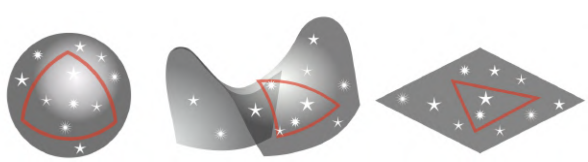
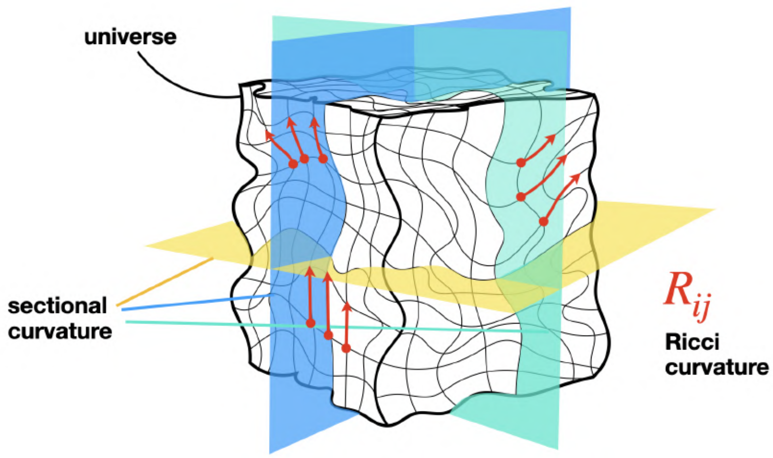
Gaussian Curvature

by DiBeos

Think of a very interesting space. A huge one – as the one depicted above – in which some parts of it look like a sphere (red). Other parts look like a saddle (blue), since they curve up in one direction and down in another. Some regions, though, are just perfectly flat (green). How can we measure these types of "warps"? How can we quantify this fundamental difference between each region? *Gaussian curvature* is the answer, and it has deep consequences, not only in pure mathematics but also in the future of our universe.



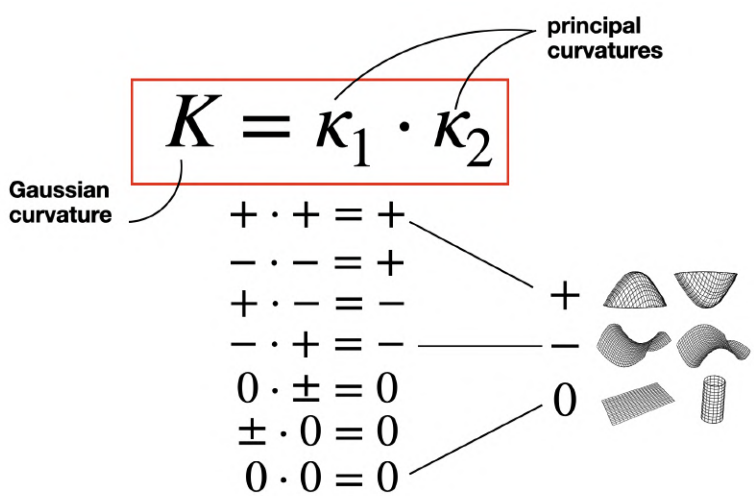
Gaussian curvature is usually denoted with a capital K. Even though Gaussian curvature is defined only for $2D$ surfaces, its core idea – so, measuring how a space bends – can be generalized to higher dimensions. In cosmology, we apply the same principles to $3D$ spaces (or $4D$ spacetime), using tools like *sectional curvature* and *Ricci curvature* to describe how the universe bends at each point.



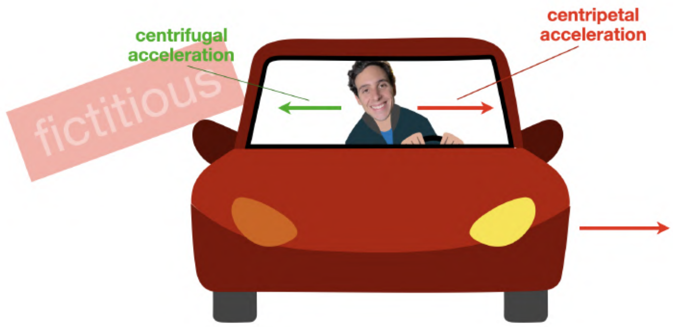
Gaussian curvature is defined as the product of two *principal curvatures*, denoted with the Greek letters κ_1 and κ_2 at a point:

$$K = \kappa_1 \cdot \kappa_2$$

The principal curvatures can be *positive*, *negative* or *zero*. Positive with positive is positive, negative with negative is positive, positive with negative is negative, and so on... At the end of the day, we notice that the Gaussian curvature can be positive, negative or zero as well.



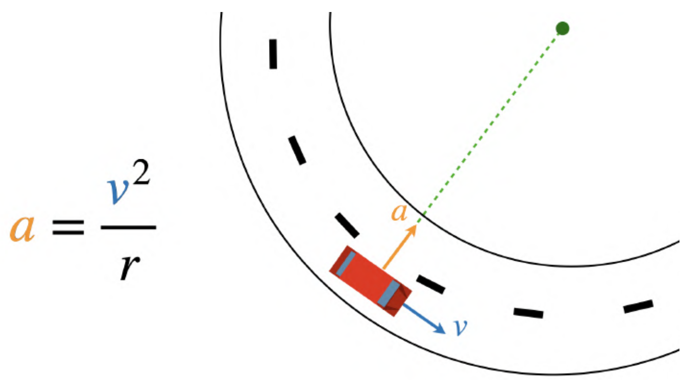
In order to understand what these principal curvatures are, and how to compute them, imagine the following scenario:



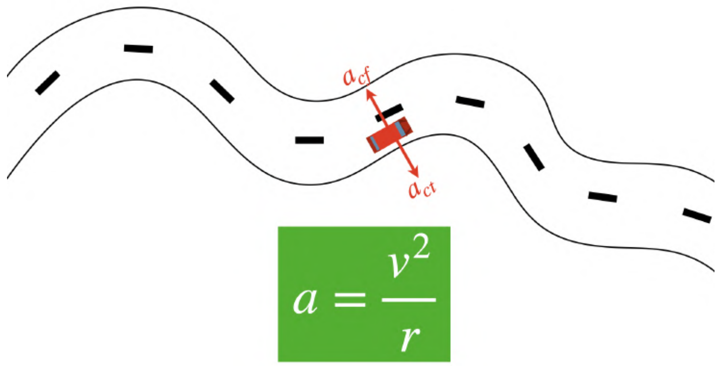
This is Luca in a car. Now, he decides to turn to the left. He will naturally feel a force pulling him to the right. The acceleration felt by Luca is called *centrifugal acceleration* in physics. This is actually a fictitious acceleration. The real acceleration here is the one inwards – the one that keeps the car, Luca, and everything inside of it, moving along the curved path. This is called *centripetal acceleration*, and is calculated as:

$$a = \frac{v^2}{r}$$

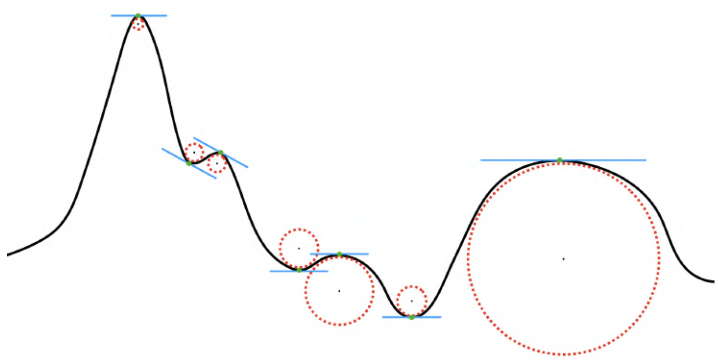
Where v here stands for *tangent velocity* and r for *radius.*



But that's the point here: the radius of what? Well, it must refer to a circle, right? But who said that Luca is moving along a circular path? It might very well be a random curved trajectory, and the formula $a = \frac{v^2}{r}$ still holds.

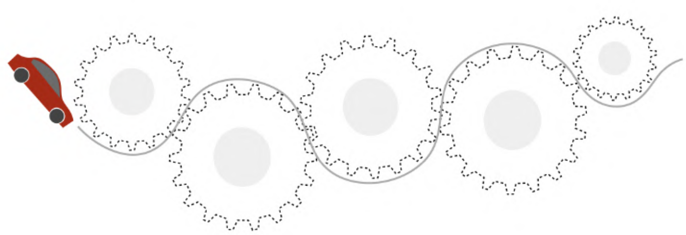


This is a very important concept in Differential Geometry, and it's called the *osculating circle*.

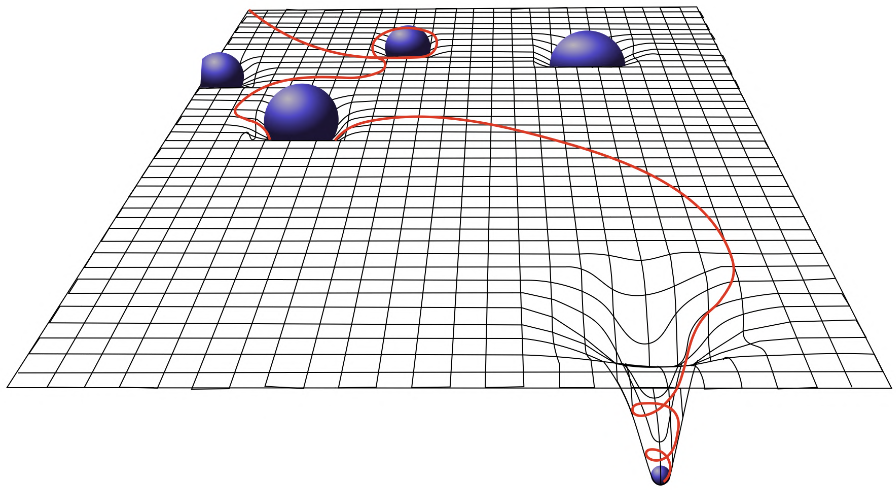


An osculating circle is the circle that best matches how a curve bends at a point – it shares the same position, the same tangent, and the same curvature. The smaller its radius, the tighter the bend – and the bigger the curvature. This connects to the formula $a = \frac{v^2}{r}$, which tells you how much acceleration is needed to stay on a curved path. Notice how the acceleration a and the radius r are inversely proportional $\left(a \propto \frac{1}{r^2}\right)$. So, the bigger the radius the less acceleration is needed, and vice-versa.

If you imagine moving along a curve, it's like constantly riding along a chain of osculating circles, and staying on track means always being pulled inward – that "pull" is what we feel as a force.

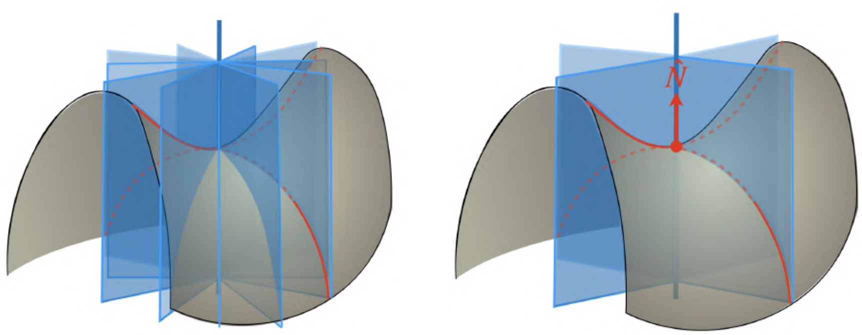


In physics, this shows up in cool ways – like in General Relativity, where gravity isn't a force but just the consequence of objects following curves in spacetime. Those curves act like osculating *hyperspheres* – and the tighter the bend, the more things feel pulled inward, just like in regular circular motion.

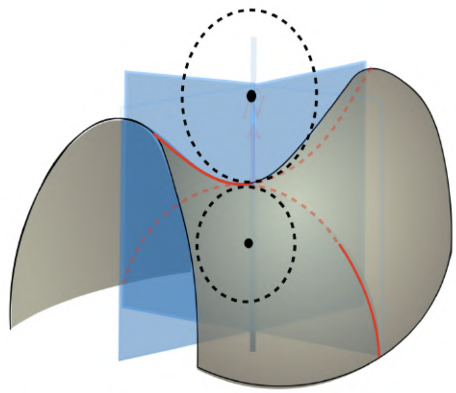


Going back now to the principal curvatures κ_1 and κ_2 of the Gaussian curvature formula ($K = \kappa_1 \cdot \kappa_2$), let's connect them to the visual concept of osculating circles.

You can cut the surface at any point with vertical planes that contain the normal vector to the surface at that point. Each slice creates a normal curve lying in that plane.



