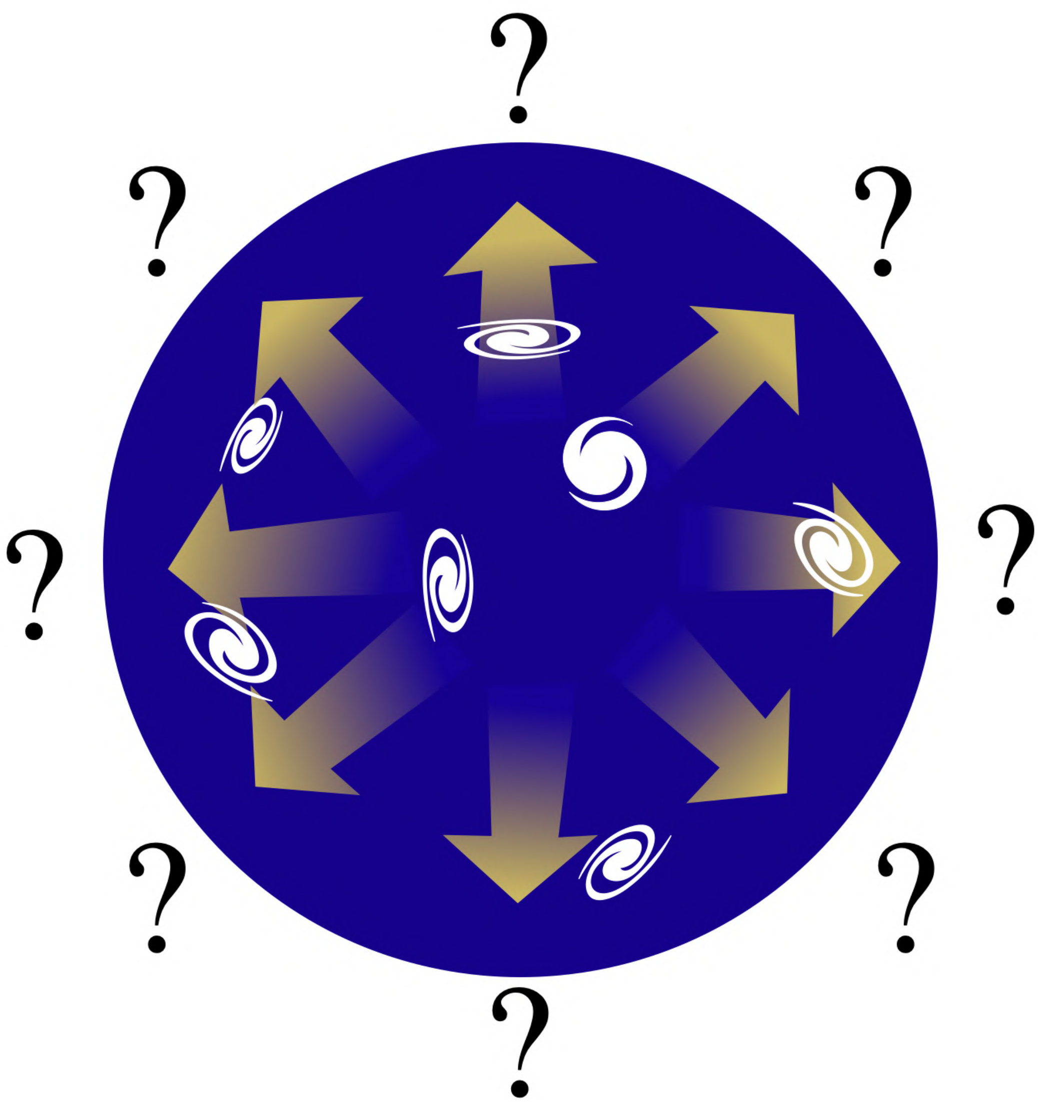
What Is The Universe Expanding INTO? (according to pure math)

by DiBeos



Introduction

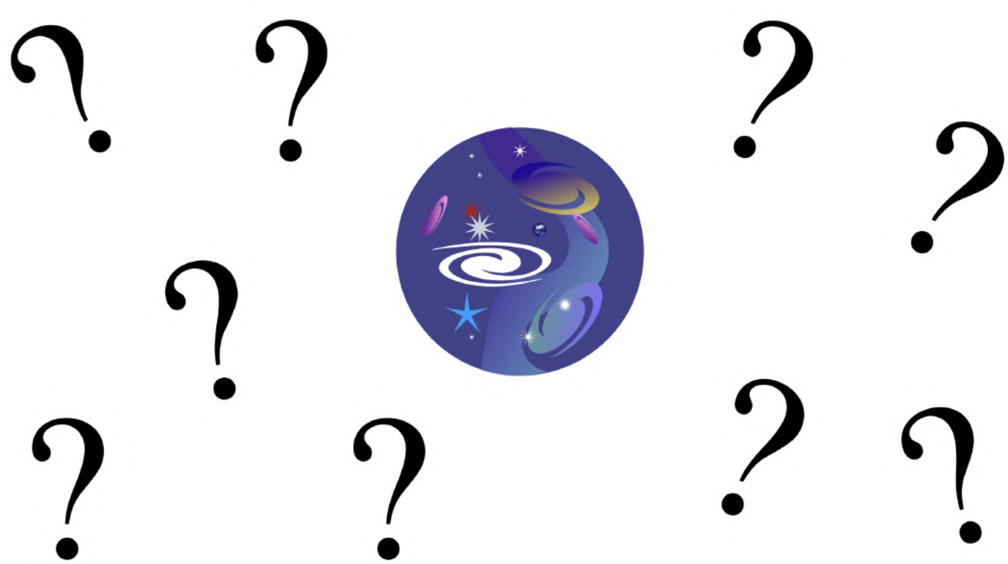
In the beginning of the 20th century, Einstein developed his theory of General Relativity (GR), which described gravity not as a force, but as the curvature of spacetime itself. When he applied his equations to the entire universe, he expected a *static cosmos* (i.e. no expansions or contractions of spacetime) and added a *cosmological constant* to prevent it from collapsing or expanding. However, Russian physicist Alexander Friedmann and later Belgian priest-physicist Georges Lemaître independently found that Einstein's equations naturally predicted a dynamic universe. Though Einstein initially dismissed these solutions, the mathematics was clear: GR allowed (and even required!) cosmic evolution.

Then came Edwin Hubble. In 1929, using *redshift* measurements of distant galaxies, Hubble discovered that galaxies were moving away from us, and that their velocity was proportional to their distance, which is now known as *Hubble's law*. This was the observational breakthrough that confirmed Friedmann and Lemaître's predictions: the universe is expanding. Importantly, this expansion is not like galaxies flying through space, but space itself stretching, increasing the distance between galaxies over time. This will be an important distinction for what comes next in this file.

Today, the expanding universe is a cornerstone of modern cosmology. The current standard model (ΛCDM model) describes a universe that began in a hot, dense state (the *Big Bang*), and is now expanding at an accelerating rate due to a mysterious component called dark energy, represented by a cosmological constant. Precise observations have confirmed this model with high precision. Yet, profound questions remain: what *dark energy* is, what happened before the Big Bang (if that question makes sense at all), and whether the universe will expand forever. But one thing is clear: we now understand that the universe is not static, and this insight emerged from a powerful synergy between mathematics, theory, and observation.

And yet, with all this progress, one question keeps coming back from students, science fans, and even physicists. If the universe is expanding, what is it expanding into?

