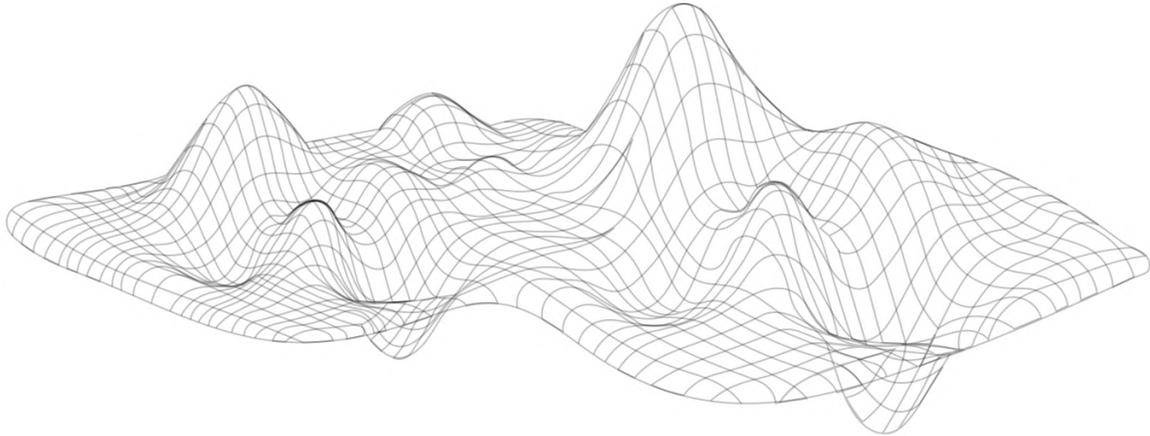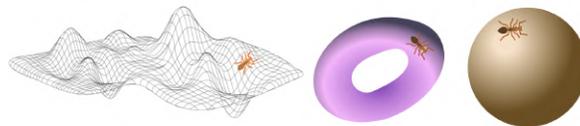# Riemannian Manifolds

by DiBeos

This is a space that looks "simple", but it has some extremely powerful properties. The first one will be expressed through the following illustration. Imagine that you are an ant walking around a giant, weirdly shaped object. Say, the surface of a doughnut, or a globe, or something funkier.

Even if that shape is complex and curvy, up close it always feels like you are walking on a flat, ordinary 2-dimensional space. A *manifold* is just a space like that, where every small patch looks like a flat space (like $\mathbb{R}^n$) if you zoom in enough. If you want to know more, check out the following video on the channel, and the PDF link below.

**How to Get to Manifolds Naturally**

Now, C^∞ means that it is *smooth* – you can do calculus on it! You can take derivatives as many times as you like, and everything behaves nicely (no sharp corners, no jumps). Think of this like saying you can draw curves and do math on them, without any weird breaks.
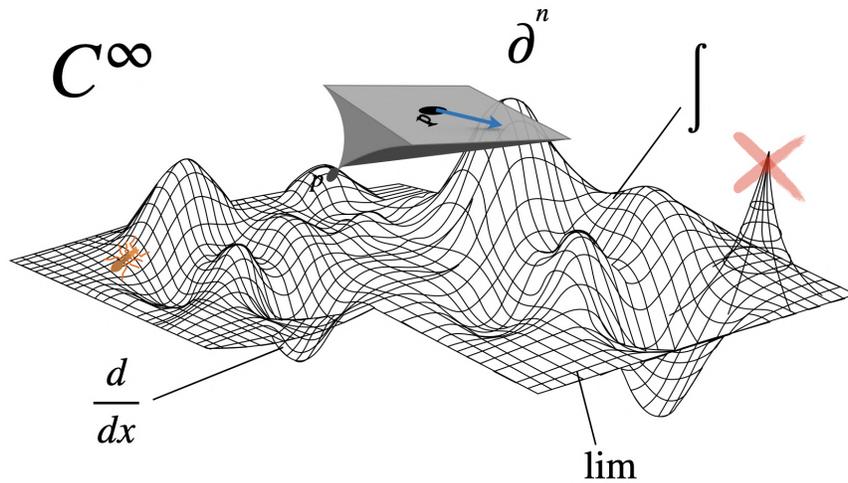