Each of these curves has an osculating circle that encodes the local curvature at that point. Out of all possible normal slices, there are two that are very special. The first is the one that produces the normal curve that bends the most (i.e., the one with the tightest osculating circle), which gives maximum curvature, and is called the "principal curvature κ_1". The second special kind of normal slice is the one that produces the normal curve that bends the least (i.e., the one with the flattest osculating circle), which gives minimum curvature, and is called the "principal curvature κ_2". These are real numbers.



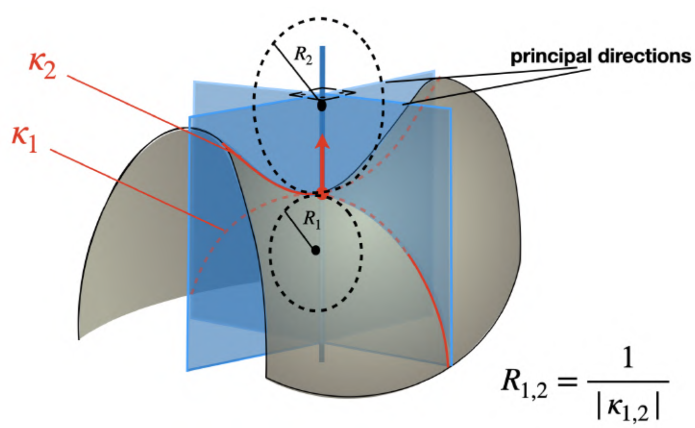
The directions in which they occur are called principal directions, and they can be proven to always be orthogonal.

At any point on the surface, in the principal directions, the surface curve through that point (i.e., the intersection of the surface with a plane containing the normal vector and the principal direction) can be locally approximated by an osculating circle.

In each principal direction, there is a plane curve lying on the surface. That curve has curvature equal to one of the $\kappa_{1,2}$, and so it has an osculating circle of radius:

$$R_{1,2} = \frac{1}{|\kappa_{1,2}|}$$

The osculating circles are the best-fitting circles to the surface in the directions of principal curvature.

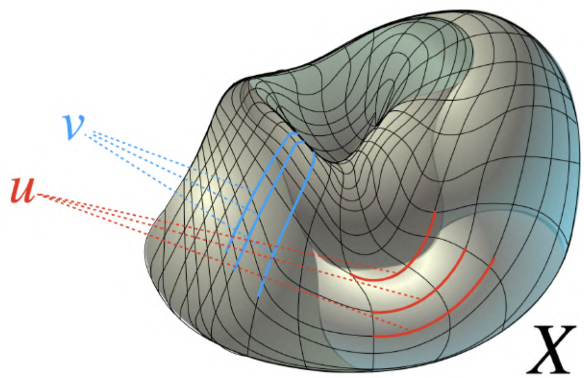


Good, but how do we calculate these real numbers κ_1 and κ_2? We are going to show you the step-by-step algorithm of how to do it. And right after that you can think of any random surface and try it out on your own! Anyway, we will also present a concrete example here so that you can better understand the procedure.

Step 1: Parametrization

Define your surface X as a function of two variables (*parametrization*):

$$X(u,v) = \{x(u,v)\,;\, y(u,v)\,;\, z(u,v)\,\}$$

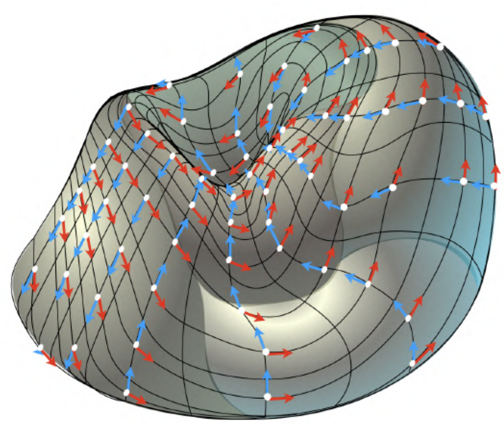


Step 2: Tangent Vectors

Compute the tangent vectors at each point by calculating the following partial derivatives:

$$\boxed{X_u = \frac{\partial X}{\partial u}} \qquad \boxed{X_v = \frac{\partial X}{\partial v}}$$

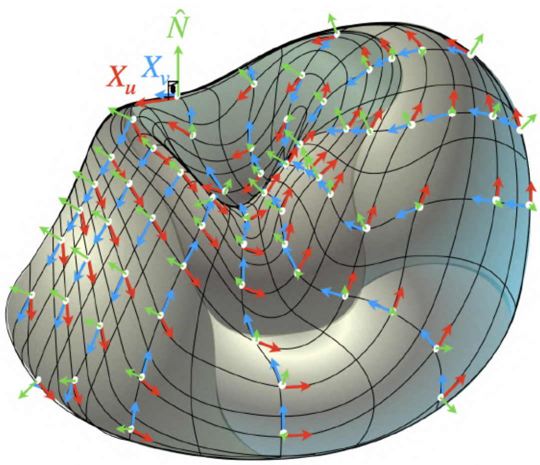
These vectors span the tangent plane at each point.



Step 3: Unit Normal Vectors

Compute the *normal vector* by taking the cross product of the tangent vectors, and then normalize (so that it has magnitude 1):

$$\hat{\mathbf{N}} = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

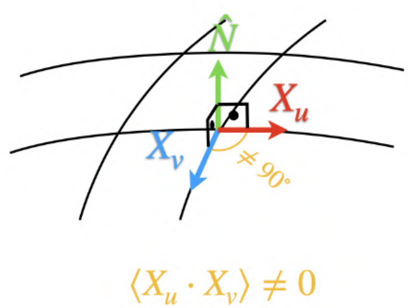


This is called the *unit normal* to the surface at each point.

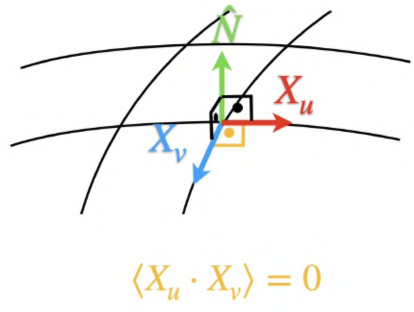
Step 4: First Fundamental Form

Compute the *first fundamental form*, denoted as I:

$$I = \begin{bmatrix} \langle X_u \, , \, X_u \rangle & \langle X_u \, , \, X_v \rangle \\ \langle X_v \, , \, X_u \rangle & \langle X_v \, , \, X_v \rangle \end{bmatrix}$$



X_u and X_v (coordinate tangent vectors) are **not** always orthogonal, i.e., it is possible that $\langle X_u \, , \, X_v \rangle \neq 0$, but when computing principal curvatures (κ_1 and κ_2), X_u and X_v form a special orthogonal basis. This means that (in this case) $\langle X_u \, , \, X_v \rangle = 0$.