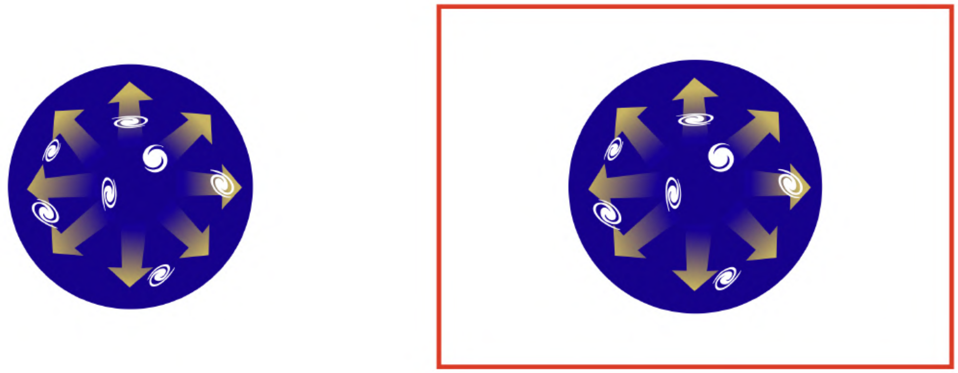
What Is the Universe Expanding Into?



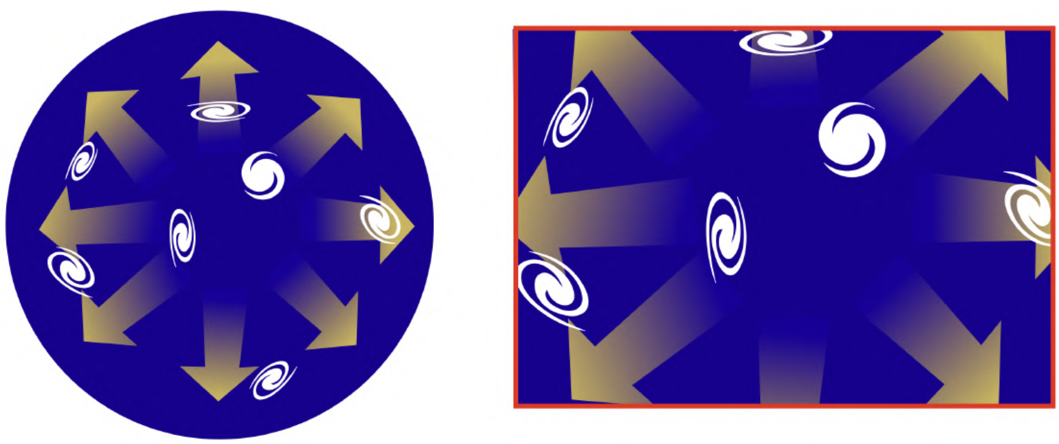
The straightforward answer is: **nobody knows**.

BUT, when people ask this question, what they actually mean (what they actually want to know) is:

"How come physicists say that the universe is expanding if they don't even know what it's expanding into?!"



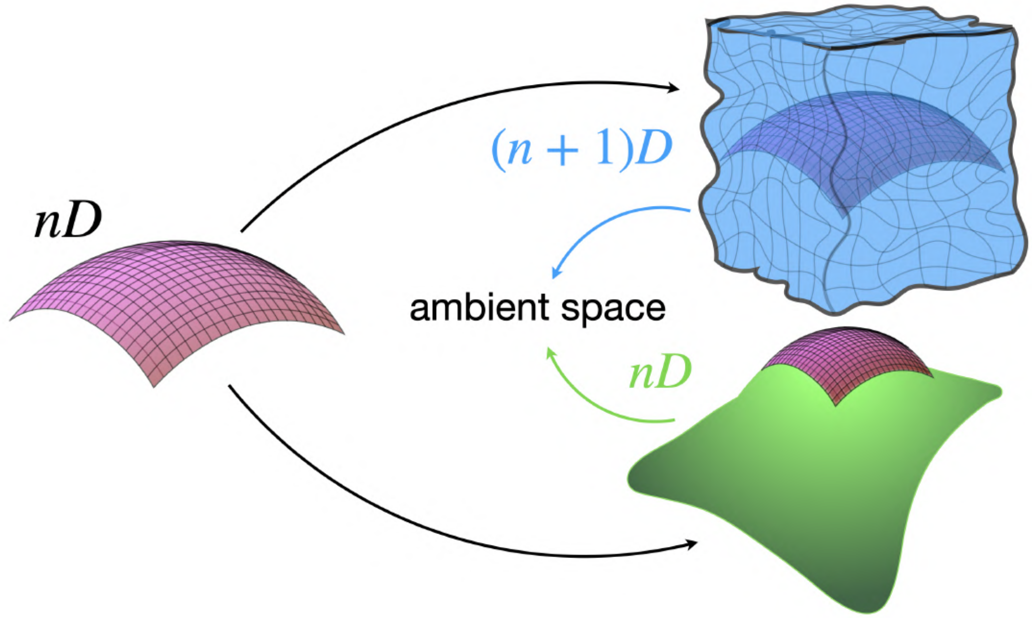
Even though this is a better way of formulating the question, it exposes a clear flawed assumption: that if something **expands**, it must expand **inside of something else**.



But here's the thing:

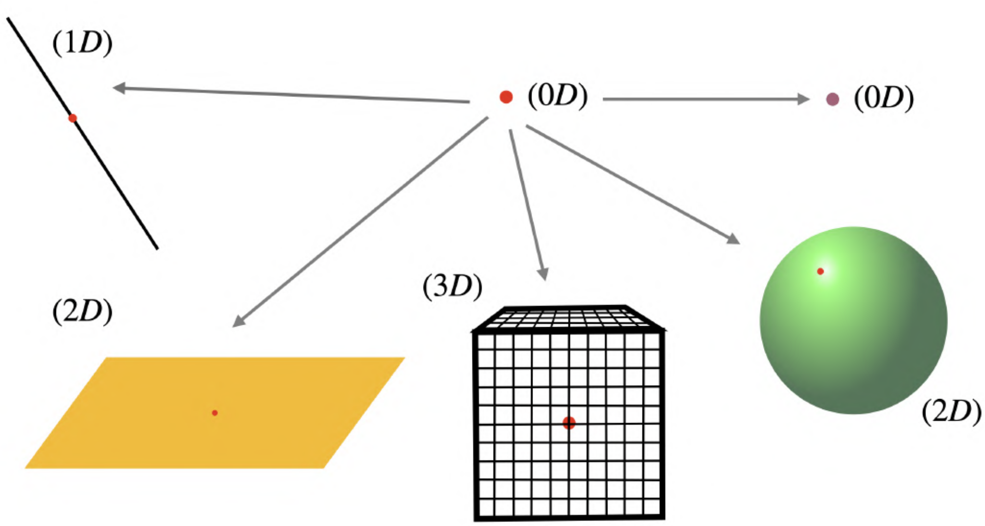
"Expansion" and "being inside of something" are two completely different concepts. They are independent from one another.

Unfortunately, physics alone (as well as intuition) is incapable of explaining this distinction, and that's where the confusion lies.

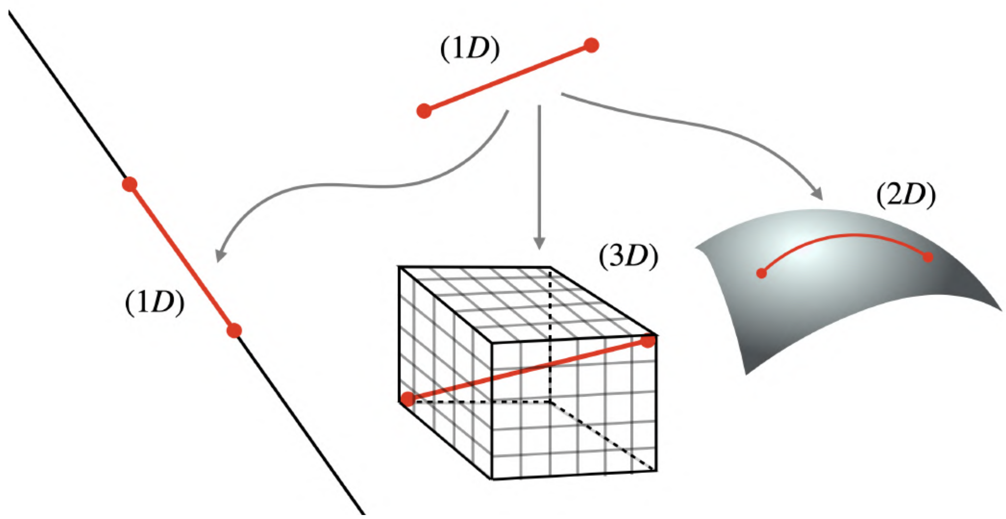


In pure mathematics, if you take a space of dimension n, you can choose to embed it in a higher-dimensional space (say $(n+1)$-dimensions, or more), or even in a space with the same dimension n. This "larger" space is called the *ambient space*.