Another important property is that this space is *Hausdorff*. This just means that you can separate two different points with their own "bubble" (that is, open sets that do not overlap). In a regular space (like the room you are probably in), if you pick two points, you can create a little ball around each, such that they don't touch. That's Hausdorff. So, it avoids "weird spaces" where points cling together in strange ways.
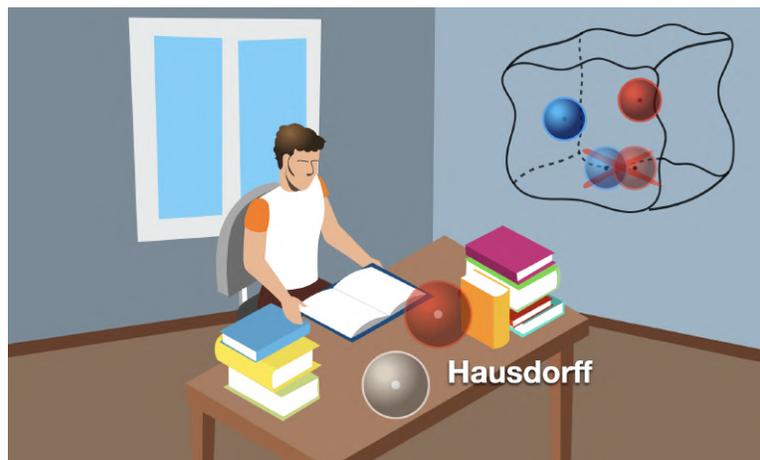

In order to appreciate this property which we very often take for granted, let's see a counterexample, such a "weird space" (so, non-Hausdorff), is called the two *touching planes*:
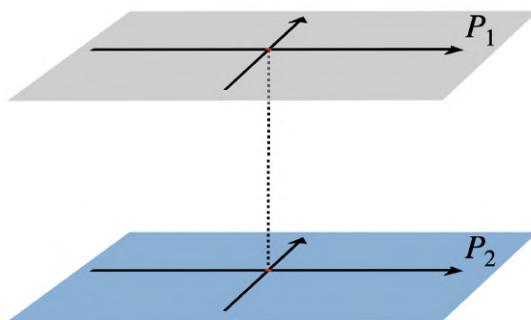
Imagine two copies of the plane $\mathbb{R}^2$, say P_1 and P_2. Now, glue them together along everything except the origin. That means all points $(x, y) \neq (0, 0)$ in P_1 and P_2 are identified – they are "the same". But the origin in each plane – $(0, 0)_1$ and $(0, 0)_2$ – are kept as separate points. So, now you have two origins, but everything else is merged into a single copy.
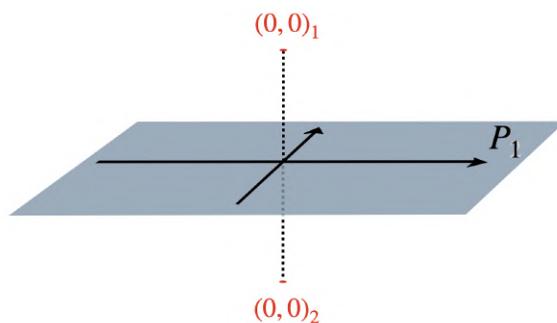