Since we are not diving into the theory of Differential Forms here, we will just think of the first fundamental form as the "matrix version of the inner product" on the surface – it tells us how to compute lengths and angles. It is the metric tensor in the surface's coordinates.

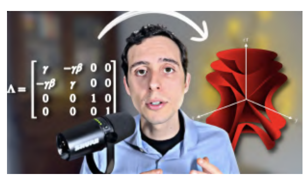
Just as, for example, in Special Relativity we represent the flat Minkowski spacetime as:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

And equivalently as a matrix:

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If you want to learn more about this specific metric, check out this video and PDF link below, where we build all these concepts from scratch in a very simple and intuitive way.



The Mathematics of Minkowski Spacetime

PDF link: Minkowski

Step 5: Second Fundamental Form

Compute the second fundamental form, denoted as II:

$$II = \begin{bmatrix} \hat{\mathbf{N}} \cdot X_{uu} & \hat{\mathbf{N}} \cdot X_{uv} \\ \hat{\mathbf{N}} \cdot X_{uv} & \hat{\mathbf{N}} \cdot X_{vv} \end{bmatrix}$$

Again, we will not get into the theory of Differential Forms here, but the second fundamental form is like "second directional derivatives". Think of it as taking a vector, differentiating it twice, and then extracting the component in the direction you are interested in – in this case this direction is given by the normal vector $\hat{\mathbf{N}}$.

Step 6: Shape Operator

Compute the *shape operator matrix*:

$$S = I^{-1} II$$

It describes how the normal vector changes as you move in different directions on the surface.

Step 7: Eigenvalues

Compute the *eigenvalues* of S:

So, if $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we calculate its eigenvalues through the following formula:

$$\kappa_{1,2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + b^2}$$

Let's see now how we got to this expression. But, before that, if you feel that you need to better understand what eigenvalues are, how to calculate them, and how to visually and intuitively interpret them, check out the video and PDF link below.

The Core of Eigenvalues & Eigenvectors

PDF link: Eigenvalues

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{\det(S - \kappa \, \mathrm{Id}) = 0} \implies \begin{vmatrix} a-\kappa & b \\ c & d-\kappa \end{vmatrix} = 0 \implies \kappa^2 - (a+d)\kappa + (ad - cb) = 0$$

$$\kappa_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-cb)}}{2} = \frac{a+d \pm \sqrt{a^2+2ad+d^2-4ad+4cb}}{2} =$$
$$= \frac{a+d \pm \sqrt{a^2-2ad+d^2+4cb}}{2} = \frac{a+d \pm \sqrt{(a-d)^2+4cb}}{2} =$$
$$= \frac{a+d \pm 2\sqrt{\left(\frac{a-d}{2}\right)^2+cb}}{2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2+cb}$$

Beyond that, there is the fact that the shape operator matrix S is *symmetric*: the off-diagonal terms (b and c) are always equal ($b = c$). This is just a direct consequence of the fact that this matrix was derived from two *symmetric bilinear forms* (I and II): $S = I^{-1}II$. It results in a *self-adjoint* (symmetric) linear map on the tangent space.

$$\therefore \quad \kappa_{1,2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2+b^2} \qquad \square$$

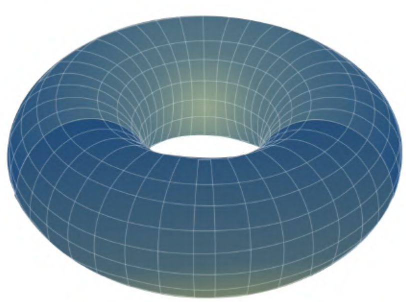
We finally did it! These eigenvalues are the *maximum* and *minimum* principal curvatures κ_1 and κ_2.

Now, all you have to do is multiply these two numbers, and then you get the famous Gaussian curvature at each point:

$$K = \kappa_1 \cdot \kappa_2$$

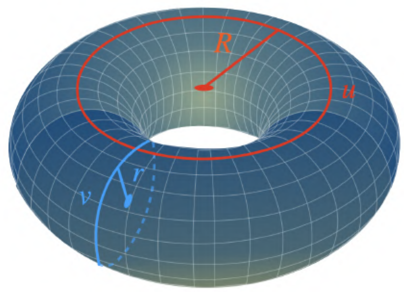
Awesome! Time to see a concrete example.

2-Torus: $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



Step 1: Parametrization

Let's parametrize the 2-torus surface in the following way:



$$T(u,v) = \begin{cases} x(u,v) = (R + r\cos v)\cos u \\ y(u,v) = (R + r\cos v)\sin u \\ z(u,v) = r\sin v \end{cases}$$

R is the *external* radius and r the *internal* one.

Attention! *Notice how this parametrization (and also the whole algorithm itself) assumes that the surface is embedded in a 3-dimensional Euclidean space. This is being used here to find a property, namely the Gaussian curvature, which is actually an intrinsic property. It is intrinsic because of the famous Gauss's Theorema Egregium (which is Latin for "Remarkable Theorem"). It basically means that Gaussian curvature shouldn't depend on the embedding. And this leads us to the following...*

Remark: Technically, you can compute Gaussian curvature without relying on an external ambient space (i.e., xyz-structure), as we are doing here. This could be done just with measurements within the surface itself (like angles, distances, and so on). However, in order to do so we would have to talk about *tensors*, and *Christoffel symbols*, which would be mathematically way more complex. So, there is a tradeoff here: it is easier to compute, and visualize, Gaussian curvature using an embedding because it avoids talking about tensors and Christoffel symbols, but unfortunately we sacrifice the core of Differential Geometry, i.e. the clear understanding of the fact that Gaussian curvature is an intrinsic property, and therefore can be calculated without an embedding – it is just mathematically harder.