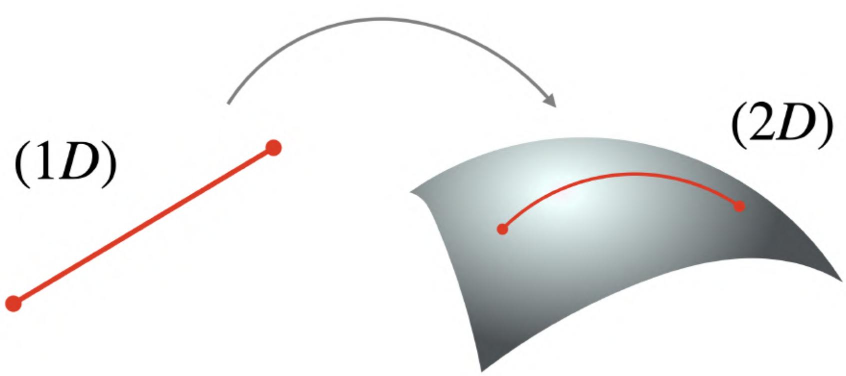
So, for example, you could embed a *point* (which is *zero*-dimensional) into a line (which is 1-dimensional). You could embed it into a plane (2D), or into a cubic region (3D), or into the surface of a sphere (2D), or even into another point (0D), and so on.



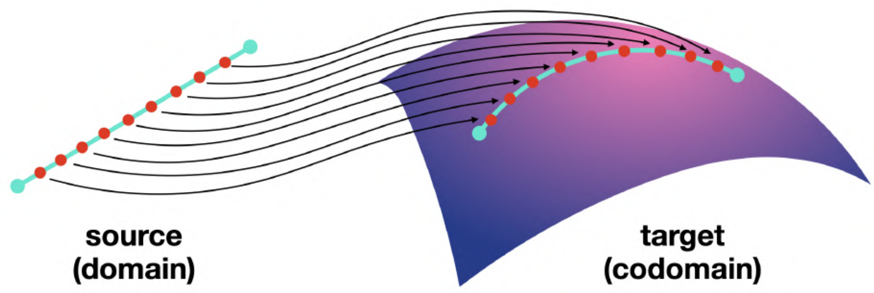
What about a *line segment*? Well, you could embed it into a line ($1D$), into a cube ($3D$), or even into a curved sheet ($2D$).



In the last case, the line segment lost one of its properties, since it's not straight anymore, right? Not necessarily...

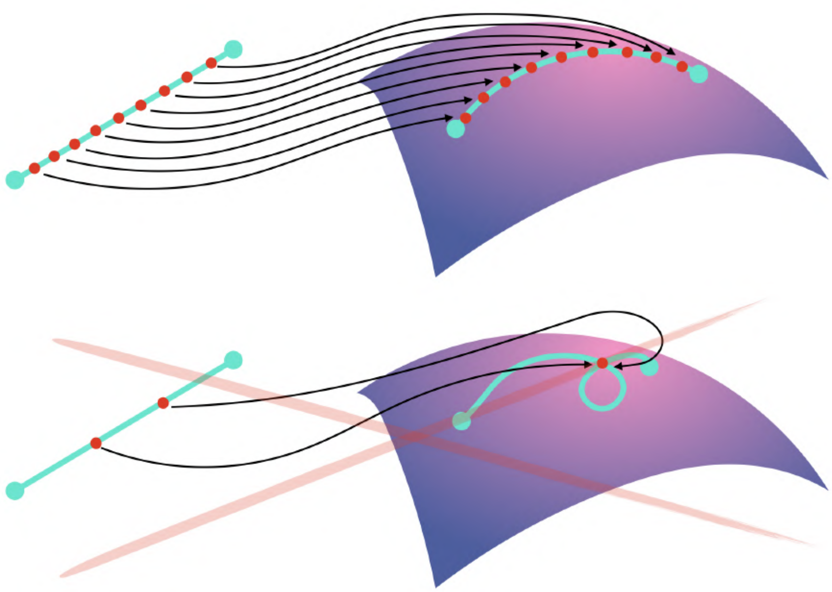


First of all, what is an **embedding**?

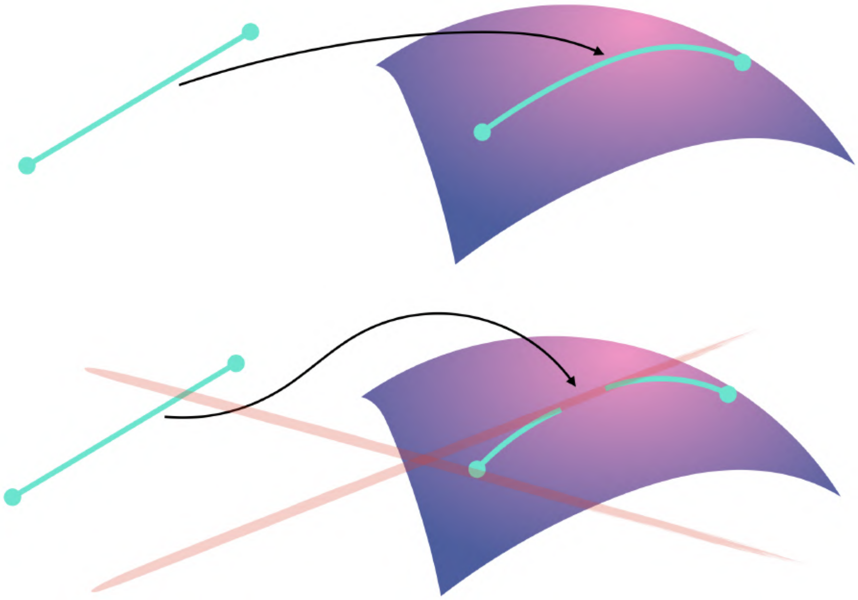


An embedding is a mapping that associates each point of a *source space* to a unique point in a *target space*, such that:

(a) The points stay distinct. So, no two different points in the source space get mapped to the same place.