Let's try to create two open disks (or *neighborhoods*): one around $(0, 0)_1$ and another around $(0, 0)_2$. The problem is, no matter how small the disks are, they will always share all the same points except the origin. So, the disks always overlap! This makes it is impossible to separate $(0, 0)_1$ and $(0, 0)_2$ with non-overlapping open sets. I.e., the space is not Hausdorff.
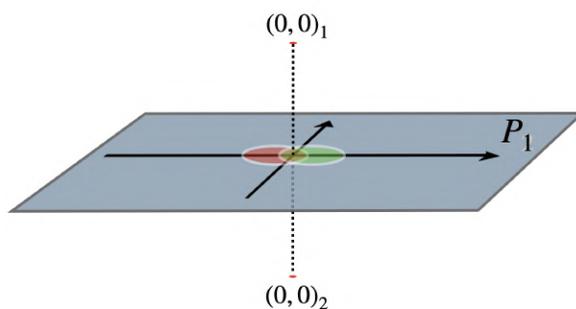
Every *Riemannian manifold* must be Hausdorff, so you have to be able to separate distinct points using open neighborhoods.
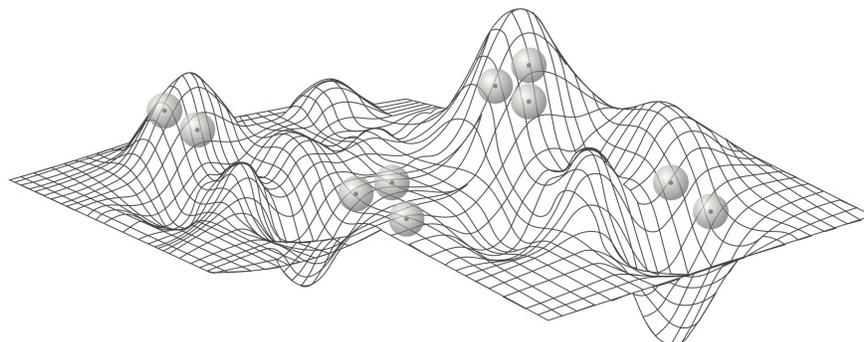
Another property that gets many people confused, but is super important, is the following: Think of a space (like the real plane $\mathbb{R}^2$) as a huge LEGO world. The open sets are like LEGO bricks that you use to build any shape or structure (any neighborhood).
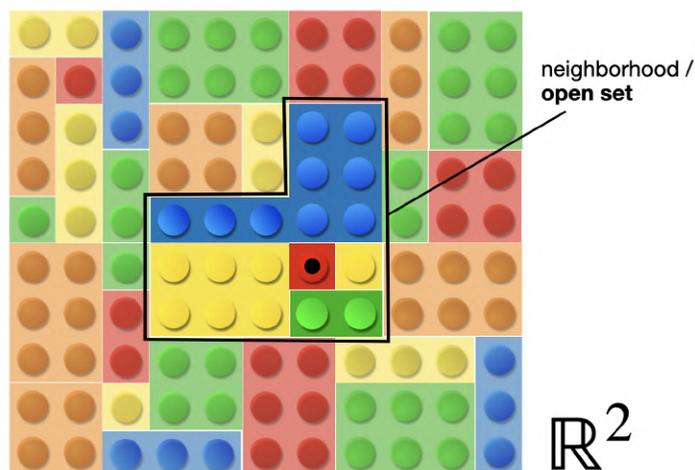
But just like in LEGO, you don't need every possible shape – you just need a collection of small pieces that you can combine. That collection is the intuition behind a *basis* in a topological space.
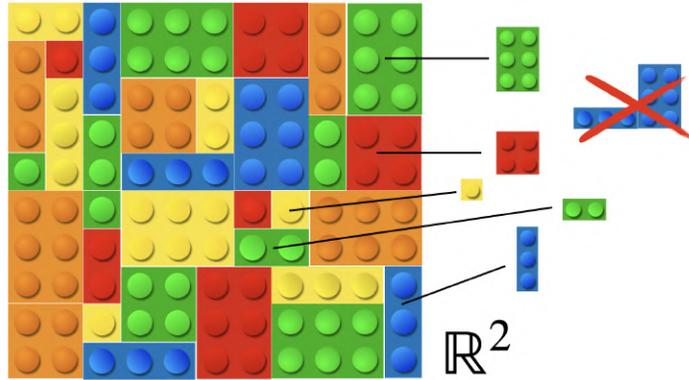
A basis is a special collection of open sets that acts like the building blocks of spaces. Now, we want this set of basic building blocks (open sets) to be *countable*. That means you could make a list of them. Even if it's infinite, the list behaves like counting $1, 2, 3, 4, ...$

If you want to dive deeper into exactly what countable means, both from the intuitive and rigorous point of view, check out the video and PDF link below.

