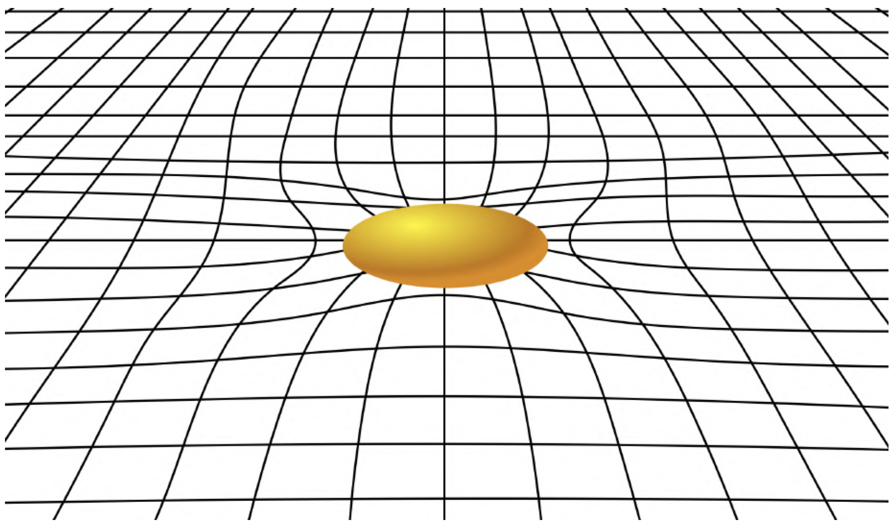
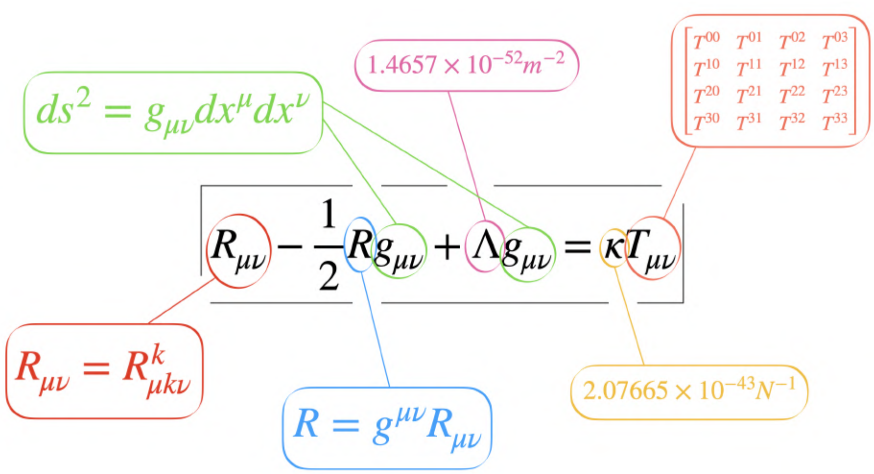
Conformal Geometry

by DiBeos

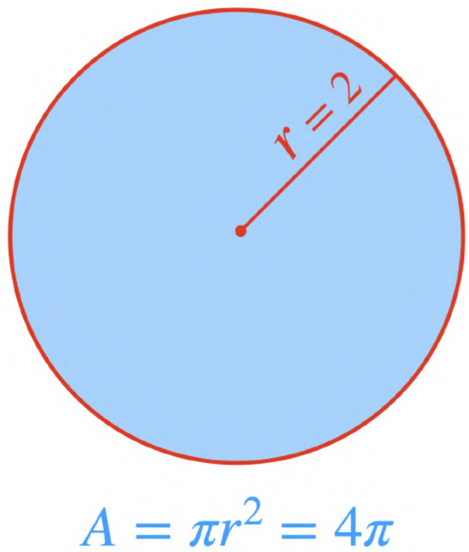


Spacetime is not necessarily curved around massive objects. There is no way to look ONLY at the mathematics of General Relativity and say (for sure) that matter produces curvature of the 4-dimensional manifold model of spacetime.

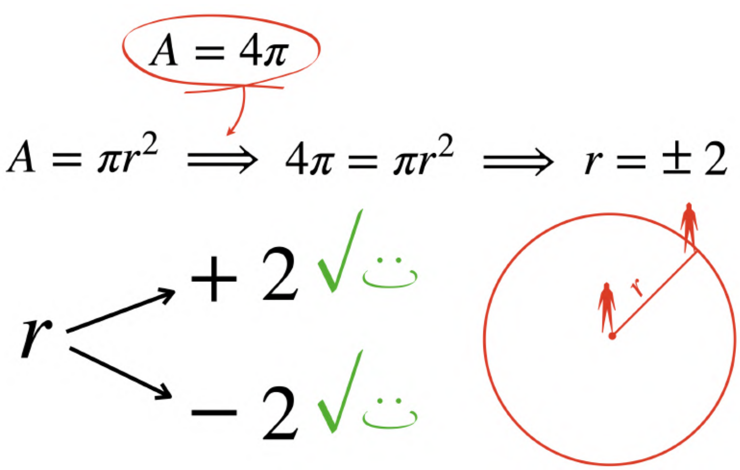


At least not using the definition of curvature adopted by most people, including some mathematicians and physicists themselves. The formalism used to describe spacetime in General Relativity is given (mostly) by Differential Geometry. But the interpretations of the purely mathematical results require more than math itself. They require physics, or even philosophy, if you will.

Let's see an illustration of what I mean. Suppose you want to measure the area of a circle of radius r. If you know the value of r (let's say $r = 2$), then you can calculate the area with the following formula: $A = \pi \cdot r^2 \implies A = 4 \cdot \pi$.

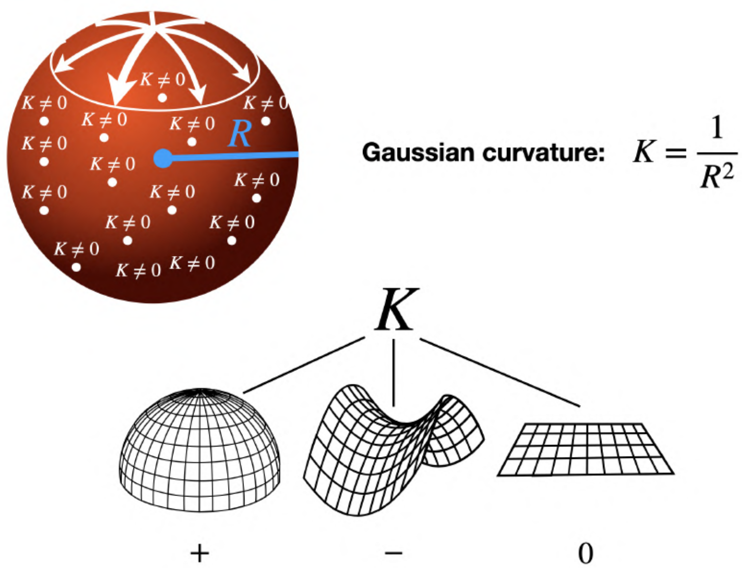


Now, let's do the opposite: you are given the value $A = 4\pi$, and you are asked to use the same formula to calculate the variable r. You can calculate its inverse: $A = \pi \cdot r^2 \implies 4\pi = \pi \cdot r^2 \implies r = \pm 2$. At this point (strictly speaking, from the purely mathematical point of view), you find two possible values for the quantity r, so $+2$ and -2 (especially in pure algebra, both results must be considered valid). Of course, if you interpret the variable r as the physical distance from the center to any point of a circle, then it is meaningless to consider the negative value $r = -2$. And that's why, at this point of our calculation, we discard it, and pick $r = +2$ as the only possible value of the radius.



This process of discarding a purely mathematical result in favor of a choice that provides physical meaning, is the bridge we cross from purely mathematical land to physical land. There is nothing wrong with it, but it does require some element of interpretation. And interpretations might be subjective sometimes. Indeed, there are specialized areas, like signed distance fields in computer graphics or oriented geometries, where negative radii are informally assigned meanings. So, the point here is to be careful when interpreting results in pure mathematics.

Let's see how this illustration connects with our context here: So, calculating purely mathematical results in Differential Geometry, and then interpreting them physically, in the case of spacetime and General Relativity as a whole.
Imagine a sphere of radius capital R. This is obviously a *curved space*, right? Well, we know that, because its Gaussian curvature is not zero everywhere ($K \neq 0$) − in fact, it is not zero anywhere! Gaussian curvature K tells us if the space has positive ($K > 0$), negative ($K < 0$) or zero ($K = 0$) curvature.