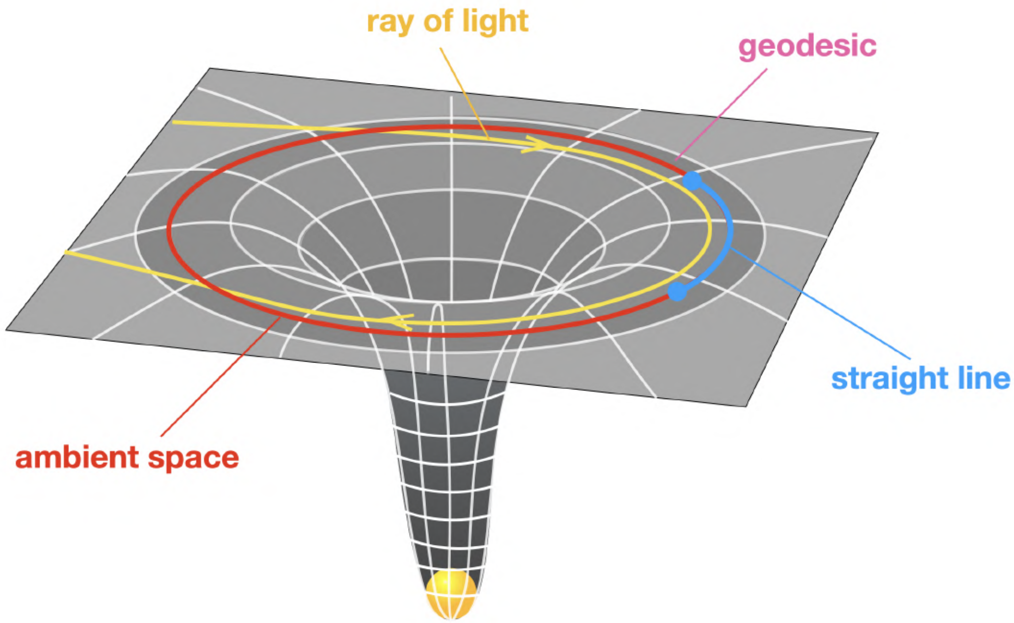
(b) The shape and structure of the source space are preserved. So, the topology does not change (like preserving continuity, genus and the openness of subsets).



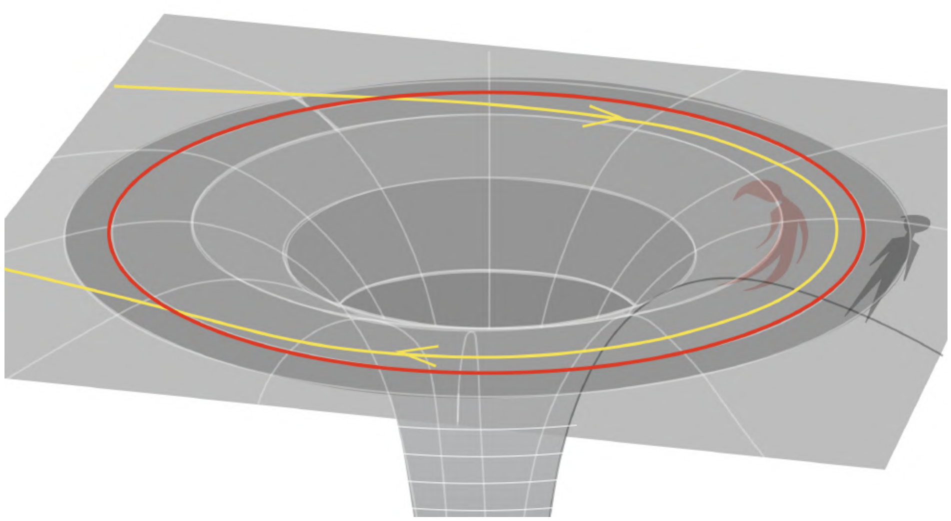
Rigorous definition (Topological Embedding): Let X and Y be topological spaces. A function $f : X \to Y$ is called an *embedding* if:

1. f is *injective* (1-to-1) ;
2. f is *continuous* ;
3. The map $f : X \to f(X) \subset Y$ is a *homeomorphism* onto its image; that is, $f(X)$ inherits the same topology from Y as X already has.

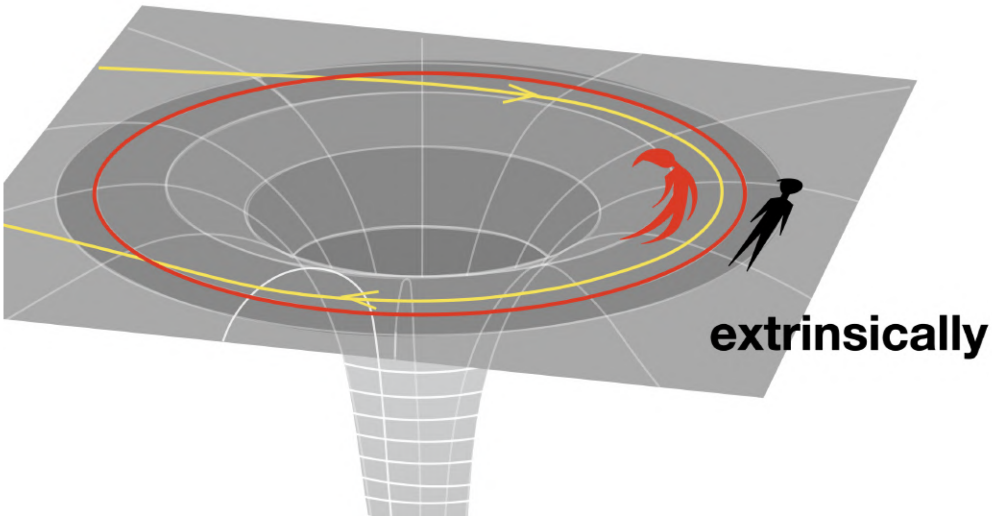
For example, in nature, when a ray of light passes near a massive object, it appears to bend, as if gravity were pulling it. But what's actually happening is that the ray of light is following a geodesic, which is the "straightest" possible path in the curved spacetime (i.e., the ambient space) around that mass. So, that's the very definition of a straight line in this particular ambient space.



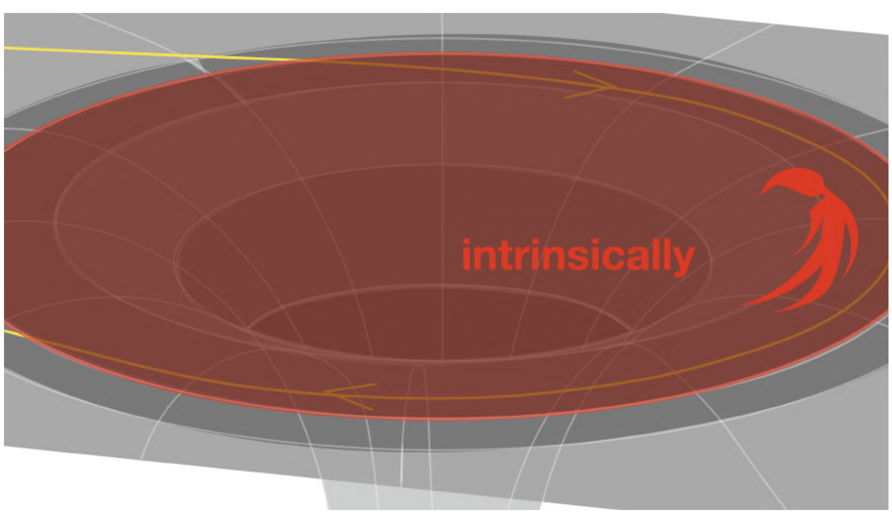
From our perspective (in flat space), the path looks curved, but to the light itself (and to any observer living within that curved region), the path is still straight.