Well, with that said, let's calculate Gaussian curvature by following the algorithm:

Step 2: Tangent Vectors

Compute the tangent vectors:

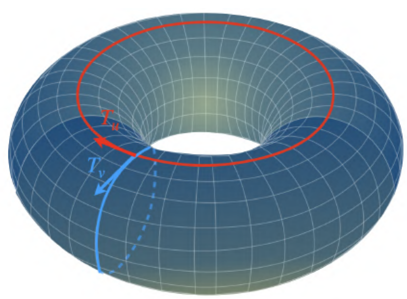
$$\boxed{T_u = \frac{\partial T}{\partial u}} \qquad \boxed{T_v = \frac{\partial T}{\partial v}}$$

$$\begin{aligned} T_u = \frac{\partial T}{\partial u} &= \left\{ \frac{\partial x}{\partial u}(x,y)\,;\ \frac{\partial y}{\partial u}(x,y)\,;\ \frac{\partial z}{\partial u}(x,y) \right\} = \\ &= \left\{ \frac{\partial}{\partial u}\left[(R + r\cos v)\cos u\right]\ ;\ \frac{\partial}{\partial u}\left[(R + r\cos v)\sin u\right]\ ;\ \frac{\partial}{\partial u}\left(r\sin v\right) \right\} = \\ &= \{-\,(R + r\cos v)\sin u\,;\ -\,(R + r\cos v)\cos u\,;\ 0\} \end{aligned}$$

$$\begin{aligned} T_v = \frac{\partial T}{\partial v} &= \left\{ \frac{\partial x}{\partial v}(x,y)\,;\ \frac{\partial y}{\partial v}(x,y)\,;\ \frac{\partial z}{\partial v}(x,y) \right\} = \\ &= \left\{ \frac{\partial}{\partial v}\left[(R + r\cos v)\cos u\right]\ ;\ \frac{\partial}{\partial v}\left[(R + r\cos v)\sin u\right]\ ;\ \frac{\partial}{\partial v}\left(r\sin v\right) \right\} = \\ &= \{-r\sin v\cos u\,;\ -r\sin v\sin u\,;\ r\cos v\} \end{aligned}$$

$$\therefore \qquad \boxed{T_u = \{-\,(R + r\cos v)\sin u\,;\ (R + r\cos v)\cos u\,;\ 0\}}$$

$$\boxed{T_v = \{-r\sin v\cos u\,;\ -r\sin v\sin u\,;\ r\cos v\}}$$



Step 3: Unit Normal Vector

Compute the unit normal vector:

$$\boxed{\hat{\mathbf{N}} = \frac{T_u \times T_v}{\|T_u \times T_v\|}}$$

$$T_u \times T_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(R + r\cos v)\sin u & (R + r\cos v)\cos u & 0 \\ -r\sin v\cos u & -r\sin v\sin u & r\cos v \end{vmatrix} =$$

$$= [(R + r\cos v)\cos u \cdot r\cos v - 0 \cdot (-r\sin v\sin u)]\ \hat{\mathbf{i}} + [0 \cdot (-r\sin v\cos u) - (-(R + r\cos v)\sin u \cdot r\cos v)]\ \hat{\mathbf{j}}$$
$$- [(R + r\cos v)\sin u \cdot (-r\sin v\sin u) - (R + r\cos v)\cos u \cdot (-r\sin v\cos u)]\ \hat{\mathbf{k}} =$$

$$= [r\cos u\cos v\,(R + r\cos v)]\ \hat{\mathbf{i}} + [r\sin u\ \cos v(R + r\cos v)]\ \hat{\mathbf{j}}$$
$$+ \left[Rr\sin v\,\underline{\sin^2 u} + r^2\cos v\,\underline{\underline{\sin^2 u}}\ \sin v + Rr\,\underline{\cos^2 u}\ \sin v + r^2\cos v\,\underline{\underline{\cos^2 u}}\ \sin v\right]\ \hat{\mathbf{k}} =$$

$$= [r\cos u\cos v\,(R + r\cos v)]\ \hat{\mathbf{i}} + [r\sin u\ \cos v(R + r\cos v)]\ \hat{\mathbf{j}} + \left[R\,r\sin v + r^2\cos v\sin v\right]\ \hat{\mathbf{k}} =$$

$$= [r\cos u\cos v\,(R + r\cos v)]\ \hat{\mathbf{i}} + [r\sin u\ \cos v(R + r\cos v)]\ \hat{\mathbf{j}} + [r\sin v\,(R + r\cos v)]\ \hat{\mathbf{k}}$$

$$\Longrightarrow \quad \boxed{T_u \times T_v = \{r\cos u\cos v\,(R + r\cos v)\ ;\ r\sin u\ \cos v(R + r\cos v)\ ;\ r\sin v\,(R + r\cos v)\}}$$

Let's denote $A := r\,(R + r\cos v)$:

$$\|T_u \times T_v\| = \sqrt{A^2\cos^2 v\cos^2 u + A^2\cos^2 v\sin^2 u + A^2\sin^2 v} =$$
$$= A\sqrt{\cos^2 v(\cos^2 u + \sin^2 u) + \sin^2 v} =$$
$$= A\sqrt{\cos^2 v + \sin^2 v} =$$
$$= A$$

$$\Longrightarrow \quad \boxed{\|T_u \times T_v\| = r\,(R + r\cos v)}$$

$$\hat{\mathbf{N}} = \frac{T_u \times T_v}{\|T_u \times T_v\|} = \frac{\{\cos u\cos v\,\cancel{r\,(R + r\cos v)}\ ;\ \sin u\ \cos v\,\cancel{r\,(R + r\cos v)}\ ;\ \sin v\,\cancel{r\,(R + r\cos v)}\}}{\cancel{r\,(R + r\cos v)}}$$

$$\therefore \qquad \hat{\mathbf{N}} = \{\cos v\cos u\ ;\ \cos v\sin u\ ;\ \sin v\}$$

Step 4: First Fundamental Form

Calculate the first fundamental form:

$$I = \begin{bmatrix} \langle T_u \, , \, T_u \rangle & \langle T_u \, , \, T_v \rangle \\ \langle T_v \, , \, T_u \rangle & \langle T_v \, , \, T_v \rangle \end{bmatrix}$$

$$\begin{aligned} \langle T_u \, , \, T_u \rangle &= \langle \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} \, , \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} \, \rangle = \\ &= (R + r\cos v)^2 \, \underline{\sin^2 u} + (R + r\cos v)^2 \, \underline{\cos^2 u} = \\ &= (R + r\cos v)^2 \end{aligned}$$

$$\begin{aligned} \langle T_u \, , \, T_v \rangle &= \langle \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} \, , \, \{-r\sin v\cos u \, ; \, -r\sin v\sin u \, ; \, r\cos v\} \, \rangle = \\ &= r \, \sin v \, \cos u \, \sin u \, (R + r\cos v) - r \, \sin v \, \cos u \, \sin u \, (R + r\cos v) = \\ &= 0 = \langle T_v \, , \, T_u \rangle \end{aligned}$$

$$\begin{aligned} \langle T_v \, , \, T_v \rangle &= \langle \, \{-r\sin v\cos u \, ; \, -r\sin v\sin u \, ; \, r\cos v\} \, , \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} \, \rangle = \\ &= r^2 \, \sin^2 v \, \underline{\cos^2 u} \, + \, r^2 \, \sin^2 v \, \underline{\sin^2 u} \, + \, r^2 \, \cos^2 v = \\ &= r^2 \, \underline{\sin^2 v} + r^2 \, \underline{\cos^2 v} = \\ &= r^2 \end{aligned}$$

$$I = \begin{bmatrix} (R + rcosv)^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