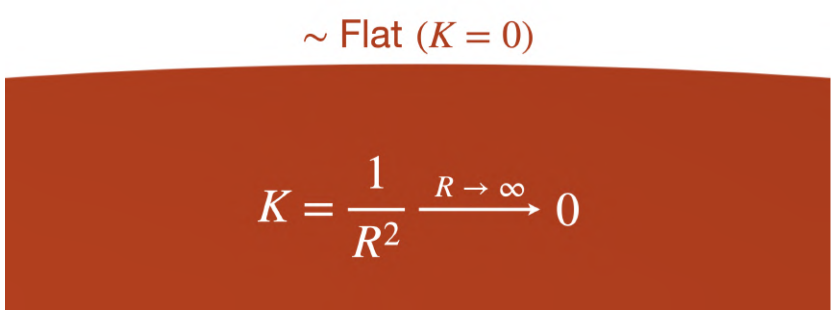
If you'd like to learn a very intuitive way of thinking about Gaussian curvature, as well as how to calculate it, check out this video and PDF:

How to Describe an Entire Surface with Just Two Numbers
PDF link: Gaussian Curvature

For this sphere, of radius R, the Gaussian curvature formula is precisely $K = \frac{1}{R^2} > 0$ everywhere, for all points. If the radius tends to infinity, the curvature tends to zero:

$$K = \frac{1}{R^2} \xrightarrow{R\to\infty} 0$$

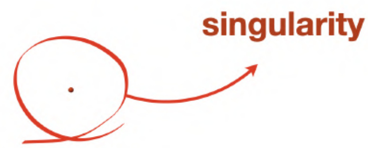
I.e., the sphere becomes flatter and flatter, approaching the geometry of an infinite plane, which has zero curvature ($K = 0$).



On the other hand, if we imagine that $R \to 0$, then the sphere shrinks to a single point (a singularity), and its Gaussian curvature, at each point, diverges to infinity:

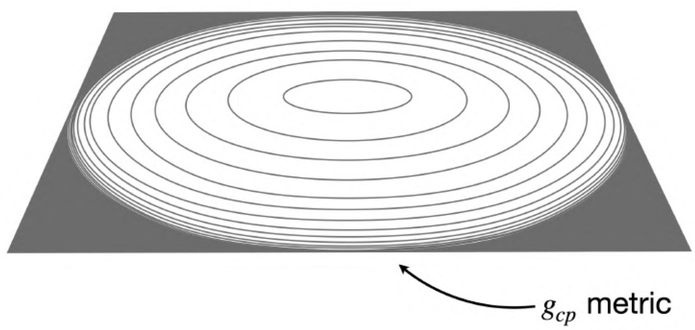
$$K = \frac{1}{R^2} \xrightarrow{R\to 0} \infty$$

This means that the notion of smooth geometry breaks down.



$$K = \frac{1}{R^2} \xrightarrow{R \to 0} \infty$$

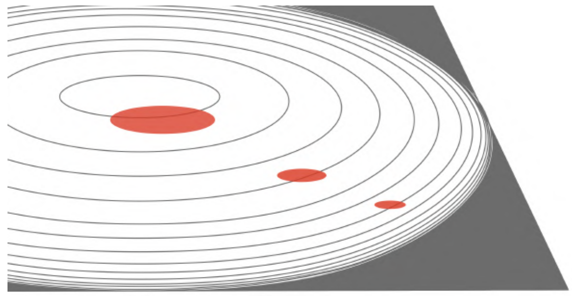
Ok, now that we have some intuition on Gaussian curvature, let's construct a "weird" infinite plane. The "weird" thing about it is that, even though it is flat, the *metric* used to measure local distances (so, the sort of ruler defined in this space) depends on the position of each point with respect to the origin.



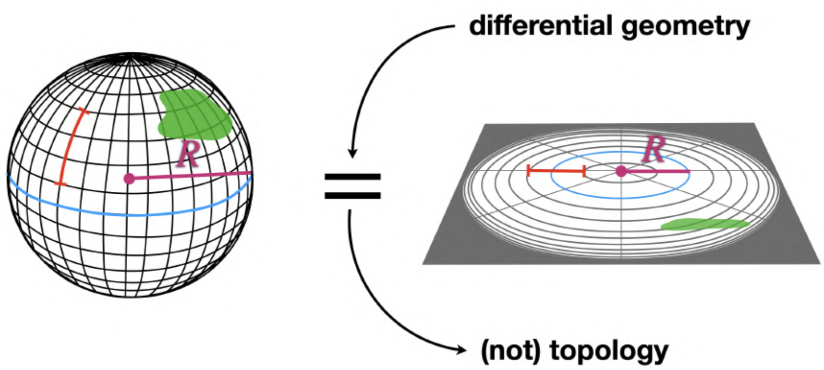
If you feel like you need a deeper understanding on what a metric is, and how to decide which metric better suits the manifold you are studying, check out this video about Riemannian manifolds, where we explain very clearly what a Riemannian metric is, since it is extremely useful in Differential Geometry:

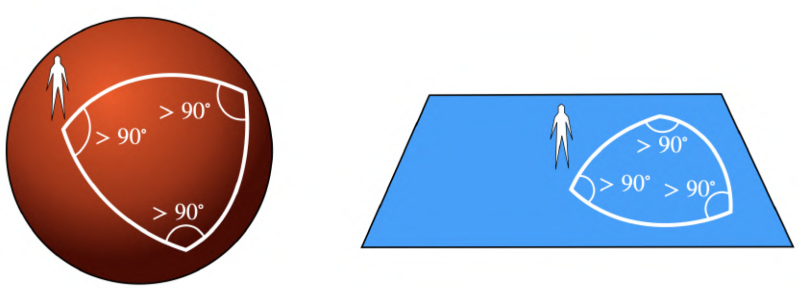
Riemannian Manifolds in 12 Minutes
PDF link: Riemannian Manifolds

Anyway, let's see an example of a $2D$ object placed near the center of this peculiar plane. After placing it there, the object undergoes almost no deformation, but as it moves away, it experiences a contraction. Its radial length progressively decreases, with the effective size tending to zero at infinity.

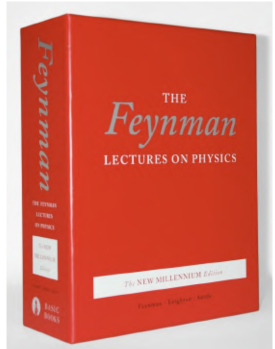


In this "elastic plane", the underlying topology remains the same as that of the flat real plane $\mathbb{R}^2$. However, the local measurements of lengths, areas, and geodesics behave exactly as if the space were intrinsically curved like a sphere of finite radius R. I.e., in a certain sense, this plane perfectly mimics the sphere in such a way that, strictly from the point of view of Differential Geometry (not taking into consideration their overall topological features), this plane is indistinguishable from the sphere. It is impossible to perform any calculation, or measurement, of intrinsic quantities related to curvature, that would allow beings in these spaces to find out whether they live on a sphere or on a plane with a position-dependent metric.





Now, before you guys start cursing me in our website, let me tell you that this originally was not my idea. I got that from Feynman, in his book "Lectures on Physics, Volume II", last chapter (number 42) – titled "Curved Space":



The Feynman Lectures on Physics

Mathematically, this effect is achieved by equipping the plane with what is called a *position-dependent conformal factor*, which is applied to the flat metric:

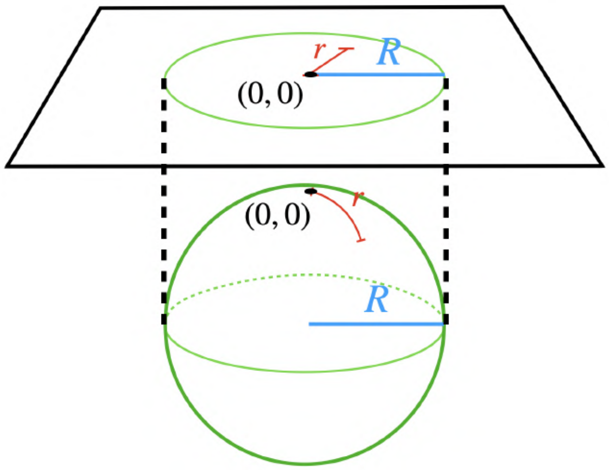
$$\Omega = \frac{2R^2}{R^2 + r^2}$$

Flat: $$ds^2 = dx^2 + dy^2$$

Conformal plane: $$ds^2 = \Omega\left(dx^2 + dy^2\right)$$

Where little r here is the radial distance from the origin, and capital R is the radius of the sphere that you want to mimic.

$$ds^2 = \left(\frac{2R^2}{R^2 + r^2}\right)^2 \left(dx^2 + dy^2\right)$$



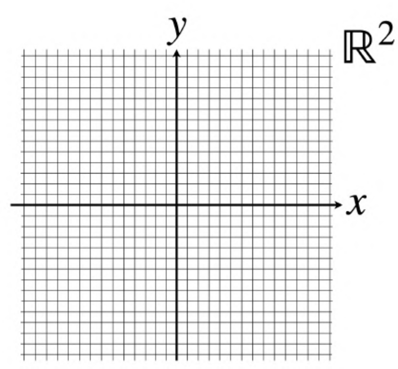
Ok, I don't know about you, but I find this fact super cool! Before continuing, though, let's take a step back and see the 3 different metrics involved here, as well as their interpretations.