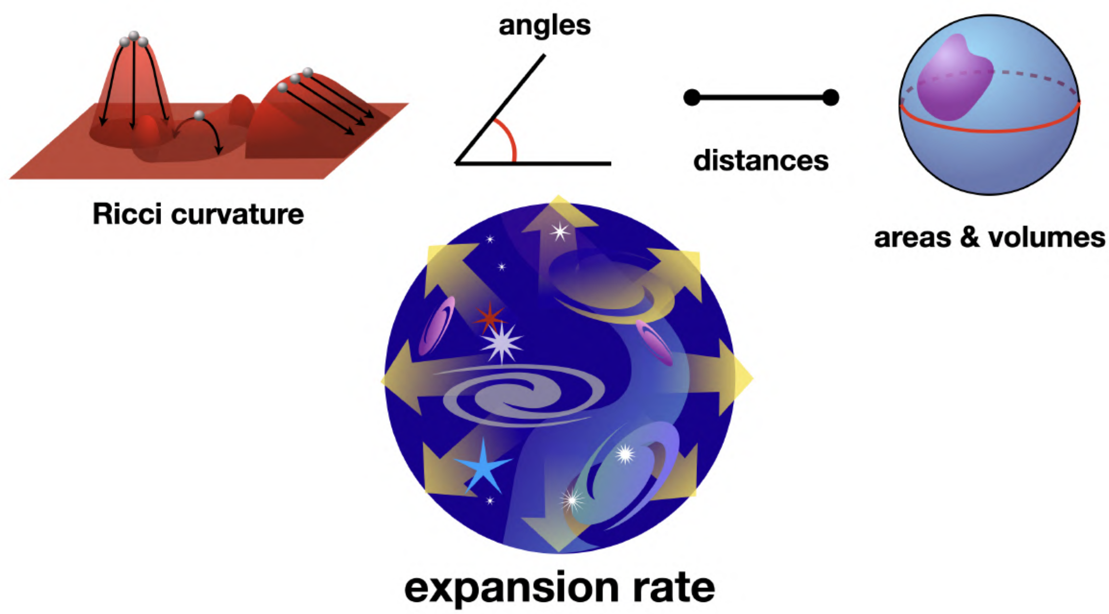
Mathematically, this is exactly what an embedding captures: the light's path remains "straight" intrinsically, even though it appears extrinsically curved when embedded into a larger reference space.



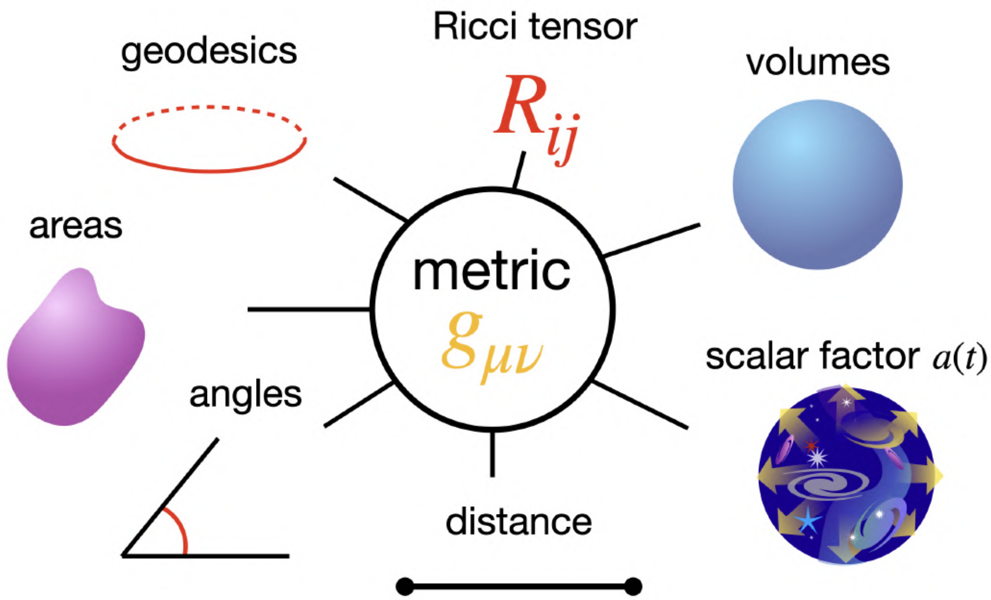
The underlying idea here is Gauss's famous **Theorema Egregium** (or "Remarkable Theorem"), which says that intrinsic properties do not depend on how the source space sits inside of an ambient (target) space.



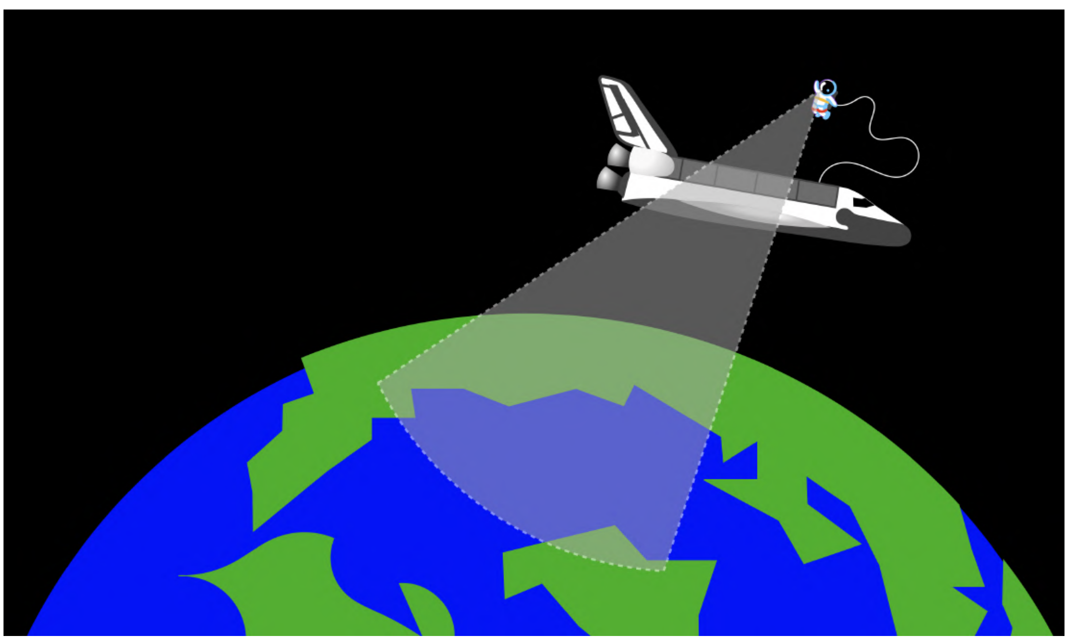
Some examples of intrinsic properties are the Ricci curvature, distances, angles, geodesics, areas, volumes, and even... EXPANSION RATES!



One of the key requirements for a property to be *intrinsic* is that it must be derivable only from the metric $(g_{\mu\nu})$ of the space, i.e. from the internal "ruler" that allows us to measure distances, angles and curvatures without relying on how the space is embedded in any higher-dimensional setting.

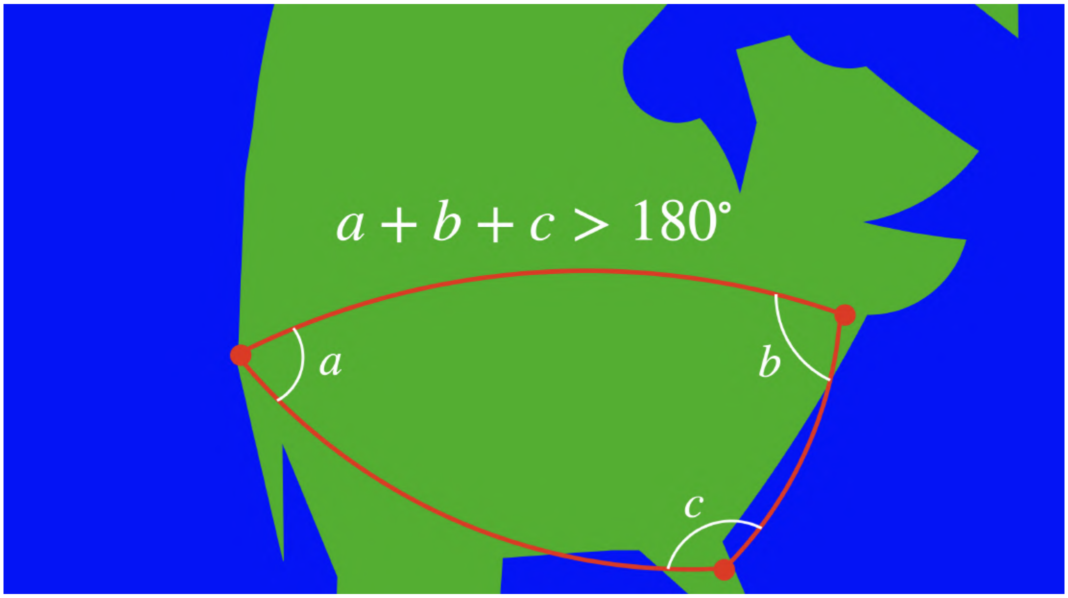


For example, we don't need to travel all the way to space and look at the Earth from its ambient space in order to see that it's curved.