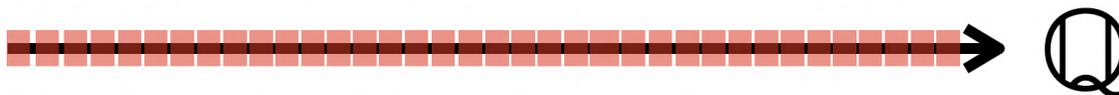

Are Dense and Uncountable Sets Different?

PDF link: *dense vs. uncountable*

Anyway, this property is better expressed by saying that the space is *second countable*. In other words, there exists a countable list of open sets such that any open set in the space can be made by "gluing" some of them together. Why does it matter? Being second countable ensures that the space is not "too big" in a topological sense. Also, it guarantees that you can use sequences instead of more complicated tools.

The classic example of a second countable space is $\mathbb{R}^n$ – the real n-dimensional space – even though the set of real numbers $\mathbb{R}$ itself is *uncountable*. You can make a countable collection (e.g., a list) of all open balls with rational center coordinates and rational radius. This is possible because, despite the fact that the set rational numbers $\mathbb{Q}$ is *dense*, it is countable.

$$B_{r_1}(q_1, \ldots q_n)$$

$$B_{r_2}(\ldots)$$

$$B_{r_3}(\ldots)$$

$$\vdots$$
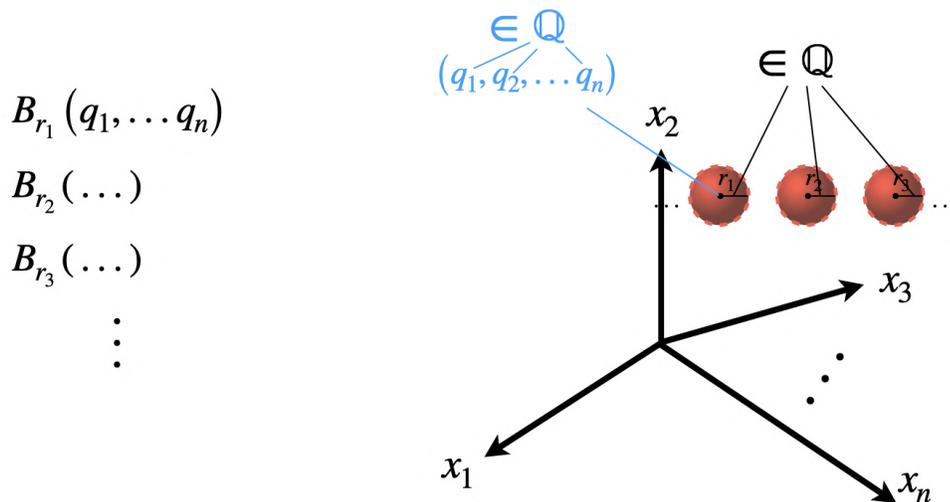
$$\mathbb{Q} = \left\{ \frac{n}{m} \in \mathbb{R} \;\middle|\; n, m \in \mathbb{Z}, m \neq 0 \right\} = \{q_1, q_2, q_3, \ldots\}$$

Of course, this depends not only on the space itself, but mainly on the kind of topology that we choose (so, the definition of open sets that we are using).

A counterexample would be the set $(\mathbb{R}, \tau)$, with the *discrete topology* τ. This topology defines open sets as subsets of $\mathbb{R}$ with only one element – called singletons:

$$\{x\} \;,\; \forall x \in \mathbb{R}\;,\; \textit{is an open set.}$$

This is certainly not countable, because if it were then the real numbers themselves ($\mathbb{R}$) would be countable as well.

$$\mathbb{R} = \left\{ 1\;,\; 7\;,\; -3\;,\; \pm\sqrt{2}\;,\; \pi\;,\; \pm e\;,\; 7.9999\ldots\;,\; \ldots \right\}$$

$\{1\}\quad\{7\}\quad\{-3\}\quad\left\{+\sqrt{2}\right\}\quad\{\pi\}\quad\{+e\}\quad\{7.9999\ldots\}$

$\{-e\}$

$\left\{-\sqrt{2}\right\}$

And we know that the set of real numbers is uncountable.