1. Euclidean Metric in $\mathbb{R}^2$

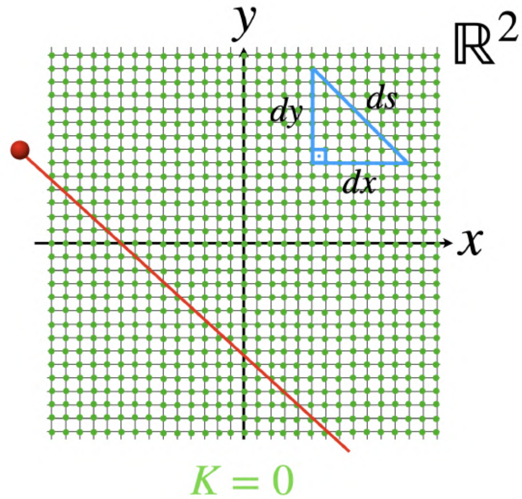
$$ds^2 = dx^2 + dy^2$$

flat metric

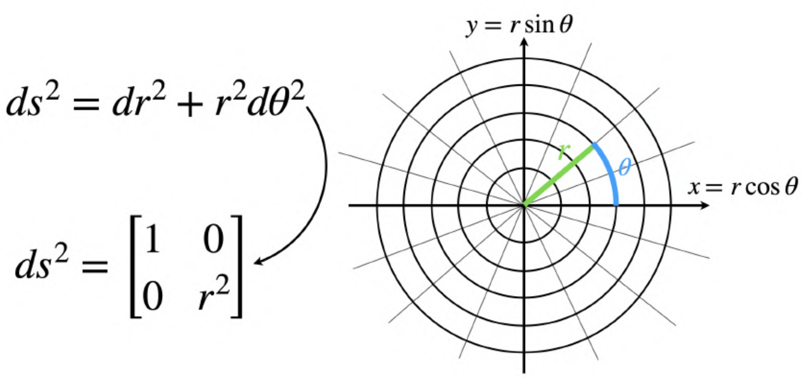
$$ds^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



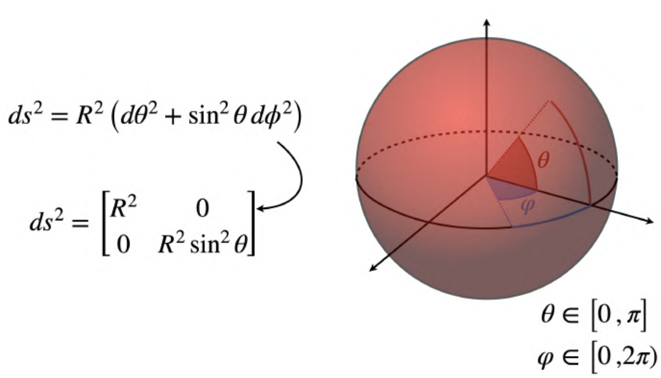
Here, distances between points are given by the Pythagorean theorem. Geodesics are straight lines and the Gaussian curvature is zero everywhere ($K = 0$). Pretty standard.



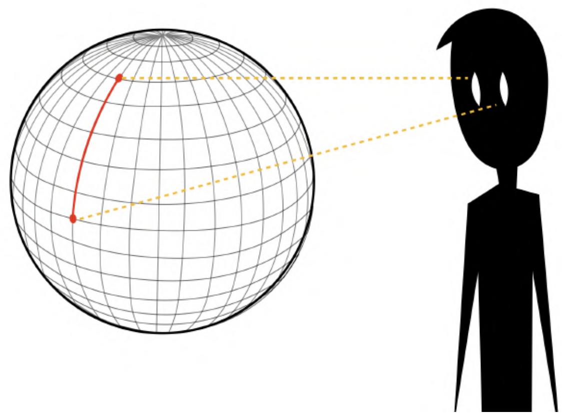
The same metric can be expressed in polar coordinates.



2. Metric on the Surface of the Sphere $\mathbb{S}^2$



Geodesics are not straight lines anymore – but only from our privileged point of view, observing the scene in $3D$. Gaussian curvature is constant and positive $K = \frac{1}{R^2}$ everywhere.

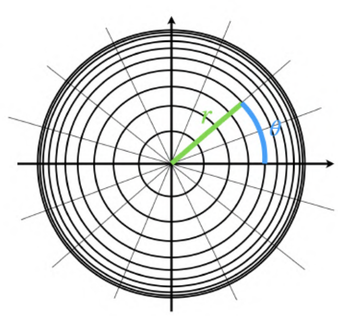


3. Conformally Flat Metric of the Sphere

This is the position-dependent stretched metric on the plane.

$$ds^2 = \left(\frac{2R^2}{R^2 + r^2}\right)^2 \left(dr^2 + r^2 d\theta^2\right)$$

$$ds^2 = \begin{bmatrix} \left(\frac{2R^2}{R^2 + r^2}\right)^2 & 0 \\ 0 & r^2 \left(\frac{2R^2}{R^2 + r^2}\right)^2 \end{bmatrix}$$



This is just like the flat metric $(dr^2 + r^2 d\theta^2)$ that we've seen before, but with each point multiplied by the conformal factor $\Omega(r) = \left(\frac{2R^2}{R^2+r^2}\right)$ depending on the radial distance r from the origin.

What about its Gaussian curvature? Is it still zero? In general, for a conformal metric like this one, the Gaussian curvature K is calculated with the following formula:
c

$$K = \frac{-1}{\Omega^2(r)} \Delta \ln \Omega(r)$$

Laplacian operator $\Delta := \frac{1}{r} \cdot \frac{d}{dr}\left(r \frac{d}{dr}\right)$

Where Δ is the *flat Laplacian operator* in polar coordinates. Now, plugging in our conformal factor in this equation, we get the following:

$$K = -\left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \frac{1}{r} \cdot \frac{d}{dr}\left\{r \frac{d}{dr}\left[\ln\left(\frac{2R^2}{R^2+r^2}\right)\right]\right\} = -\left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \frac{1}{r} \cdot \frac{d}{dr}\left(r \cdot \frac{-2r}{R^2+r^2}\right) =$$

$$= \left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \frac{1}{r} \cdot \frac{d}{dr}\left(\frac{2r^2}{R^2+r^2}\right) = \left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \frac{1}{r} \cdot \left(\frac{4r \cdot (R^2+r^2) - 2r^2 \cdot 2r}{(R^2+r^2)^2}\right) =$$

$$= \left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \frac{1}{\cancel{r}} \cdot \left(\frac{4\cancel{r}R^2 + \cancel{4r^3} - \cancel{4r^3}}{(R^2+r^2)^2}\right) =$$

$$= \frac{\cancel{(R^2+r^2)^2}}{\cancel{4}R^4} \cdot \frac{\cancel{4}R^2}{\cancel{(R^2+r^2)^2}} = \frac{R^2}{R^4} =$$

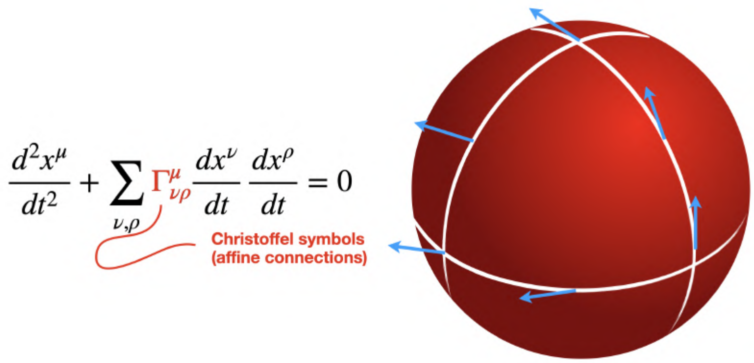
$$= \frac{1}{R^2}$$

Which is identical to the Gaussian curvature in the sphere!

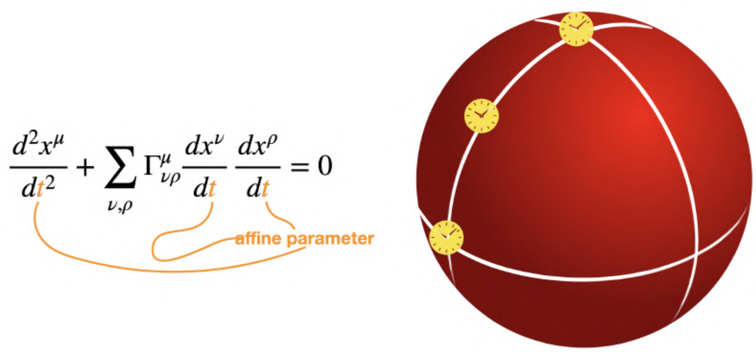
Beyond that, we can compare geodesic paths on the sphere with the ones on the conformal plane. In order to find geodesics we need to solve second-order, nonlinear, autonomous ordinary differential equations (known as ODEs) that are defined on our smooth manifolds – in our case the sphere and this conformal plane:

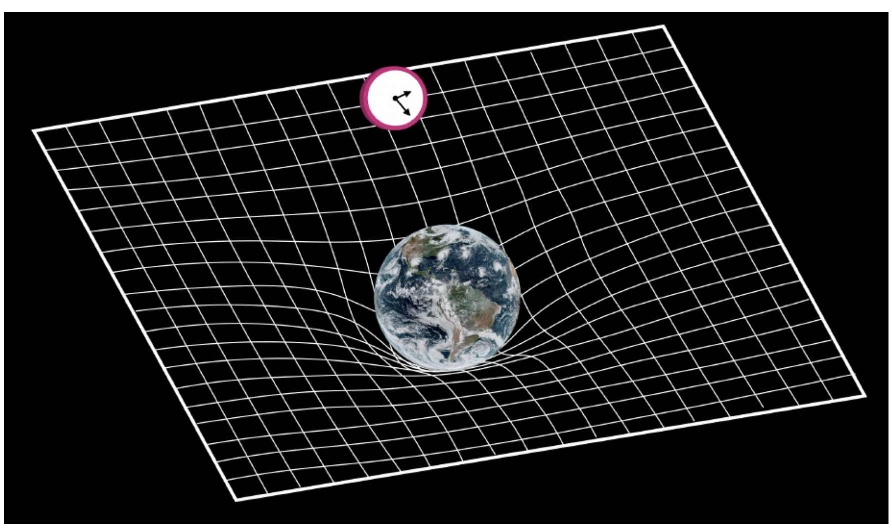
$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0$$

$\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols, or the coordinate representation of an affine connection. They tell us how to slide a vector forward while staying "parallel" to itself on a curved surface.



t here is the so-called affine parameter. Think of it as the natural "clock" along a geodesic. In classical mechanics, it is often interpreted as Newtonian time. In General Relativity, for example, it becomes proper time τ for timelike paths. This interpretation is not always valid, depending on the context, but, just to get a feeling of what we are trying to measure here, this "intuitive frame" will be enough.