Step 5: Second Fundamental Form

Calculate the second fundamental form:

$$II = \begin{bmatrix} \langle \hat{\mathbf{N}} \, , \, T_{uu} \rangle & \langle \hat{\mathbf{N}} \, , \, T_{uv} \rangle \\ \langle \hat{\mathbf{N}} \, , \, T_{uv} \rangle & \langle \hat{\mathbf{N}} \, , \, T_{vv} \rangle \end{bmatrix}$$

$$\begin{aligned} T_{uu} &= \frac{\partial \, T_u}{\partial \, u} = \frac{\partial}{\partial \, u} \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} = \\ &= \{-\,(R + r\cos v)\cos u \, ; \, -\,(R + r\cos v)\sin u \, ; \, 0\} \end{aligned}$$

$$\begin{aligned} T_{uv} &= \frac{\partial \, T_u}{\partial \, v} = \frac{\partial}{\partial \, v} \, \{-\,(R + r\cos v)\sin u \, ; \, (R + r\cos v)\cos u \, ; \, 0\} = \\ &= \{r \, \sin v \, \sin u \, ; \, -r \, \sin v \, \cos u \, ; \, 0\} \end{aligned}$$

$$\begin{aligned} T_{vv} &= \frac{\partial \, T_v}{\partial \, v} = \frac{\partial}{\partial \, v} \, \{-r\sin v\cos u \, ; \, -r\sin v\sin u \, ; \, r\cos v\} = \\ &= \{-r \, \cos v \, \cos u \, ; \, -r \, \cos v \, \sin u \, ; \, -r \, \sin v\} \end{aligned}$$

$$
\begin{aligned}
\langle \hat{\mathbf{N}} \,,\, T_{uu} \rangle &= \langle \{\cos v \cos u \;;\; \cos v \sin u \;;\; \sin v\} \;,\; \{-(R + r\cos v)\cos u \,;\, -(R + r\cos v)\sin u \,;\, 0\} \rangle = \\
&= -(R + r\cos v)\cos u \cdot \cos v \cos u - (R + r\cos v)\sin u \cdot \cos v \sin u = \\
&= -(R + r\cos v)\cos v(\cos^2 u + \sin^2 u) = \\
&= -(R + r\cos v)\cos v
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{\mathbf{N}} \,,\, T_{uv} \rangle &= \langle \{\cos v \cos u \;;\; \cos v \sin u \;;\; \sin v\} \;,\; \{r\,\sin v\,\sin u \,;\, -r\,\sin v\,\cos u \,;\, 0\} \rangle = \\
&= r\sin v \sin u \cdot \cos v \cos u - r\sin v \cos u \cdot \cos v \sin u = \\
&= r\sin v \cos v(\sin u \cos u - \sin u \cos u) = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{\mathbf{N}} \,,\, T_{vv} \rangle &= \langle \{\cos v \cos u \;;\; \cos v \sin u \;;\; \sin v\} \;,\; \{-r\,\cos v\,\cos u \,;\, -r\,\cos v\,\sin u \,;\, -r\,\sin v\} \rangle = \\
&= -r\cos v \cos u \cdot \cos v \cos u - r\cos v \sin u \cdot \cos v \sin u - r\sin v \cdot \sin v = \\
&= -r\cos^2 v(\cos^2 u + \sin^2 u) - r\sin^2 v = -r(\cos^2 v + \sin^2 v) = -r
\end{aligned}
$$

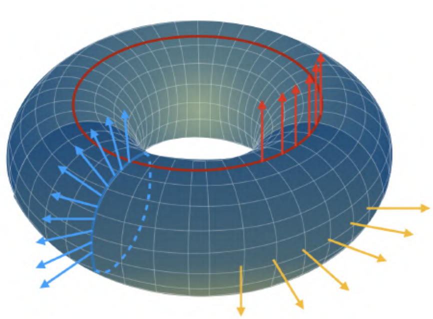
$$
II = \begin{bmatrix} -(R + r\cos v)\cos v & 0 \\ 0 & -r \end{bmatrix}
$$

Step 6: Shape Operator

Calculate the shape operator:

$$
S = I^{-1} II
$$

$$
S = \begin{bmatrix} \frac{1}{(R+r\cos v)^2} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix} \cdot \begin{bmatrix} -(R + r\cos v)\cos v & 0 \\ 0 & -r \end{bmatrix} = \begin{bmatrix} \frac{-\cos v}{R+r\cos v} & 0 \\ 0 & -\frac{1}{r} \end{bmatrix}
$$



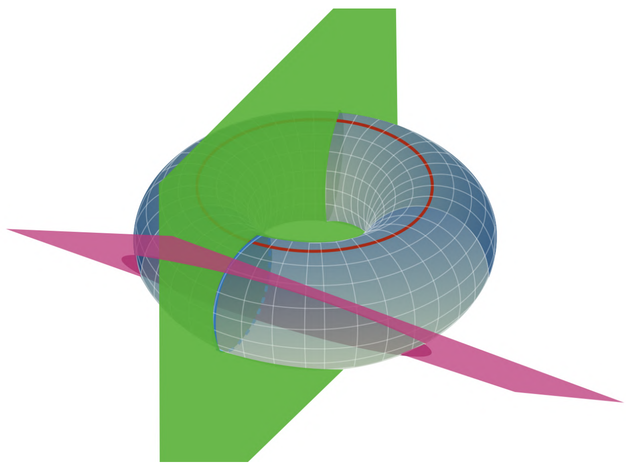
Step 7: Eigenvalues

Find the eigenvalues of the shape operator matrix:

$$S = \begin{bmatrix} \frac{-\cos v}{R+r\cos v} & 0 \\ 0 & -\frac{1}{r} \end{bmatrix}$$

Since it is already diagonalized, the eigenvalues correspond to the terms in the diagonal.

$$\begin{cases} \kappa_1 = \frac{-\cos v}{R+r\cos v} \\ \kappa_2 = -\frac{1}{r} \end{cases}$$

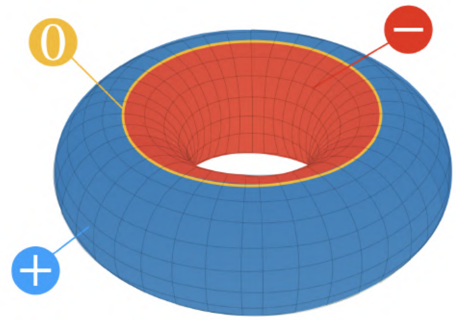


These are the principal curvatures!