So, these are the properties that we've seen so far – the space is:

- Locally Euclidean (**manifold**);

- **Smooth** – you can do calculus in it (C^∞);

- Well-behaved – no points stuck together in a "weird" way (**Hausdorff**);

- Not "too big" or "too messy" (**Second countable**).

So far, this is the precise definition of a *smooth manifold*.

Remark: *In Differential Geometry, the definition of a manifold implicitly assumes the space to be Hausdorff and second countable.*

However, in order to build a Riemannian manifold, we need one more thing: a *Euclidean inner product* on each tangent space of the manifold. But, first of all, what is an inner product? It's a mathematical tool that allows you to measure lengths of vectors, angles between them, and say how much one vector "points in the direction" of another. It's like giving your space a way to talk about geometry – even if it's curved or abstract.
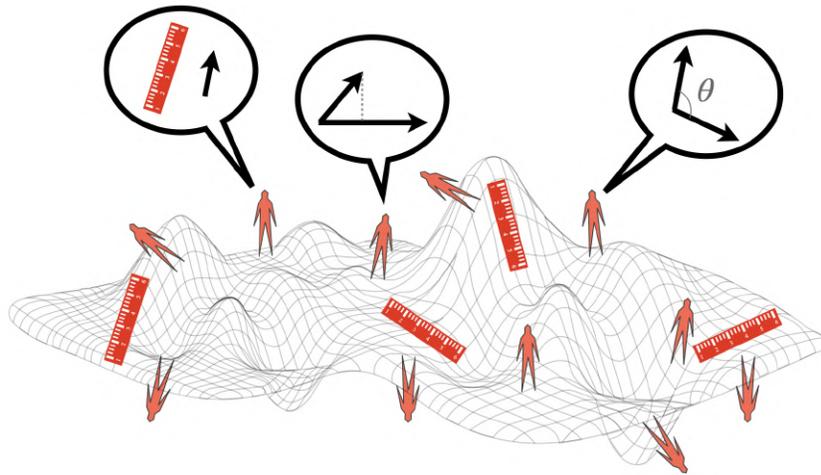

You should think of the inner product as a "directional ruler". It doesn't just tell you how long something is. It also tells you how aligned two things are. Imagine two vectors in space. If they point in the same direction, it produces a *large positive* inner product. If they are at 90°, the inner product is *zero*. If they point in opposite directions, the inner product is very *negative*.
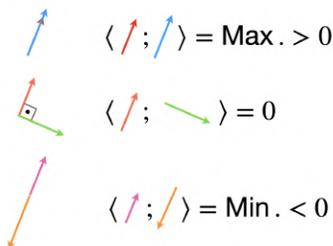
Let's introduce a little bit more rigor to the discussing with a specific example:

The two vectors are $\vec{v} = (x_1, y_1)$ and $\vec{w} = (x_2, y_2)$. Then, the standard inner product (which in this particular case is also called *dot product*) is:

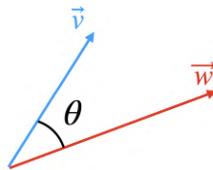$$\langle \vec{v}, \vec{w} \rangle = \langle (x_1; y_1), (x_2; y_2) \rangle = x_1 x_2 + y_1 y_2$$
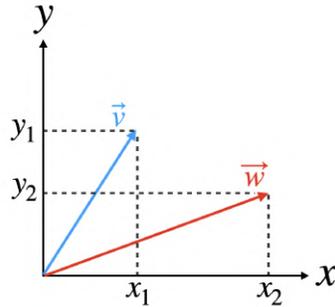
Which implies that the length of the vector $\vec{v}$ can be calculated as:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\langle (x_1; y_1), (x_1; y_1) \rangle}$$
$$= \sqrt{x_1^2 + y_1^2} \quad \text{(Pythagorean theorem)}$$

And the angle formed between $\vec{v}$ and $\vec{w}$ is calculated using the cosine formula:

$$cos\theta = \frac{adjacent\ side}{hypotenuse} = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$$
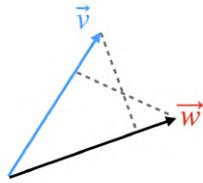

But in a Riemannian manifold, we also require the inner product to be Euclidean. Thus, three conditions must be satisfied – the inner product must be:

(1) **Symmetric**:

$$\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$$

This means that projecting the first vector's length onto the second, or vice-versa, produces the same result.

$$\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$$

(2) **Bilinear**:

$$\langle a\vec{v} + b\vec{w}, \vec{z} \rangle = a \langle \vec{v}, \vec{z} \rangle + b \langle \vec{w}, \vec{z} \rangle \quad , \quad a, b \in \mathbb{R}$$

In words, if you scale one vector by a and another by b, then add them, and project the result onto a third vector $\vec{z}$, you get the same outcome as projecting each vector onto $\vec{z}$ first, scaling those projections, and then adding them together.
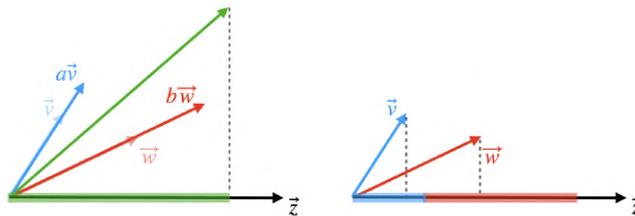
(3) **Positive-definite**:

$$\langle \vec{v}, \vec{v} \rangle > 0 \ , \quad \forall \vec{v} \neq \vec{0}$$

$$\sqrt{\langle \vec{v}, \vec{v} \rangle} \in \mathbb{R}^+ \ \text{(length)}$$
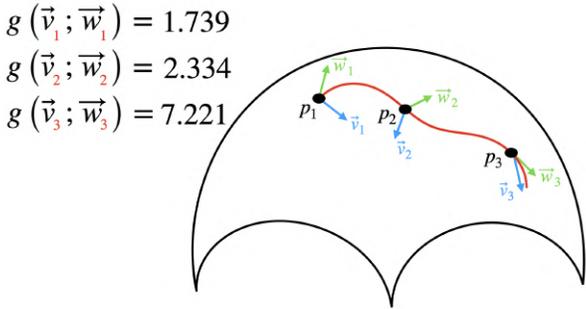
Think of the Euclidean inner product as a "normal ruler". It works just like measuring things on flat paper, or in the real-world $3D$ space. Non-Euclidean inner products are those that fail at least one of these three conditions, like in pseudo-geometries (as in *General Relativity*).

Since the Euclidean inner product only works naturally in flat spaces, Differential Geometry extends it to curved spaces by defining it locally – specifically, on the tangent space at each point. This is possible because manifolds are locally Euclidean and (can be made) smooth, which ensures that a well-defined, unique tangent space exists at every point (no sharp corners or discontinuities here). This way, we can still enjoy the benefits of the Euclidean inner product – like measuring lengths, angles and curvature – while applying it to more general, curved spaces.
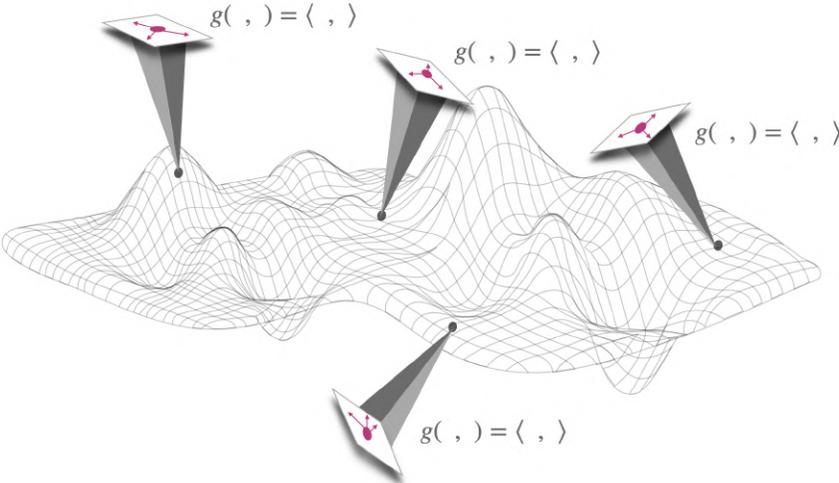