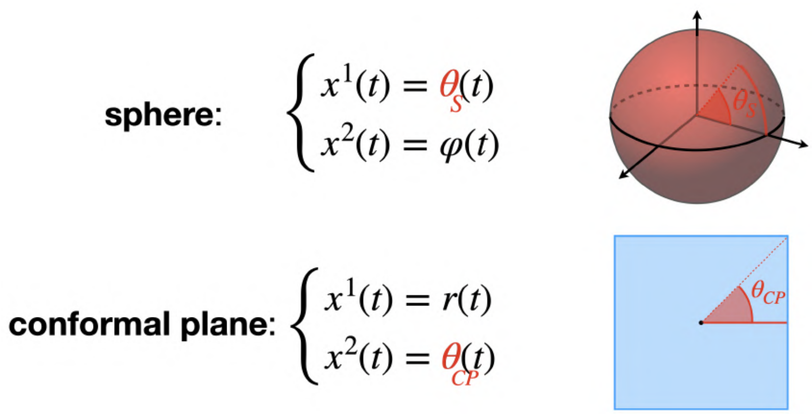
x^μ are the coordinate basis, which, for the sphere, will be:

$$x^{\mu} \quad (\mu \in \{1,2\}) : \begin{cases} x^1(t) = \theta(t) \\ x^2(t) = \phi(t) \end{cases}$$

And for the conformal plane, will be:

$$x^{\mu} \quad (\mu \in \{1,2\}) : \begin{cases} x^1(t) = r(t) \\ x^2(t) = \theta(t) \end{cases}$$

Attention! Both of them have variables denoted as θ, but they track different angles in each space.



$$\frac{d^2x^{\mu}}{dt^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{dt}\frac{dx^{\rho}}{dt} = 0$$

This equation is, actually, a *coupled system* of 2 equations for the function $x^{\mu}(t)$.

From the theory of ordinary differential equations, each geodesic equation admits a *unique solution* for any given initial point $x^{\mu}(t_0)$ and initial derivative show: "velocity" $\frac{dx^{\mu}}{dt}(t_0)$. This result is just the famous *Cauchy problem*, which means that geodesic paths are uniquely determined by their initial position and direction.

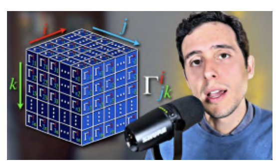
Now, let's use the standard metric defined on the sphere (in polar coordinates) in order to find the geodesic equations of this space. To start we notice that we need to calculate the Christoffel symbols first. But how do we do it?

Well, following the order given by the "Matryoshka effect" here, we see that we need to explicitly write down the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$:

$$g = g_{\mu\nu} = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix} \implies$$

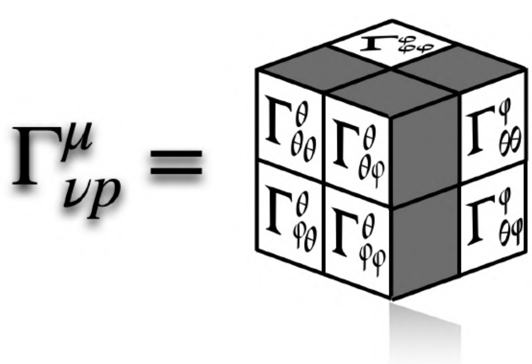
$$\implies g^{-1} = g^{\mu\nu} = \begin{bmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2\theta} \end{bmatrix}$$

If you don't know what we mean by "Matryoshka effect", you might want to watch this video and PDF (below), where we give the step-by-step algorithm of how to calculate the most important mathematical objects in Differential Geometry and General Relativity:



How a Single Real Number Determines Curvature
PDF link: Scalar Curvature

Anyway, this is the matrix form of the metric in the sphere. And as a consequence we find its inverse as well.



Now we can plug them in the Christoffel symbols formula, taking into consideration that we are working in 2 dimensions and the indices can have only the following two values: θ, ϕ.

$$\textbf{(1)}\ \ \Gamma^{\theta}_{\theta\theta} = \frac{1}{2} \sum_{\sigma \in \{\theta, \phi\}} g^{\theta\sigma} \left(\frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} + \frac{\partial g_{\theta\sigma}}{\partial x^{\theta}} - \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}} \right) =$$

(We can use Einstein sum convention from now on, and the notations $\frac{\partial}{\partial x^a} = \partial_a$ *and* $x^{\theta} = \theta$ *,* $x^{\phi} = \phi$ *, and so on)*