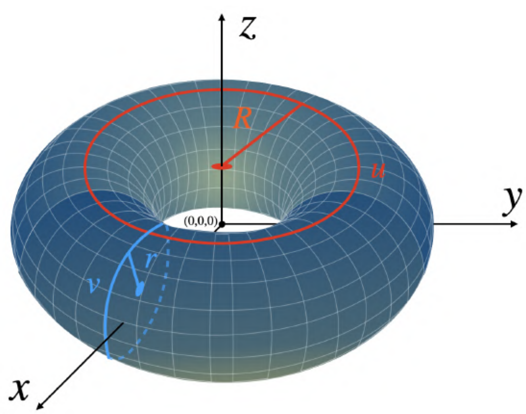
Finally, we can calculate the Gaussian curvature at each point (u, v) on the 2-torus:

$$K = \kappa_1 \cdot \kappa_2 = \frac{\cos v}{r\,(R + r\cos v)}$$

So, for example, for points "inside" of the doughnut shape, the Gaussian curvature is *negative*. For points "outside" it is *positive*. And the Gaussian curvature is *zero* (the surface is considered flat) only along these lines on top and on the bottom dividing the "inside" from the "outside" regions.



Dive deeper

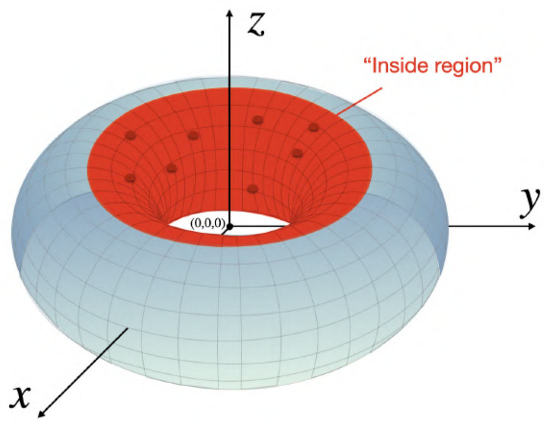


$$T(u,v) = \begin{cases} x(u,v) = (R + r\cos v)\cos u \\ y(u,v) = (R + r\cos v)\sin u \\ z(u,v) = r\sin v \end{cases}$$

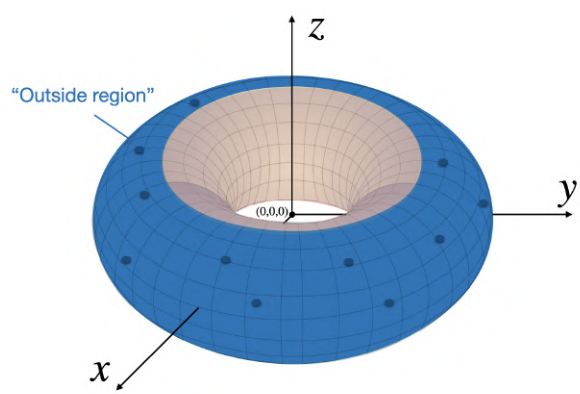
$$0 < r < R \qquad\qquad u, v \in [0, 2\pi)$$

$\rho(u,v) = \sqrt{x(u,v)^2 + y(u,v)^2}$ is the radial distance from the origin to any point on the surface, ignoring the z-component. Therefore, we can write it as $\rho(u,v) = R + r\cos v$.

In order to calculate the Gaussian curvature for a point in the "inside region" of the torus (i.e., the one closer to the origin), let's minimize the function $\rho(u,v) = R + r\cos v$.

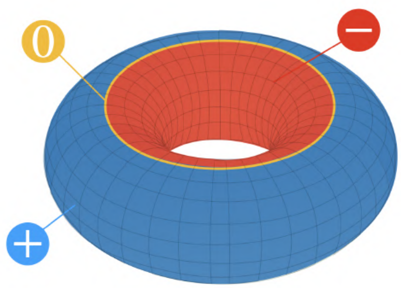


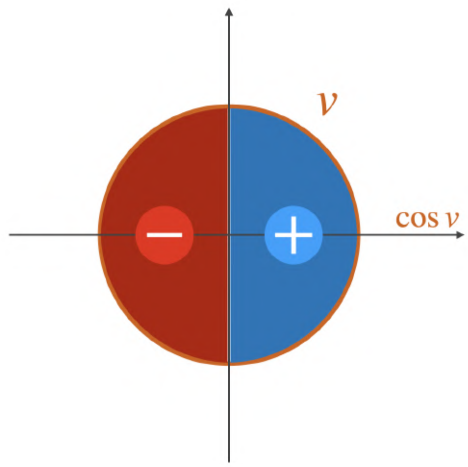
Since we are supposing that the radii R and r are fixed (i.e., constant) the minimal values of $\rho(u,v)$ take place when $\cos v = -1$, i.e. $v = \pi$ $(\rho(u,\pi) = R - r)$. When $\cos v = 0$ $\left(v = \frac{\pi}{2}\right)$, we have that $\rho(u, \frac{\pi}{2}) = R$. And when $\cos v = 1$ $(v = 0)$, we have that $\rho(u,0) = R + r$ – maximum distance from the origin, i.e. the "outside region" of the torus.



