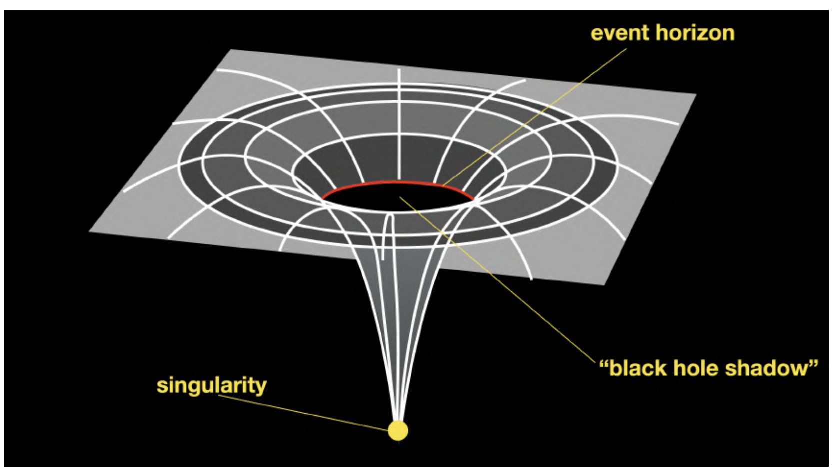
Algebraic Geometry



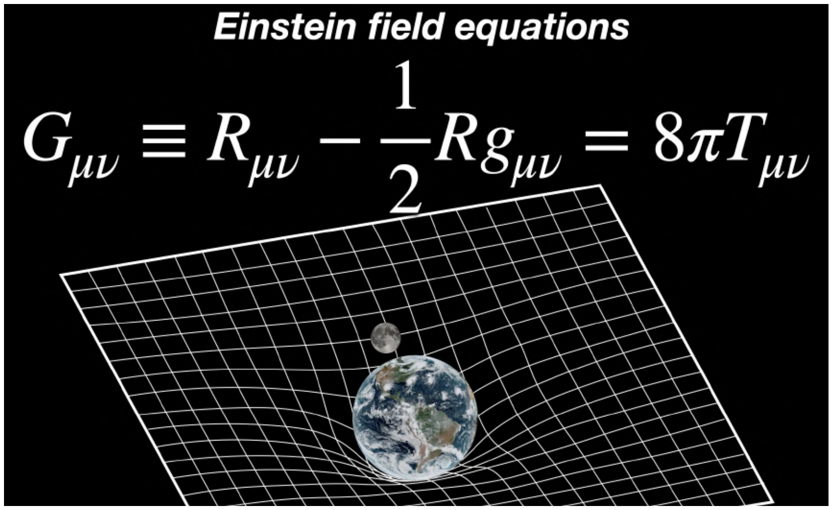
Though this is traditionally more of a pure mathematics field, it becomes applied through tools like Gröbner bases, which let you systematically simplify systems of polynomial equations. They're used in robotics and kinematics to figure out possible positions or movements.

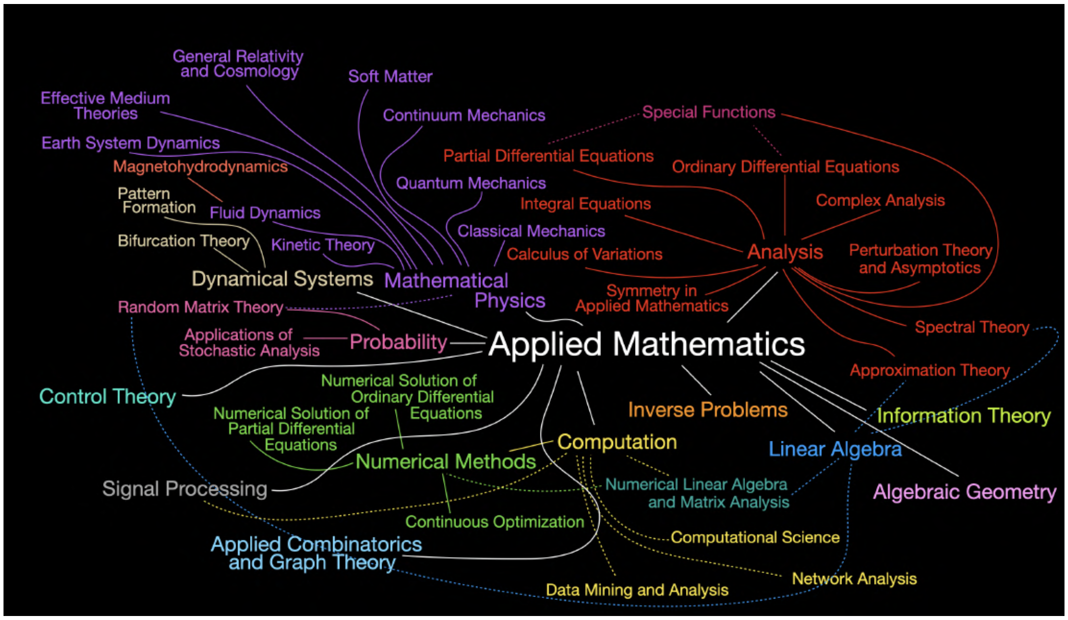


General Relativity and Cosmology

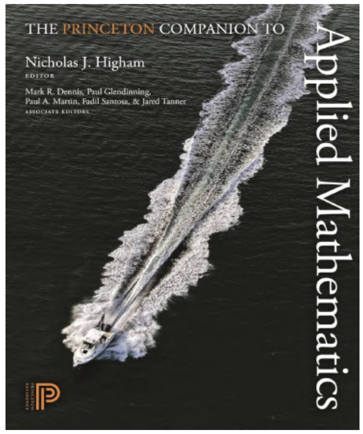


General relativity and cosmology study the large-scale structure and dynamics of the universe. This is done through Einstein's field equations, which relate spacetime geometry to matter and energy. The theory predicts phenomena like black holes, gravitational waves, and the expansion of the universe.


Einstein field equations
$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$


General Relativity and Cosmology
Soft Matter
Effective Medium Theories
Continuum Mechanics
Special Functions
Earth System Dynamics
Partial Differential Equations
Ordinary Differential Equations
Magnetohydrodynamics
Quantum Mechanics
Pattern Formation
Integral Equations
Complex Analysis
Fluid Dynamics
Classical Mechanics
Bifurcation Theory
Kinetic Theory
Calculus of Variations
Analysis
Perturbation Theory and Asymptotics
Dynamical Systems
Mathematical Physics
Symmetry in Applied Mathematics
Random Matrix Theory
Spectral Theory
Applications of Stochastic Analysis
Probability
Applied Mathematics
Approximation Theory
Numerical Solution of Ordinary Differential Equations
Control Theory
Inverse Problems
Information Theory
Numerical Solution of Partial Differential Equations
Computation
Linear Algebra
Numerical Methods
Algebraic Geometry
Signal Processing
Numerical Linear Algebra and Matrix Analysis
Continuous Optimization
Applied Combinatorics and Graph Theory
Computational Science
Network Analysis
Data Mining and Analysis

This file was based on the book "*The Princeton Companion to Applied Mathematics*"



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