$$\begin{aligned}
&= \frac{1}{2} g^{\theta\sigma} \left(2\partial_{\theta}\, g_{\theta\sigma} - \partial_{\sigma} g_{\theta\theta} \right) = \\
&= g^{\theta\sigma} \partial_{\theta}\, g_{\theta\sigma} - \frac{1}{2} g^{\theta\sigma} \partial_{\sigma} g_{\theta\theta} = \\
&= g^{\theta\theta} \partial_{\theta}\, g_{\theta\theta} + \cancel{g^{\theta\phi}} \partial_{\theta}\, \cancel{g_{\theta\phi}} - \frac{1}{2} g^{\theta\theta} \partial_{\theta} g_{\theta\theta} - \frac{1}{2} \cancel{g^{\theta\phi}} \partial_{\phi} g_{\theta\theta} = \\
&= \frac{1}{R^2} \cancel{\partial_{\theta} \left(R^2 \right)} - \frac{1}{2} \cdot \frac{1}{R^2} \cdot \cancel{\partial_{\theta} \left(R^2 \right)} = \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\textbf{(2)}\ \ \Gamma^{\theta}_{\theta\phi} &= \frac{1}{2} g^{\theta\sigma} \left(\partial_{\theta} g_{\phi\sigma} + \partial_{\phi} g_{\theta\sigma} - \partial_{\sigma} \cancel{g_{\theta\phi}} \right) = \\
&= \frac{1}{2} g^{\theta\theta} \left(\partial_{\theta}\, \cancel{g_{\phi\theta}} + \partial_{\phi} g_{\theta\theta} \right) + \frac{1}{2} \cancel{g^{\theta\phi}} \left(\partial_{\theta}\, g_{\phi\phi} + \partial_{\phi} \cancel{g_{\theta\phi}} \right) = \\
&= \frac{1}{2} g^{\theta\theta} \partial_{\phi} g_{\theta\theta} = \\
&= \frac{1}{2} \cdot \frac{1}{R^2} \cdot \cancel{\partial_{\phi} \left(R^2 \right)} = \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\textbf{(3)}\ \ \Gamma^{\theta}_{\phi\theta} &= \frac{1}{2} g^{\theta\sigma} \left(\partial_{\phi} g_{\theta\sigma} + \partial_{\theta} g_{\phi\sigma} - \partial_{\sigma} \cancel{g_{\phi\theta}} \right) = \\
&= \frac{1}{2} g^{\theta\theta} \left(\partial_{\phi}\, g_{\theta\theta} + \partial_{\theta} \cancel{g_{\phi\theta}} \right) + \frac{1}{2} \cancel{g^{\theta\phi}} \left(\partial_{\phi}\, \cancel{g_{\theta\phi}} + \partial_{\theta} g_{\phi\phi} \right) = \\
&= \frac{1}{2} g^{\theta\theta} \partial_{\phi} g_{\theta\theta} = \\
&= \frac{1}{2} \cdot \frac{1}{R^2} \cdot \cancel{\partial_{\phi} \left(R^2 \right)} = \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\textbf{(4)}\ \Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}\, g^{\theta\sigma}\left(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\phi\sigma} - \partial_\sigma g_{\phi\phi}\right) = \\
&= \frac{1}{2}\, g^{\theta\theta}\left(\partial_\phi \cancel{g_{\phi\theta}} + \partial_\phi \cancel{g_{\phi\theta}} - \partial_\theta g_{\phi\phi}\right) + \frac{1}{2}\, \cancel{g^{\theta\phi}}\left(\partial_\phi\, g_{\phi\phi} + \cancel{\partial_\phi g_{\phi\phi}} - \cancel{\partial_\phi g_{\phi\phi}}\right) = \\
&= -\frac{1}{2}\, g^{\theta\theta}\partial_\theta g_{\phi\phi} = \\
&= -\frac{1}{2}\cdot\frac{1}{R^2}\cdot\partial_\theta\left(R^2\sin^2\theta\right) = \\
&= -\frac{R^2\cdot 2\cos\theta\sin\theta}{2R^2} = \\
&= -\sin\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(5)}\ \Gamma^{\phi}_{\theta\theta} &= \frac{1}{2}\, g^{\phi\sigma}\left(\partial_\theta g_{\theta\sigma} + \partial_\theta g_{\theta\sigma} - \partial_\sigma g_{\theta\theta}\right) = \\
&= \frac{1}{2}\, \cancel{g^{\phi\theta}}\left(\partial_\theta g_{\theta\theta} + \cancel{\partial_\theta g_{\theta\theta}} - \cancel{\partial_\theta g_{\theta\theta}}\right) + \frac{1}{2} g^{\phi\phi}\left(\partial_\theta \cancel{g_{\theta\phi}} + \partial_\theta \cancel{g_{\theta\phi}} - \partial_\phi g_{\theta\theta}\right) = \\
&= -\frac{1}{2}\, g^{\phi\phi}\partial_\phi g_{\theta\theta} = -\frac{1}{2}\cdot\frac{1}{R^2\sin^2\theta}\cdot\cancel{\partial_\phi\left(R^2\right)} = \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(6)}\ \Gamma^{\phi}_{\theta\phi} &= \frac{1}{2}\, g^{\phi\sigma}\left(\partial_\theta g_{\phi\sigma} + \partial_\phi g_{\theta\sigma} - \partial_\sigma \cancel{g_{\theta\phi}}\right) = \\
&= \frac{1}{2}\, \cancel{g^{\phi\theta}}\left(\partial_\theta \cancel{g_{\phi\theta}} + \partial_\phi g_{\theta\theta}\right) + \frac{1}{2} g^{\phi\phi}\left(\partial_\theta g_{\phi\phi} + \partial_\phi \cancel{g_{\theta\phi}}\right) = \\
&= \frac{1}{2}\, g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{1}{2}\cdot\frac{1}{R^2\sin^2\theta}\cdot\partial_\theta\left(R^2\sin^2\theta\right) = \\
&= \frac{R^2\cdot 2\sin\theta\cos\theta}{2R^2\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \\
&= \cot\theta
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(7)}\ \Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}\, g^{\phi\sigma}\left(\partial_\phi g_{\theta\sigma} + \partial_\theta g_{\phi\sigma} - \partial_\sigma \cancel{g_{\phi\theta}}\right) = \\
&= \frac{1}{2}\, \cancel{g^{\phi\theta}}\left(\partial_\phi g_{\theta\theta} + \partial_\theta \cancel{g_{\phi\theta}}\right) + \frac{1}{2} g^{\phi\phi}\left(\partial_\phi \cancel{g_{\theta\phi}} + \partial_\theta g_{\phi\phi}\right) = \\
&= \frac{1}{2}\, g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{1}{2}\cdot\frac{1}{R^2\sin^2\theta}\cdot\partial_\theta\left(R^2\sin^2\theta\right) = \\
&= \frac{R^2\cdot 2\sin\theta\cos\theta}{2R^2\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \\
&= \cot\theta
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(8)}\ \Gamma^{\phi}_{\phi\phi} &= \frac{1}{2}\, g^{\phi\sigma}\left(\partial_\phi g_{\phi\sigma} + \partial_\phi g_{\phi\sigma} - \partial_\sigma g_{\phi\phi}\right) = \\
&= \frac{1}{2}\, \cancel{g^{\phi\theta}}\left(\partial_\phi \cancel{g_{\phi\theta}} + \partial_\phi \cancel{g_{\phi\theta}} - \partial_\theta g_{\phi\phi}\right) + \frac{1}{2} g^{\phi\phi}\left(\partial_\phi g_{\phi\phi} + \cancel{\partial_\phi g_{\phi\phi}} - \cancel{\partial_\phi g_{\phi\phi}}\right) = \\
&= \frac{1}{2}\cdot\frac{1}{R^2\sin^2\theta}\cdot\cancel{\partial_\phi\left(R^2\sin^2\theta\right)} = \\
&= 0
\end{aligned}
$$

$\therefore$ These are all the Christoffel symbols of the metric in the sphere:

(1) $\Gamma^{\theta}_{\theta\theta} = 0$	**(5)** $\Gamma^{\phi}_{\theta\theta} = 0$
(2) $\Gamma^{\theta}_{\theta\phi} = 0$	**(6)** $\Gamma^{\phi}_{\theta\phi} = \cot\theta$
(3) $\Gamma^{\theta}_{\phi\theta} = 0$	**(7)** $\Gamma^{\phi}_{\phi\theta} = \cot\theta$
(4) $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$	**(8)** $\Gamma^{\phi}_{\phi\phi} = 0$

Since these Christoffel symbols can be represented as a $2 \times 2 \times 2$ matrix (remember, we are working in 2 dimensions), then there are in total $2 \times 2 \times 2 = 8$ entries.

Finally, plugging them in the geodesic formula, and recalling that $\begin{cases} x^1(t) = \theta(t) \\ x^2(t) = \phi(t) \end{cases}$, we get the following:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \begin{cases} \frac{d^2x^1}{dt^2} + \Gamma^{1}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \\ \frac{d^2x^2}{dt^2} + \Gamma^{2}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \end{cases} \implies$$

$$\implies \begin{cases} \frac{d^2\theta}{dt^2} + \Gamma^{\theta}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \quad (I) \\ \frac{d^2\phi}{dt^2} + \Gamma^{\phi}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \quad (II) \end{cases}$$

$$(I) \implies \frac{d^2\theta}{dt^2} + \Gamma^{\theta}_{\theta\rho}\frac{d\theta}{dt}\frac{dx^\rho}{dt} + \Gamma^{\theta}_{\phi\rho}\frac{d\phi}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \frac{d^2\theta}{dt^2} + \cancel{\Gamma^{\theta}_{\theta\theta}}\frac{d\theta}{dt}\frac{d\theta}{dt} + \cancel{\Gamma^{\theta}_{\theta\phi}}\frac{d\theta}{dt}\frac{d\phi}{dt} + \cancel{\Gamma^{\theta}_{\phi\theta}}\frac{d\phi}{dt}\frac{d\theta}{dt} + \Gamma^{\theta}_{\phi\phi}\frac{d\phi}{dt}\frac{d\phi}{dt} = 0 \implies$$

$$\implies \boxed{\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta \cdot \left(\frac{d\phi}{dt}\right)^2 = 0} \quad (I')$$

$$(II) \implies \frac{d^2\phi}{dt^2} + \Gamma^{\phi}_{\theta\rho}\frac{d\theta}{dt}\frac{dx^\rho}{dt} + \Gamma^{\phi}_{\phi\rho}\frac{d\phi}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \frac{d^2\phi}{dt^2} + \cancel{\Gamma^{\phi}_{\theta\theta}}\frac{d\theta}{dt}\frac{d\theta}{dt} + \Gamma^{\phi}_{\theta\phi}\frac{d\theta}{dt}\frac{d\phi}{dt} + \Gamma^{\phi}_{\phi\theta}\frac{d\phi}{dt}\frac{d\theta}{dt} + \cancel{\Gamma^{\phi}_{\phi\phi}}\frac{d\phi}{dt}\frac{d\phi}{dt} = 0 \implies$$

$$\implies \frac{d^2\phi}{dt^2} + \cot\theta \cdot \frac{d\theta}{dt} \cdot \frac{d\phi}{dt} + \cot\theta \cdot \frac{d\phi}{dt} \cdot \frac{d\theta}{dt} = 0 \implies$$

$$\implies \boxed{\frac{d^2\phi}{dt^2} + 2\cot\theta\frac{d\theta}{dt} \cdot \frac{d\phi}{dt} = 0} \quad (II')$$

$\therefore$ These are the geodesic equations of the sphere:

$$\boxed{\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta \cdot \left(\frac{d\phi}{dt}\right)^2 = 0} \quad (I')$$

$$\boxed{\frac{d^2\phi}{dt^2} + 2\cot\theta\frac{d\theta}{dt} \cdot \frac{d\phi}{dt} = 0} \quad (II')$$

Solving them simultaneously gives us all possible geodesic paths on the sphere!

Let's do the same process with the conformal plane.