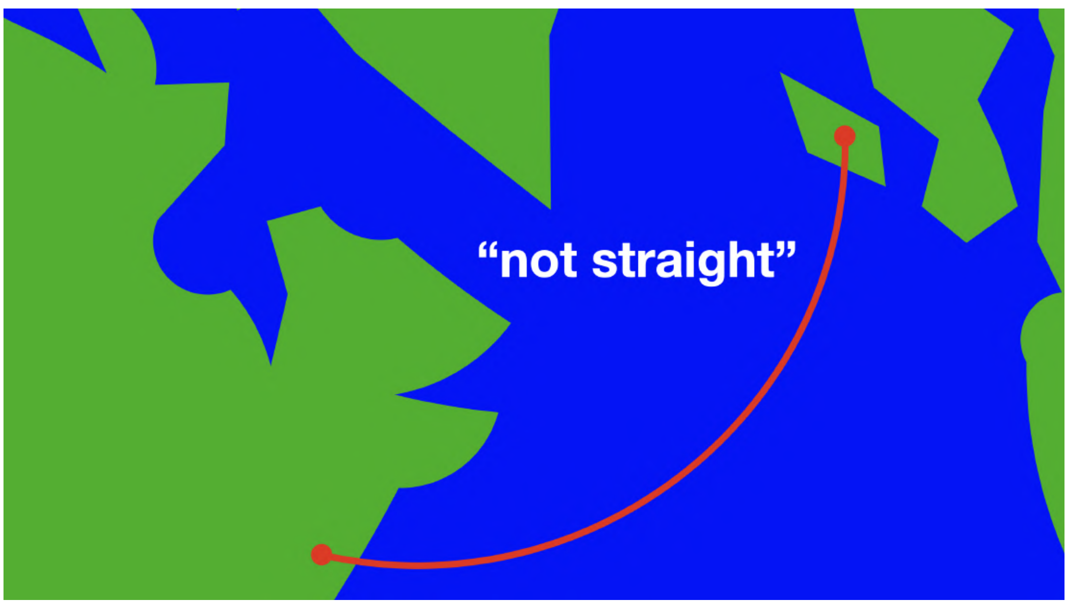


We can simply observe intrinsic phenomena that take place on its surface, like:

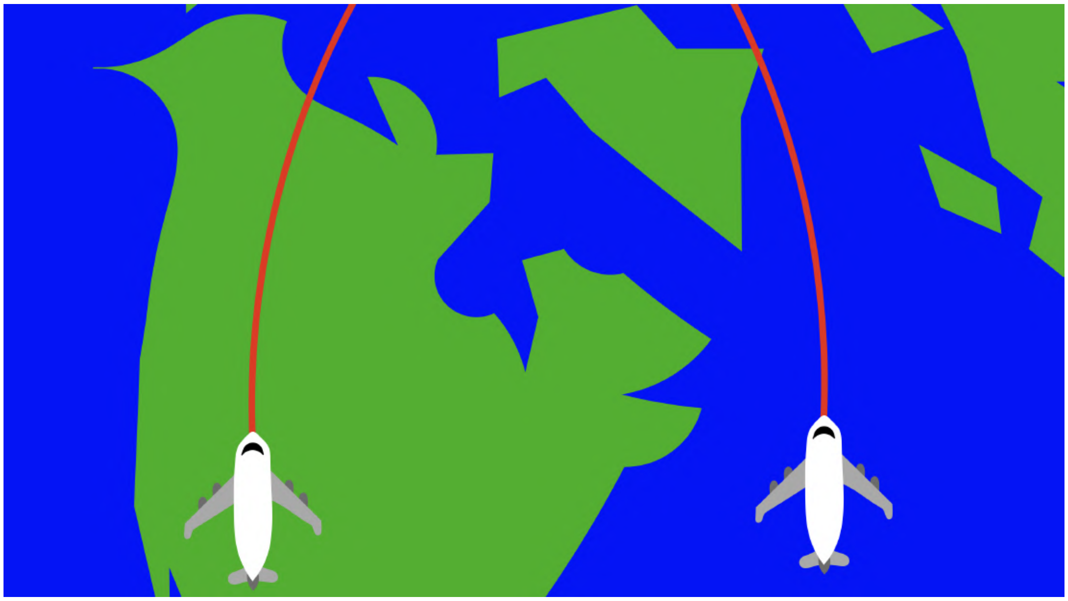
∗ The *sum of angles* of a triangle over large distances is always greater than 180°.



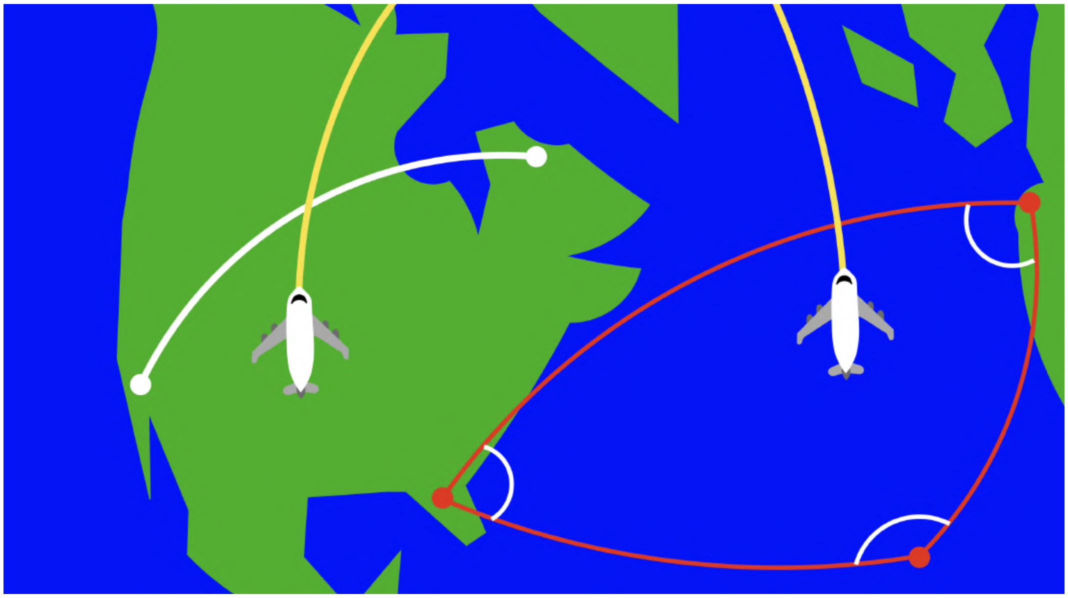
∗ The *shortest routes* are always great-circle paths, not what we would call "straight line" anymore.



∗ *Geodesic deviation*, i.e. two initially parallel paths eventually converge at the poles.