So, there you go, the complete definition of a Riemannian manifold:

A space (M, g), which is a C^∞ manifold M (i.e., Hausdorff and second countable), equipped with a Euclidean inner product g_p on each of the tangent spaces $T_p M$ of M, $\forall$ points $p \in M$. In addition, g has the nice property of varying smoothly from point to point.
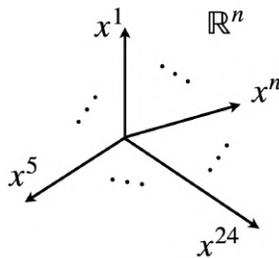
$$g\left(\vec{v}_1; \overrightarrow{w}_1\right) = 1.739$$
$$g\left(\vec{v}_2; \overrightarrow{w}_2\right) = 2.334$$
$$g\left(\vec{v}_3; \overrightarrow{w}_3\right) = 7.221$$

These are some quick concrete examples of Riemannian manifolds:

(1) $\left(\mathbb{R}^n \, ; \, g_{\mathbb{R}^n} \right)$
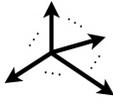
(2) $\left(V \, ; \, g \right)$
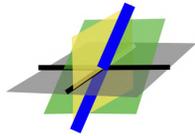
(3) $\mathbb{S}^n (R)$

(4) $\mathbb{H}^n$
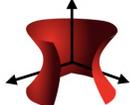
(5) $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

(Let us know if you guys would like a video and PDF summary only discussing in detail each of these examples, or maybe just one of them in particular. You can contact us in dibeos.contact@gmail.com)
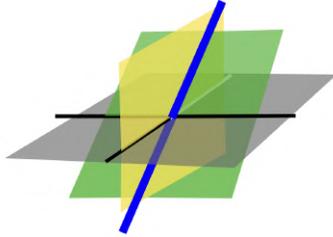
(1) **Euclidean space** $(\mathbb{R}^n, g_{\mathbb{R}^n})$

The simplest and most fundamental Riemannian manifold is the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n})$, where:

$$g_{\mathbb{R}^n} \left((p, \vec{v}), (p, \vec{w}) \right) := \vec{v} \cdot \vec{w} \, , \quad p \in \mathbb{R}^n$$

(2) **Vector space** (V, g)

Any finite-dimensional vector space with an inner product such that:

$$g\left((p; \vec{v}), (p; \vec{w})\right) := \vec{v} \cdot \vec{w}$$

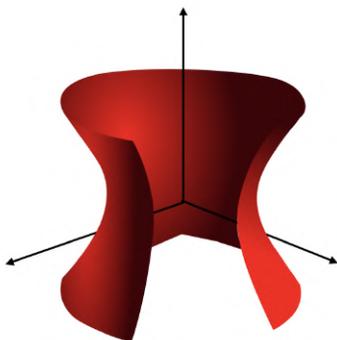(3) **Euclidean sphere $\mathbb{S}^n(R)$**

The Euclidean sphere of radius R:

$$\mathbb{S}^n(R) = \left\{x \in \mathbb{R}^{n+1} \,\middle|\, |x| = R\right\}$$

The inner product induced from the embedding $\mathbb{S}^n(R) \hookrightarrow \mathbb{R}^{n+1}$ is the canonical inner product of $\mathbb{S}^n(R)$, i.e. $g_{\mathbb{R}^n}$.

(4) **Hyperbolic space $\mathbb{H}^n$**

The hyperboloid model of hyperbolic space $\mathbb{H}^n$:

(The 2-dimensional hyperboloid above is obviously cut open so that you can see it inside and understand its shape – the "correct" one will actually look like this: 2D hyperboloid)