These are the metric and its inverse:

$$ds^2 = \Omega^2(r)\,\left(dr^2 + r^2 d\theta^2\right) \quad , \ \Omega(r) = \frac{2R^2}{R^2 + r^2} \quad (\textit{conformal factor})$$

$$g = \Omega^2(r)\begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix} = \left(\frac{2R^2}{R^2+r^2}\right)^2 \begin{bmatrix}1 & 0\\ 0 & r^2\end{bmatrix}$$

$$g^{-1} = \left(\frac{R^2+r^2}{2R^2}\right)^2 \begin{bmatrix}1 & 0\\ 0 & \frac{1}{r^2}\end{bmatrix}$$

With them we calculate the Christoffel symbols:

$$\begin{aligned}
\textbf{(1)}\ \Gamma^r_{rr} &= \frac{1}{2}g^{r\sigma}\left(\frac{\partial g_{r\sigma}}{\partial r} + \frac{\partial g_{r\sigma}}{\partial r} - \frac{\partial g_{rr}}{\partial x^\sigma}\right) = \\
&= \frac{1}{2}g^{r\sigma}\left(2\partial_r g_{r\sigma} - \partial_\sigma g_{rr}\right) = \\
&= \frac{1}{2}g^{rr}\left(2\partial_r g_{rr} - \partial_r g_{rr}\right) + \frac{1}{2}\cancel{g^{r\theta}}\left(2\partial_r \cancel{g_{r\theta}} - \cancel{\partial_\theta g_{rr}}\right) = \\
&= \frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{1}{\cancel{2}}\left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \cancel{2}\left(\frac{2R^2}{R^2+r^2}\right)\cdot\frac{2R^2\cdot(-1)\cdot 2r}{\left(R^2+r^2\right)^2} = \\
&= -\frac{\cancel{\left(R^2+r^2\right)^2}\cdot\cancel{\left(2R^2\right)\cdot\left(2R^2\right)}\cdot 2r}{\cancel{\left(2R^2\right)^2}\cdot\left(R^2+r^2\right)\cdot\cancel{\left(R^2+r^2\right)^2}} = \\
&= -\frac{2r}{R^2+r^2}
\end{aligned}$$

$$\begin{aligned}
\textbf{(2)}\ \Gamma^r_{r\theta} &= \frac{1}{2}g^{r\sigma}\left(\frac{\partial g_{\theta\sigma}}{\partial r} + \frac{\partial g_{r\sigma}}{\partial \theta} - \frac{\partial \cancel{g_{r\theta}}}{\partial x^\sigma}\right) = \\
&= \frac{1}{2}g^{rr}\left(\partial_r \cancel{g_{\theta r}} + \partial_\theta g_{rr}\right) + \frac{1}{2}\cancel{g^{r\theta}}\left(\partial_r g_{\theta\theta} + \partial_\theta \cancel{g_{r\theta}}\right) = \\
&= \frac{1}{2}g^{rr}\partial_\theta g_{rr} = \frac{1}{2}\left(\frac{R^2+r^2}{2R^2}\right)^2 \cdot \cancel{\partial_\theta\left(\left(\frac{2R^2}{R^2+r^2}\right)^2\right)} = \\
&= 0
\end{aligned}$$

$$\textbf{(3)}\ \Gamma^r_{\theta r} = \frac{1}{2} g^{r\sigma} \left(\frac{\partial g_{r\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^\sigma} \right) =$$

$$= \frac{1}{2} g^{rr} \left(\partial_\theta g_{rr} + \partial_r \cancel{g_{\theta r}} \right) + \frac{1}{2} \cancel{g^{r\theta}} \left(\partial_\theta \cancel{g_{r\theta}} + \partial_r g_{\theta\theta} \right) =$$

$$= \frac{1}{2} g^{rr} \partial_\theta g_{rr} = \frac{1}{2} \left(\frac{R^2 + r^2}{2R^2} \right)^2 \cdot \cancel{\partial_\theta \left(\left(\frac{2R^2}{R^2 + r^2} \right)^2 \right)} =$$

$$= 0$$

$$\textbf{(4)}\ \Gamma^r_{\theta\theta} = \frac{1}{2} g^{r\sigma} \left(\frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^\sigma} \right) =$$

$$= \frac{1}{2} g^{rr} \left(2\, \partial_\theta \cancel{g_{\theta r}} - \partial_r g_{\theta\theta} \right) + \frac{1}{2} \cancel{g^{r\theta}} \left(2\, \partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta} \right) =$$

$$= -\frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -\frac{1}{2} \cdot \frac{\left(R^2 + r^2\right)^2}{\left(2R^2\right)^2} \cdot \partial_r \left(\frac{r^2 \left(2R^2\right)^2}{\left(R^2 + r^2\right)^2} \right) =$$

$$= -\frac{1}{\cancel{2}} \cdot \frac{\cancel{\left(R^2 + r^2\right)^2}}{\left(2R^2\right)^2} \cdot \frac{\cancel{2} \left(2R^2\right)^2 \cdot r \cdot \left(R^2 + r^2\right)^2 - r^2 \left(2R^2\right)^2 \cdot \cancel{2} \left(R^2 + r^2\right) \cdot 2r}{\left(R^2 + r^2\right)^{\cancel{4}\, 2}} =$$

$$= -\frac{\cancel{\left(2R^2\right)^2} \cdot \cancel{\left(R^2 + r^2\right)} \cdot \left(r \cdot \left(R^2 + r^2\right) - 2r^3\right)}{\cancel{\left(2R^2\right)^2} \cdot \left(R^2 + r^2\right)^{\cancel{2}}} =$$

$$= -\frac{r \cdot \left(R^2 + r^2\right) - 2r^3}{\left(R^2 + r^2\right)} = -\frac{r \cdot \cancel{\left(R^2 + r^2\right)}}{\cancel{\left(R^2 + r^2\right)}} + \frac{2r^3}{\left(R^2 + r^2\right)} =$$

$$= -r + \frac{2r^3}{R^2 + r^2}$$

$$\textbf{(5)}\ \Gamma^\theta_{rr} = \frac{1}{2} g^{\theta\sigma} \left(\frac{\partial g_{r\sigma}}{\partial r} + \frac{\partial g_{r\sigma}}{\partial r} - \frac{\partial g_{rr}}{\partial x^\sigma} \right) =$$

$$= \frac{1}{2} \cancel{g^{\theta r}} \left(\partial_r g_{rr} + \cancel{\partial_r g_{rr}} - \cancel{\partial_r g_{rr}} \right) + \frac{1}{2} g^{\theta\theta} \left(\partial_r \cancel{g_{r\theta}} + \partial_r \cancel{g_{r\theta}} - \partial_\theta g_{rr} \right) =$$

$$= -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{rr} = -\frac{1}{2} \cdot \left(\frac{R^2 + r^2}{2R^2} \right)^2 \cdot \frac{1}{r^2} \cdot \cancel{\partial_\theta \left(\left(\frac{2R^2}{R^2 + r^2} \right)^2 \right)}$$

$$= 0$$

$$\textbf{(6)}\ \Gamma^{\theta}_{r\theta} = \frac{1}{2} g^{\theta\sigma}\left(\frac{\partial g_{\theta\sigma}}{\partial r} + \frac{\partial g_{r\sigma}}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial x^{\sigma}}\right) =$$

$$= \frac{1}{2}\cancel{g^{\theta r}}\left(\partial_r \cancel{g_{\theta r}} + \partial_\theta g_{rr}\right) + \frac{1}{2} g^{\theta\theta}\left(\partial_r g_{\theta\theta} + \partial_\theta \cancel{g_{r\theta}}\right) =$$

$$= \frac{1}{2} g^{\theta\theta}\partial_r g_{\theta\theta} = \frac{1}{2}\cdot\frac{\left(R^2+r^2\right)^2}{\left(2R^2\right)^2}\cdot\frac{1}{r^2}\partial_r\left(\frac{\left(2R^2\right)^2}{\left(R^2+r^2\right)^2} r^2\right) =$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{\left(R^2+r^2\right)^2}}{r^2\cancel{\left(2R^2\right)^2}}\cdot\left(-\frac{\cancel{\left(2R^2\right)^2}\cdot\cancel{4}^{2}\, r}{\left(R^2+r^2\right)^{\cancel{4}\,3}\cdot r^2 + \frac{\cancel{\left(2R^2\right)^2}}{\cancel{\left(R^2+r^2\right)^2}}\cdot\cancel{2}r}\right) =$$

$$= \frac{1}{r^2}\left(-\frac{2r^3}{R^2+r^2} + r\right) =$$

$$= \frac{1}{r} - \frac{2r}{R^2+r^2}$$

$$\textbf{(7)}\ \Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\sigma}\left(\frac{\partial g_{r\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^{\sigma}}\right) =$$

$$= \frac{1}{2}\cancel{g^{\theta r}}\left(\partial_\theta g_{rr} + \partial_r g_{\theta r}\right) + \frac{1}{2} g^{\theta\theta}\left(\partial_\theta \cancel{g_{r\theta}} + \partial_r g_{\theta\theta}\right) =$$

$$= \frac{1}{2} g^{\theta\theta}\partial_r g_{\theta\theta} =$$

$$= \frac{1}{r} - \frac{2r}{R^2+r^2}$$

$$\textbf{(8)}\ \Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\sigma}\left(\frac{\partial g_{\theta\sigma}}{\partial \theta} + \frac{\partial g_{\theta\sigma}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial x^{\sigma}}\right) =$$

$$= \frac{1}{2}\cancel{g^{\theta r}}\left(2\,\partial_\theta \cancel{g_{\theta r}} - \partial_r g_{\theta\theta}\right) + \frac{1}{2} g^{\theta\theta}\left(2\,\partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta}\right) =$$

$$= \frac{1}{2} g^{\theta\theta}\cancel{\partial_\theta g_{\theta\theta}} =$$

$$= 0$$

$\therefore$ These are all the Christoffel symbols of the metric in the conformal plane:

(1) $\Gamma^{r}_{rr} = -\frac{2r}{R^2+r^2}$	**(5)** $\Gamma^{\theta}_{rr} = 0$
(2) $\Gamma^{r}_{r\theta} = 0$	**(6)** $\Gamma^{\theta}_{r\theta} = \frac{1}{r} - \frac{2r}{R^2+r^2}$
(3) $\Gamma^{r}_{\theta r} = 0$	**(7)** $\Gamma^{\theta}_{\theta r} = \frac{1}{r} - \frac{2r}{R^2+r^2}$
(4) $\Gamma^{r}_{\theta\theta} = -r + \frac{2r^3}{R^2+r^2}$	**(8)** $\Gamma^{\theta}_{\theta\theta} = 0$

And finally, using these results, together with the fact that $\begin{cases} x^1(t) = r(t) \\ x^2(t) = \theta(t) \end{cases}$, we get the geodesic equations:

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \begin{cases} \frac{d^2x^1}{dt^2} + \Gamma^1_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \\ \frac{d^2x^2}{dt^2} + \Gamma^2_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 \end{cases} \implies$$

$$\implies \begin{cases} \frac{d^2r}{dt^2} + \Gamma^r_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 & (A) \\ \frac{d^2\theta}{dt^2} + \Gamma^\theta_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0 & (B) \end{cases}$$

$$(A) \implies \frac{d^2r}{dt^2} + \Gamma^r_{r\rho}\frac{dr}{dt}\frac{dx^\rho}{dt} + \Gamma^r_{\theta\rho}\frac{d\theta}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \frac{d^2r}{dt^2} + \Gamma^r_{rr}\left(\frac{dr}{dt}\right)^2 + \cancel{\Gamma^r_{r\theta}}\frac{dr}{dt}\frac{d\theta}{dt} + \cancel{\Gamma^r_{\theta r}}\frac{d\theta}{dt}\frac{dr}{dt} + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{dt}\right)^2 = 0 \implies$$

$$\implies \boxed{\frac{d^2r}{dt^2} - \frac{2r}{R^2+r^2}\cdot\left(\frac{dr}{dt}\right)^2 + \left(-r + \frac{2r^3}{R^2+r^2}\right)\cdot\left(\frac{d\theta}{dt}\right)^2 = 0} \quad (A')$$

$$(B) \implies \frac{d^2\theta}{dt^2} + \Gamma^\theta_{r\rho}\frac{dr}{dt}\frac{dx^\rho}{dt} + \Gamma^\theta_{\theta\rho}\frac{d\theta}{dt}\frac{dx^\rho}{dt} = 0 \implies$$

$$\implies \frac{d^2\theta}{dt^2} + \cancel{\Gamma^\theta_{rr}}\left(\frac{dr}{dt}\right)^2 + \Gamma^\theta_{r\theta}\frac{dr}{dt}\frac{d\theta}{dt} + \Gamma^\theta_{\theta r}\frac{d\theta}{dt}\frac{dr}{dt} + \cancel{\Gamma^\theta_{\theta\theta}}\left(\frac{d\theta}{dt}\right)^2 = 0 \implies$$

$$\implies \boxed{\frac{d^2\theta}{dt^2} + 2\left(\frac{1}{r} - \frac{2r}{R^2+r^2}\right)\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} = 0} \quad (B')$$

∴ These are the geodesic equations of this conformal plane:

$$\boxed{\frac{d^2r}{dt^2} - \frac{2r}{R^2+r^2}\cdot\left(\frac{dr}{dt}\right)^2 + \left(-r + \frac{2r^3}{R^2+r^2}\right)\cdot\left(\frac{d\theta}{dt}\right)^2 = 0} \quad (A')$$

$$\boxed{\frac{d^2\theta}{dt^2} + 2\left(\frac{1}{r} - \frac{2r}{R^2+r^2}\right)\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} = 0} \quad (B')$$

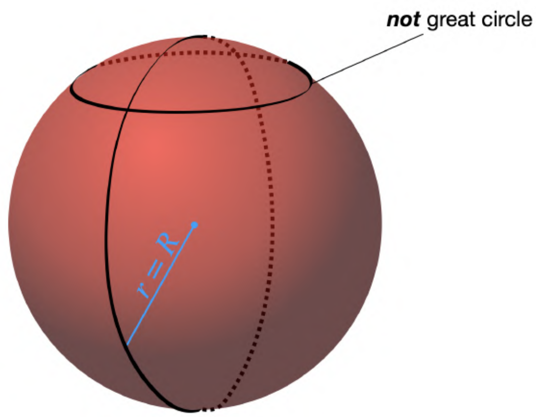
Ok, to say the least, these two systems of non-linear ODEs are very hard to solve in general...

These are the geodesic equations of the sphere and of the conformal plane, respectively:

$$\underline{\text{Sphere}}: \begin{cases} \frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 = 0 & (I') \\ \frac{d^2\phi}{dt^2} + 2\cot\theta\,\frac{d\theta}{dt}\frac{d\phi}{dt} = 0 & (II') \end{cases}$$

$$\underline{\text{Conformal plane}}: \begin{cases} \frac{d^2r}{dt^2} - \frac{2r}{R^2+r^2}\left(\frac{dr}{dt}\right)^2 + \left(-r + \frac{2r^3}{R^2+r^2}\right)\left(\frac{d\theta}{dt}\right)^2 = 0 & (A') \\ \frac{d^2\theta}{dt^2} + 2\left(\frac{1}{r} - \frac{2r}{R^2+r^2}\right)\cdot\frac{dr}{dt}\frac{d\theta}{dt} = 0 & (B') \end{cases}$$

Solutions for the sphere are just great circles on the sphere. Great circles have their radius identical to the sphere's radius (R).



Solutions for the conformal plane, on the other hand, are a bit more difficult to find, even though the sphere and the conformal plane are "equivalent" (as far as curvature is concerned), except for a point at infinity in the conformal plane, which corresponds to the south pole in the sphere.

These are some solutions of the conformal plane:

Uniform Circular Motion:

$$\begin{cases} r(t) = r_0 & \text{(constant)} \\ \theta(t) = \omega t \ , & \forall \omega \in \mathbb{R} \setminus \{0\} \end{cases}$$

Where $\omega \in \mathbb{R}$ corresponds to what physicists would call *angular velocity.*

$$(A') \implies \cancel{\frac{d^2}{dt^2}(r_0)} - \frac{2\,r_0}{R^2 + r^2}\left(\cancel{\frac{d}{dt}(r_0)}\right)^2 + \left(-r_0 + \frac{2\,r_0^3}{R^2 + r_0^2}\right)\left(\frac{d}{dt}(\omega t)\right)^2 = 0 \implies$$

$$\implies \left(-r_0 + \frac{2\,r_0^3}{R^2 + r^2}\right)\omega^2 = 0 \implies (\text{since } \omega \neq 0) \implies \left(-r_0 + \frac{2\,r_0^3}{R^2 + r^2}\right) = 0 \implies r_0\left(R^2 + r_0^2\right) = 2\,r_0^3 \implies$$

$$\implies \boxed{r_0 = 0} \ \ (\text{trivial}) \ \ \text{or} \ \ R^2 + r_0^2 = 2\,r_0^2 \implies \boxed{r_0 = R}$$

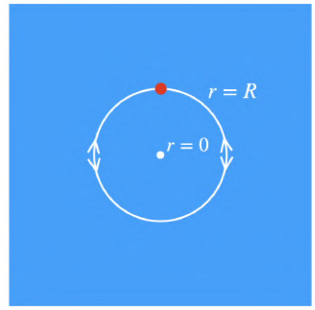
Now, let's use the non-trivial solution ($r = R$) in the second geodesic equation and see what happens:

$$(B') \implies \cancel{\frac{d^2}{dt^2}(\omega t)} + 2\left(\frac{1}{R} - \frac{2R}{R^2 + R^2}\right) \cdot \cancel{\frac{d}{dt}(R)}\frac{d}{dt}(\omega t) = 0 \implies 0 = 0 \quad \checkmark$$

Thus, the second geodesic equation doesn't give us any new information. It's just always satisfied as long as $r \neq 0$ (constant) and $\theta \propto t$ (directly proportional).

1. Uniform Circular Motion

$$\begin{cases} r(t) = 0 \text{ or } R \\ \theta(t) = \omega t \ , \forall \omega \in \mathbb{R} \backslash \{0\} \end{cases}$$



Radial Motion (No Angular Change):

$$\begin{cases} r = r(t) \\ \theta(t) = \theta_0 \quad \text{(constant)} \end{cases}$$

$$(I') \implies \frac{d^2 r}{dt^2}(t) - \frac{2\,r(t)}{R^2 + r^2(t)} \left(\frac{dr}{dt}(t)\right)^2 + \left(-r(t) + \frac{2\,r^3(t)}{R^2 + r^2(t)}\right) \left(\cancel{\frac{d}{dt}(\theta_0)}\right)^2 = 0 \implies$$

$$\implies \boxed{\ddot{r} - \left(\frac{2r}{R^2 + r^2}\right) \dot{r}^2 = 0} \qquad \star$$

In order to find solutions of $\star$, we can apply a well-known method for solving *ODEs* and *PDEs* called "separation of variables".