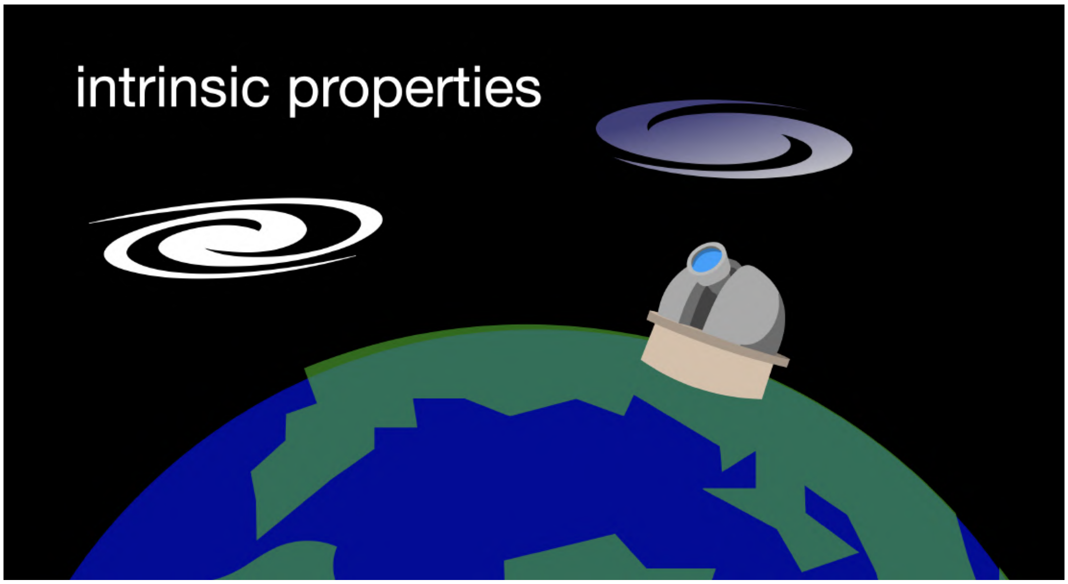


This implies that the Earth's surface could very well be all there is in the universe (effectively being the universe itself), and , even then, we would still be able to note that it has intrinsic curvature.

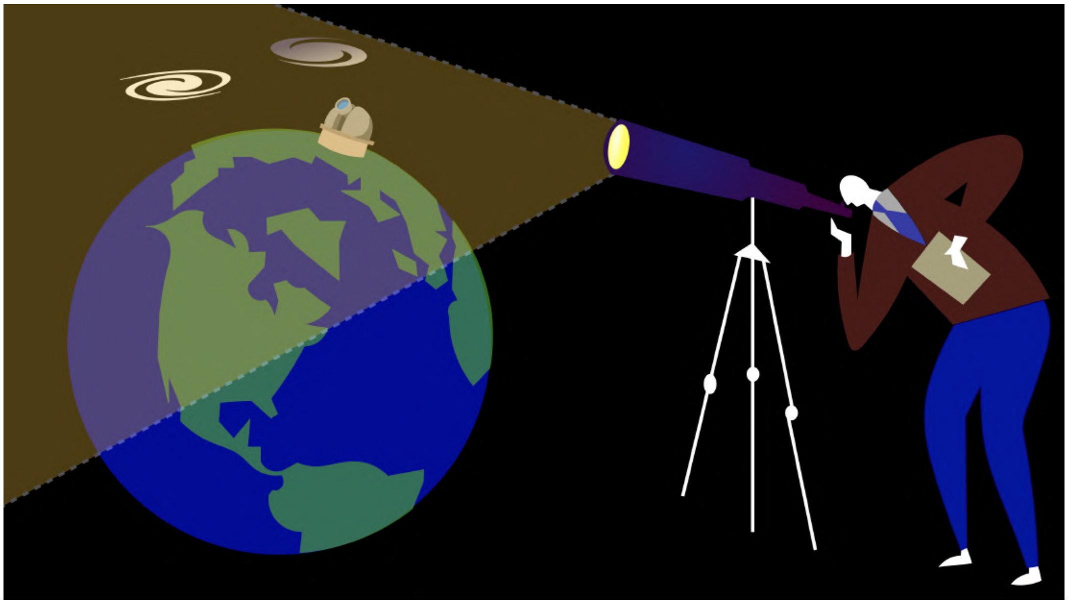


That's exactly what happens to what we currently call: the known universe. We can measure its intrinsic curvature (and all the other intrinsic properties) without ever assuming that it is embedded in a larger ambient space.

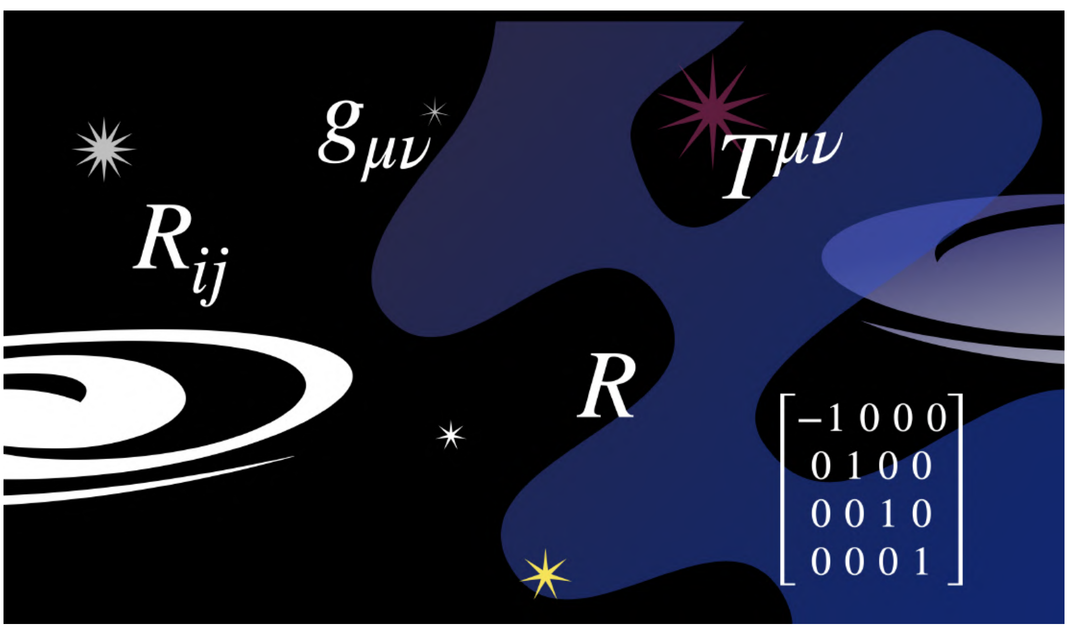
By the way, intrinsic properties are very important in physics because they represent physical characteristics of spacetime that are independent of coordinate systems or embedding choices. In other words, they're invariant under *reparametrization* and are the same for all observers, which makes them perfect for expressing physical laws.



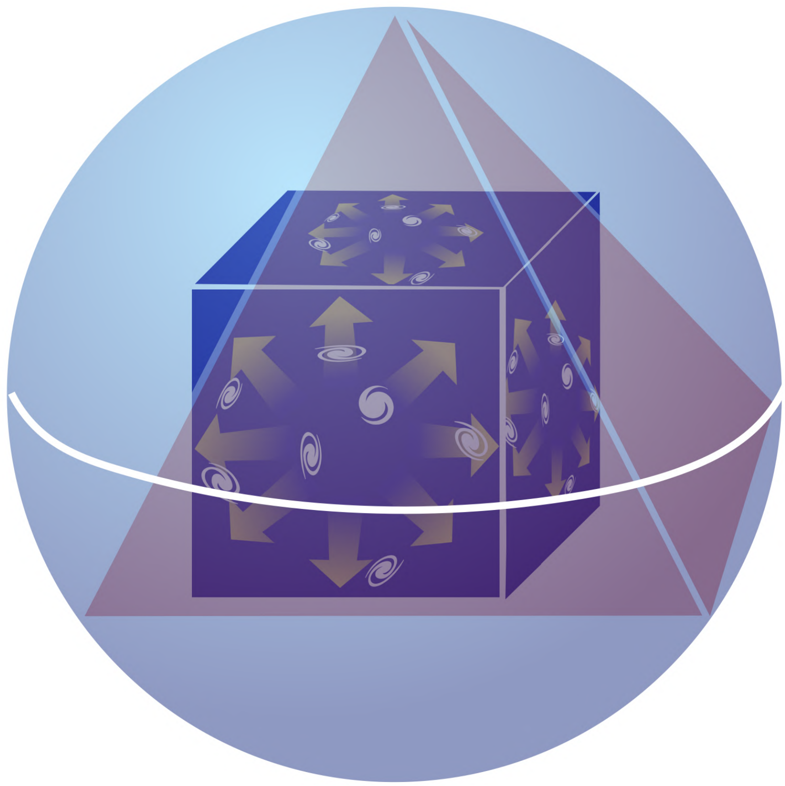
As we said earlier, expansion rate is one of these intrinsic properties of a space.