First we define the ambient space $\mathbb{R}^{n+1}$, with the appropriate inner product, where the hyperboloid $\mathbb{H}^n$ is embedded. Points of $\mathbb{R}^{n+1}$ are $(x_1, x_2, ..., x_n, x_{n+1})$, equipped with the inner product: $(x, y \in \mathbb{R}^{n+1})$

$$g_M(x, y) = x_1\, y_1 + x_2\, y_2 + ... + x_n\, y_n - x_{n+1}\, y_{n+1}$$

This specific inner product is called the *Minkowski metric*, and it's not positive-definite since it allows for negative "square lengths" – i.e., $g_M(x, x)$ might be negative. We say that this inner product is indefinite, and thus $(\mathbb{R}^{n+1}, g_M)$ is NOT a Riemannian manifold, but instead it is a *pseudo-Riemannian* manifold, or *Lorentzian* manifold. In special relativity, for example, if $g_M(x, x) < 0$ in the spacetime diagram, then x is a spacelike vector – which corresponds to a path that would require faster-than-light travel (region outside of the light-cone below).
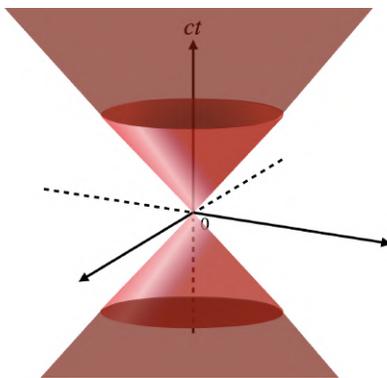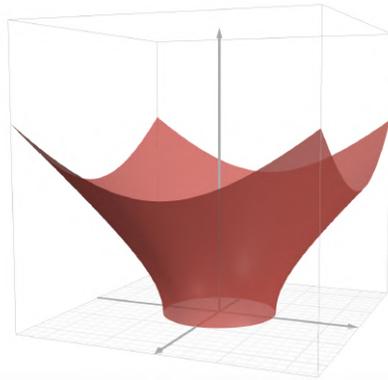

If you guys want to learn more about Minkowski geometry, which is the foundation of special relativity, check out the video and PDF link below:

The Mathematics of Minkowski Spacetime

The hyperboloid, though, will be defined just as the upper sheet:

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x_1^2 + x_2^2 + \, ... \, + x_n^2 - x_{n+1}^2 = -1 \,,\; x_{n+1} > 0 \right\}$$

The metric g_M on $\mathbb{H}^n$ is the metric induced from the ambient Minkowski space. At each point $p \in \mathbb{H}^n$, define the inner product on the tangent space $T_p\mathbb{H}^n$ by restricting the Minkowski inner product to $T_p\mathbb{H}^n$.

Despite the fact that the ambient space has an indefinite inner product, the restriction to the tangent spaces of the hyperboloid yields a positive-definite inner product – making $\mathbb{H}^n$ a Riemannian manifold.
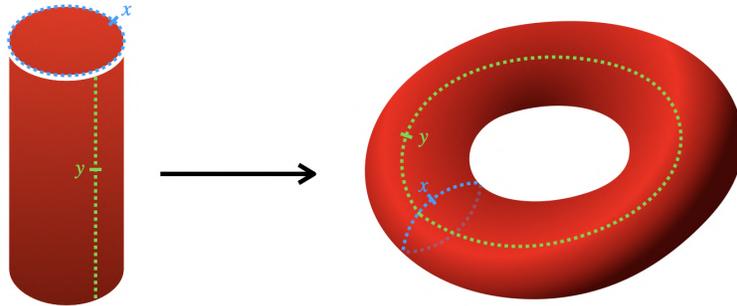
(5) **Torus** $\mathbb{T}^2$
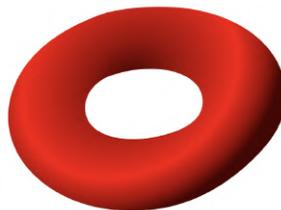
The manifold is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$: "the 2-torus, defined as the quotient of $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$".

Basically, we take $\mathbb{R}^2$ and identify points that differ by integer vectors:

$$(x,y) \sim (x+m, y+n) \ , \quad \forall m, n \in \mathbb{Z}$$

We define a Riemannian metric g on $\mathbb{T}^2$ as:

$$g = dx^2 + dy^2$$

This inner product on $\mathbb{T}^2$ is inherited from the standard Euclidean inner product on $\mathbb{R}^2$, and this process is formally described by the concept of taking the metric and *pushing it forward* via the quotient map.

Check out our website for more: https://dibeos.net