Separation of variables:

A 1^{st} order ordinary differential equation (ODE) can be solved by *separation of variables* if it can be written in the form:

$$\frac{dy}{dx} = f(y)\,g(x) \tag{1}$$

Then, separating variables gives:

$$\frac{dy}{f(y)} = g(x)\,dx \tag{2}$$

Notice a few things:

(a) The functions involved must be *continuous* (at least locally);

(b) $f(y) \neq 0\ ,\ \ \forall y \in \mathrm{dom}(f)$;

(c) In eq. (2), the left-hand side (LHS) is a function of y alone (explicitly), while the right-hand side (RHS) is a function of x alone. This allows us to integrate both sides separately:

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

In our case:

$$\boxed{x := t} \qquad \boxed{y := \frac{dr}{dt}}$$

$$\boxed{f(y) := \frac{dr}{dt}} \neq 0\,,\ \ \forall t \in \mathbb{R}$$

$$\boxed{g(x) := \left(\frac{2r}{R^2 + r^2}\right) \frac{dr}{dt}}$$

Substituting in the eq. (2), we ge the following:

$$(2) \implies \frac{dy}{f(y)} = g(x)\,dx \implies \frac{d\,(\dot{r})}{(\dot{r})} = \left(\frac{2r}{R^2 + r^2}\right) \frac{dr}{dt}\,dt \implies$$

$$\implies \int \frac{d\,(\dot{r})}{(\dot{r})} = \int \left(\frac{2r}{R^2 + r^2}\right) d(r) \implies$$

(C ≡ constant)

$$\implies \ln|\dot{r}| = \ln\left(R^2 + r^2\right) + C \implies$$

Applying the exponential function to both sides:

$$\implies e^{\ln|\dot{r}|} = e^{\ln\left(R^2 + r^2\right) + C} \implies |\dot{r}| = \left(R^2 + r^2\right) \cdot e^C \implies$$

Finally, denoting the constant as $C_1 := e^C$:

$$\implies \boxed{\dot{r} = C_1\left(R^2 + r^2\right)}$$

Once again, we separate variables and integrate:

$$\frac{dr}{dt} = C_1\left(R^2 + r^2\right) \implies \int \frac{dr}{R^2 + r^2} = \int C_1\, dt \implies$$
$$\implies \tan^{-1}\left(\frac{r}{R}\right) = C_1 R\, t + C_2 \implies \boxed{r(t) = R\,\tan\left(C_1 R\, t + C_2\right)}$$

($C_2 \equiv$ constant)

And that's the second solution we found from the first geodesic equation of the conformal plane. Now, let's see what the second equation has to say:

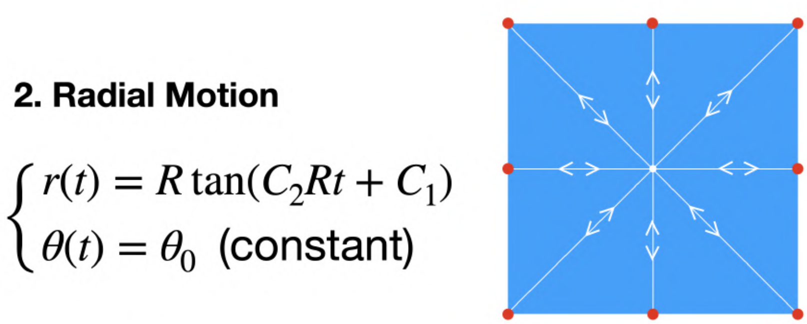
$$(1) \implies \cancel{\frac{d^2}{dt^2}(\theta_0)} + 2\left(\frac{1}{r(t)} - \frac{2\,r(t)}{R^2 + r^2(t)}\right) \cdot \frac{d}{dt}(r(t))\cancel{\frac{d}{dt}(\theta_0)} = 0 \implies 0 = 0 \quad \checkmark$$

So, once again, no new information. But at least the equation is satisfied.

$\therefore$ the solution is:

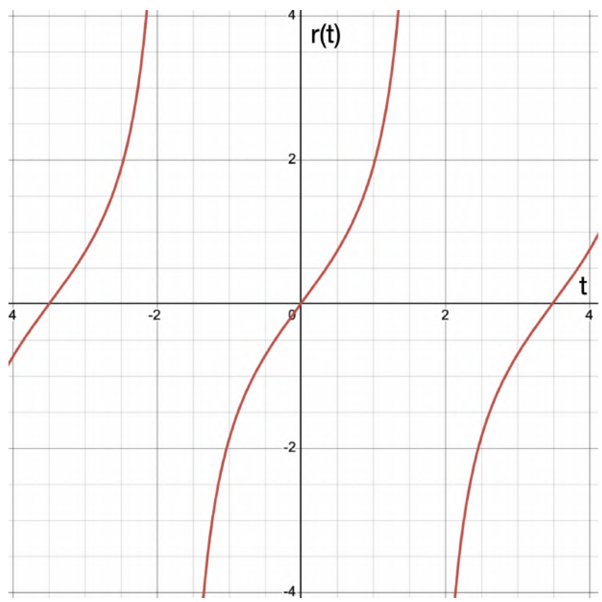
$$\begin{cases} r(t) = R\,\tan\left(C_1 R\, t + C_2\right) \\ \theta(t) = \theta_0 \end{cases}$$

The parameter C_1 controls the speed of growth. These are radial lines from the origin.



The function $r(t) = R\,\tan\left(C_1 R\,t + C_2\right)$ has vertical asymptotes at

$$C_1 R\,t + C_2 = \pm\frac{\pi}{2} \implies \boxed{t = \pm\frac{\pi/2 - C_2}{C_1 R}}$$



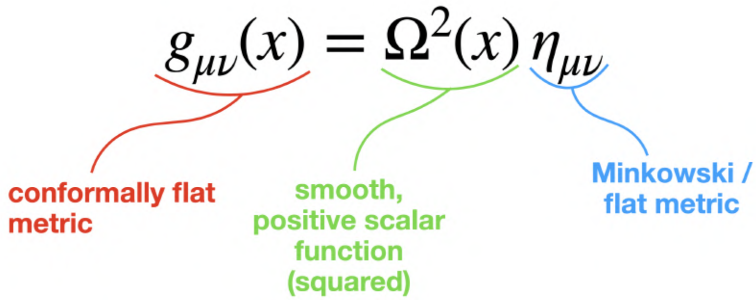
Near these points, $r(t) \to \infty$. So a particle in free fall from the origin moves radially outwards and reaches infinite radial distance in finite proper time (time τ registered in the conformal plane).

As the particle moves along this geodesic path, it accelerates. From the point of view of coordinates $r(t)$, the second derivative $\ddot{r}$ is positive, and grows – this is *radial acceleration.* The geometry "pulls" the particle outward faster and faster, even though no force acts on it; it's all a consequence of the structure of the geodesic, i.e. the intrinsic curvature of this plane.

In conclusion, in General Relativity we say that spacetime is "curved", when there is matter. But, what we actually mean by curvature is that the spacetime manifold has nonzero Riemann curvature, and this implies that vector fields fail to commute under parallel transport, and that geodesics are nontrivial. But there's a big difference between saying that a space has *curvature* and saying that it literally *bends* in some higher-dimensional direction. And honestly that's the non-rigorous picture that most people have in mind when they talk about curvature. This literal bending is just a visual metaphor, but not mathematically accurate, from the strict point of view of Differential Geometry. So, using the definition of curvature in Differential Geometry, spacetime is definitely curved. But using the loose definition of most people which assume curvature as a sort of "bending in an extra dimension", then spacetime cannot be considered curved in the presence of matter.

It is important to notice that not all geometries are *conformally flat.* A manifold is conformally flat if there exists a coordinate system in which the metric is a scalar multiple of the flat metric. In two dimensions, every surface is locally conformally flat because of something called the *Uniformization*

Theorem. But in higher dimensions, the situation is way more complex.



But, the key questions here are: is this curvature of spacetime just a mathematical effect or could it be mimicked by placing objects in a flat, or conformally flat, background but with a position-dependent rescaling of geometry?

This is exactly what gave birth to many alternative theories of gravity. For example, conformal gravity says that the spacetime metric is only defined up to a scaling – i.e., the true geometry is conformally invariant. *Einstein-Weyl geometry* is an example of such theories that generalizes General Relativity by introducing a separate connection that can scale vectors during parallel transport. Another one is *conformal teleparallel gravity*, where the role of curvature is replaced by torsion, and a scalar field is introduced to restore conformal symmetry to the torsion-based action. Then there is the *twistor theory*, developed by Roger Penrose, which treats spacetime as emerging from more fundamental conformally invariant complex structures. And there are others...

Of course, if conformal gravity, or alternative theories, will replace or generalize General Relativity, then we need experiments to validate them. This is out of the question. One would need to show that a non-trivial Riemann tensor does not necessarily imply intrinsic curvature, in the physical ("bending") sense. Until then, conformal models remain very interesting from the mathematical point of view.

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