This naturally let's us separate the surface into 3 regions:

$$\begin{cases} \cos v > 0 \iff v \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \\ \cos v = 0 \iff v \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \\ \cos v < 0 \iff v \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$



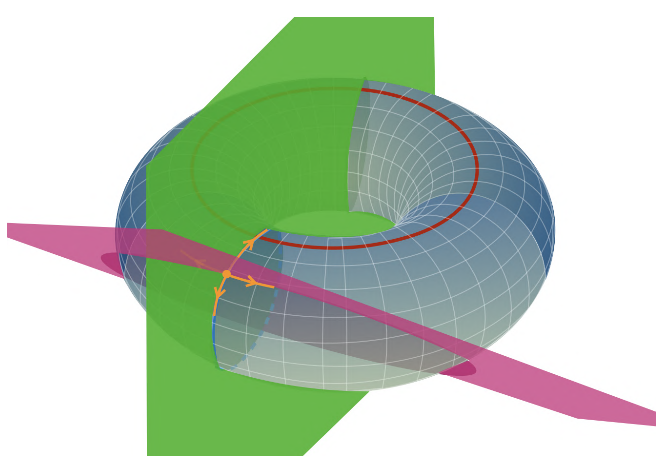


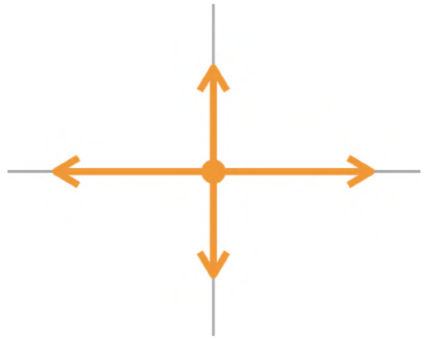
Indeed! If we calculate the Gaussian curvature using the formula that we've found earlier we get:

$$K = \frac{\cos v}{r(R + r\cos v)}$$

- $\cos v > 0$ → $K > 0$
- $\cos v = 0$ → $K = 0$
- $\cos v < 0$ → $K < 0$

This behavior can also be explained using the principal curvatures k_1 and k_2, which (for a point in the "outside region") looks like this:





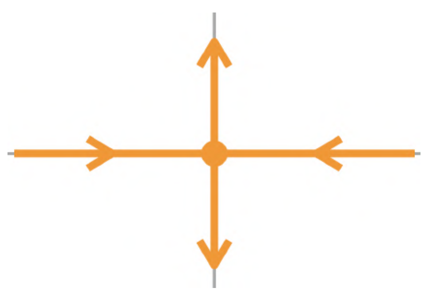
Notice how it curves "down" in both directions. This representation shows arrows pointing in the principal directions, which align with the axes of the osculating circles at that point. This is called a *cap point*, or an *elliptic point*. One direction follows the tightest bend (smaller radius) – which produces maximum curvature. The other direction (perpendicular to the previous one), on the other hand, follows the flattest bend (large radius) – which gives minimum curvature. These two curvatures have the same sign:

$$\kappa_1 = -\frac{\cos v}{R + r\cos v} < 0 \quad (\text{since } \cos v > 0 \text{ and } R + r\cos v > 0)$$

$$\kappa_2 = -\frac{1}{r} < 0 \quad (\text{since } r > 0)$$

$$\therefore \; K = \kappa_1 \cdot \kappa_2 > 0$$

And that's why the surface has positive Gaussian curvature at cap points. For a point in the "inside region", the planes of principal curvatures look like this:



Once again, the principal directions align with the axes of the osculating circles. This is called a *saddle point*. The principal curvatures have opposite signs:

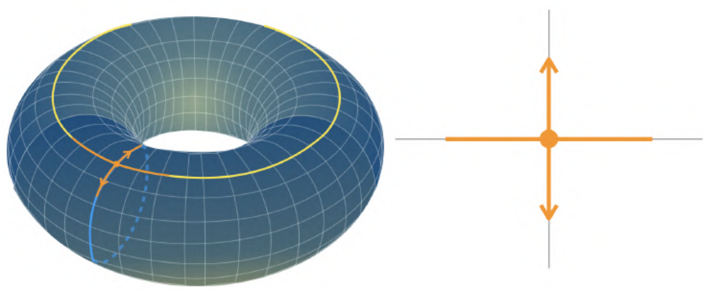
$$\kappa_1 = -\frac{\cos v}{R + r\cos v} > 0 \quad (\text{since } \cos v < 0 \text{ and } R + r\cos v > 0)$$

$$\kappa_2 = -\frac{1}{r} < 0 \quad (\text{since} \quad r > 0)$$

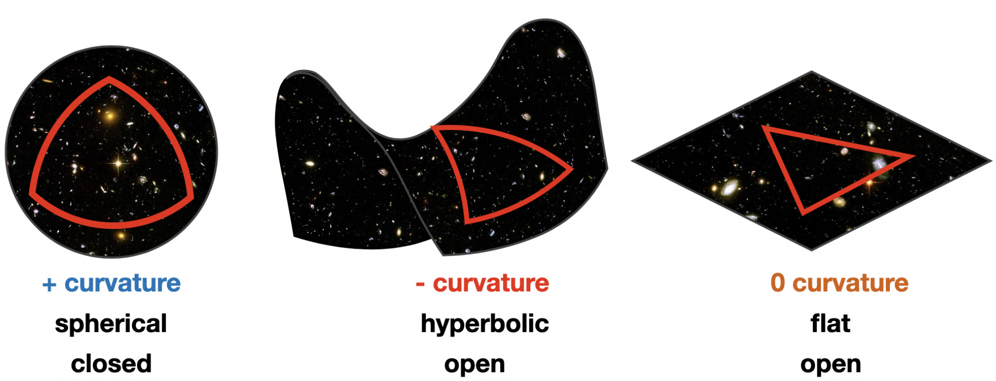
$$\therefore \; K = \kappa_1 \cdot \kappa_2 < 0$$

And that's why the surface has negative Gaussian curvature at saddle points.

Points in these circular "yellow" regions (on top and on the bottom) are called *parabolic*, or *flat direction points*, where the surface bends in one direction but stays flat in the other.



The sign of curvature isn't just mathematical curiosity — it has profound implications for the geometry of our universe. In this context, we move beyond Gaussian curvature, which applies only to $2D$ surfaces, and instead use higher-dimensional generalizations to describe the curvature of $3D$ space or $4D$ spacetime. These include Ricci curvature, sectional curvature, and scalar curvature – each capturing different aspects of how space bends.



In cosmology, the large-scale shape of space itself could be spherical (positive curvature) – which would be closed – or hyperbolic (negative curvature) – which is open – or flat (zero curvature) – also open. This is important because it will determine whether the universe will eventually collapse (spherical), expand forever (hyperbolic), or eventually find equilibrium (flat). Current observations (like those from the cosmic microwave background) suggest that the universe is very close to flat, but we still can't say with certainty, because, as you can imagine, this is very difficult to measure. But in any case, understanding curvature isn't just "math for fun" – it's about understanding the fate of the universe.

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