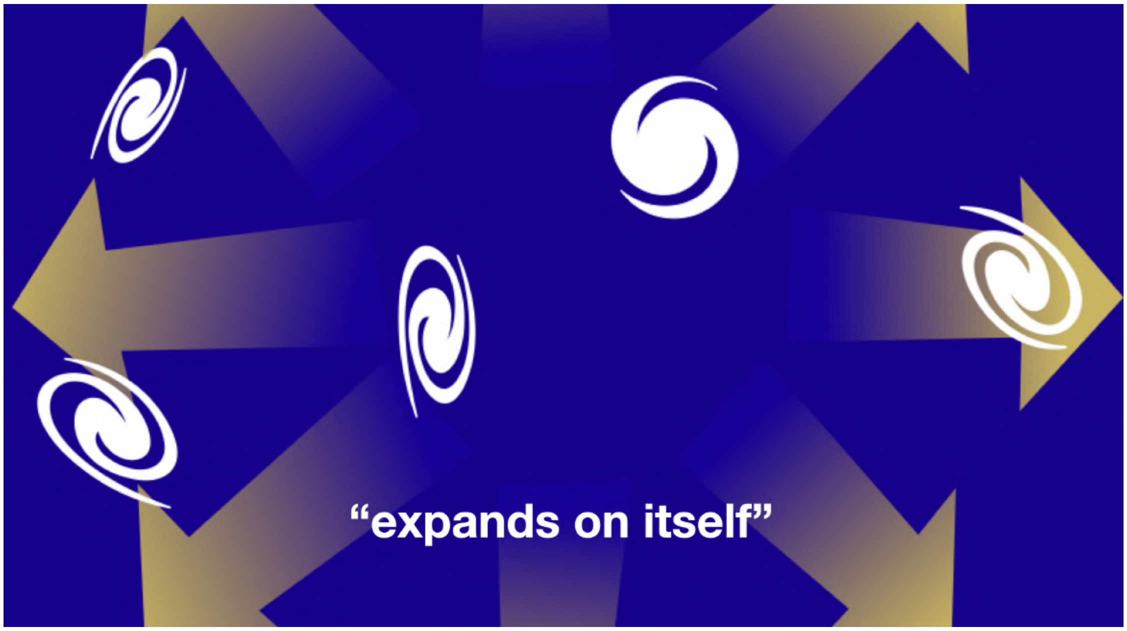
When astronomers observe galaxies moving away from each other, they're measuring an intrinsic property of the universe. So, even if we have no idea whether the universe is embedded in a higher-dimensional reality (or whether there is no "outside" at all), we can still say, with confidence, that it is expanding.



That’s the power of mathematics: it allows us to describe the behavior of the universe from within, even when we can’t possibly visualize it.

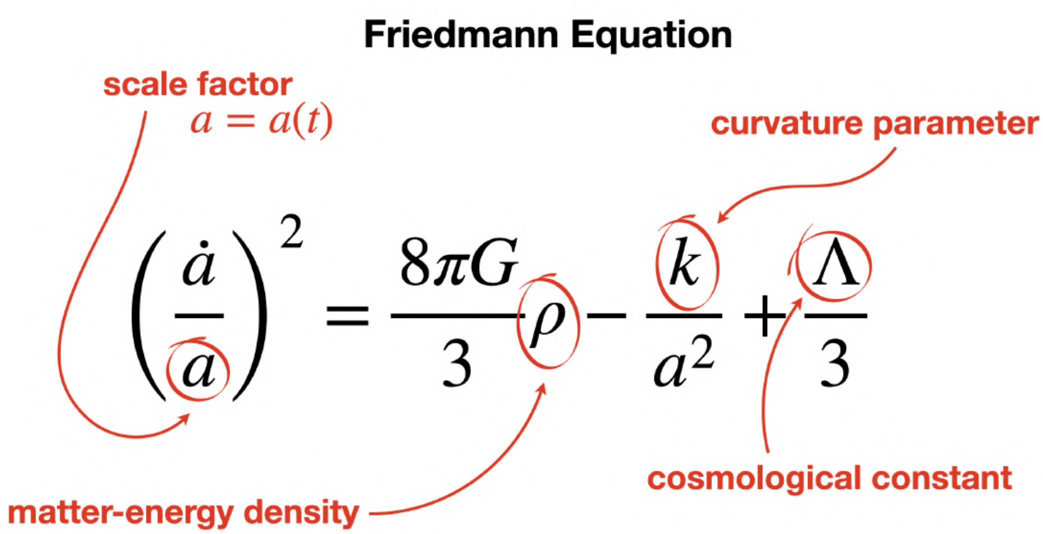


Of course, our brains are so used to perceiving the 3-dimensional world around us that we have a hard time trying to imagine a universe that (sort of) “expands on itself”.



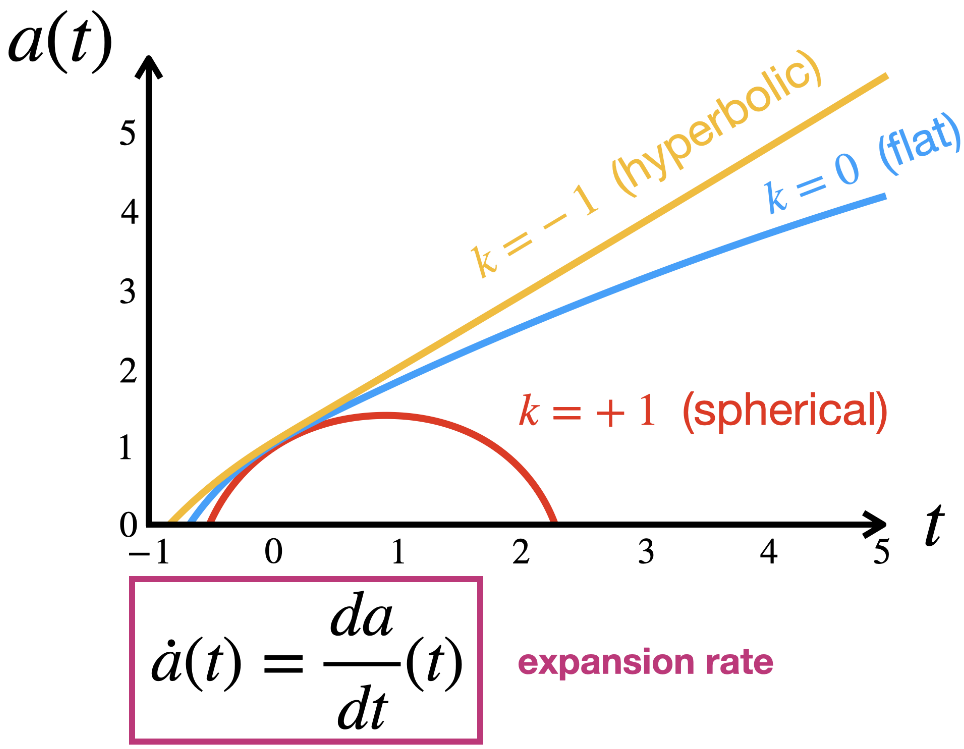
The Math of Expansion

This is the mathematical formula that describes the expansion of our universe:



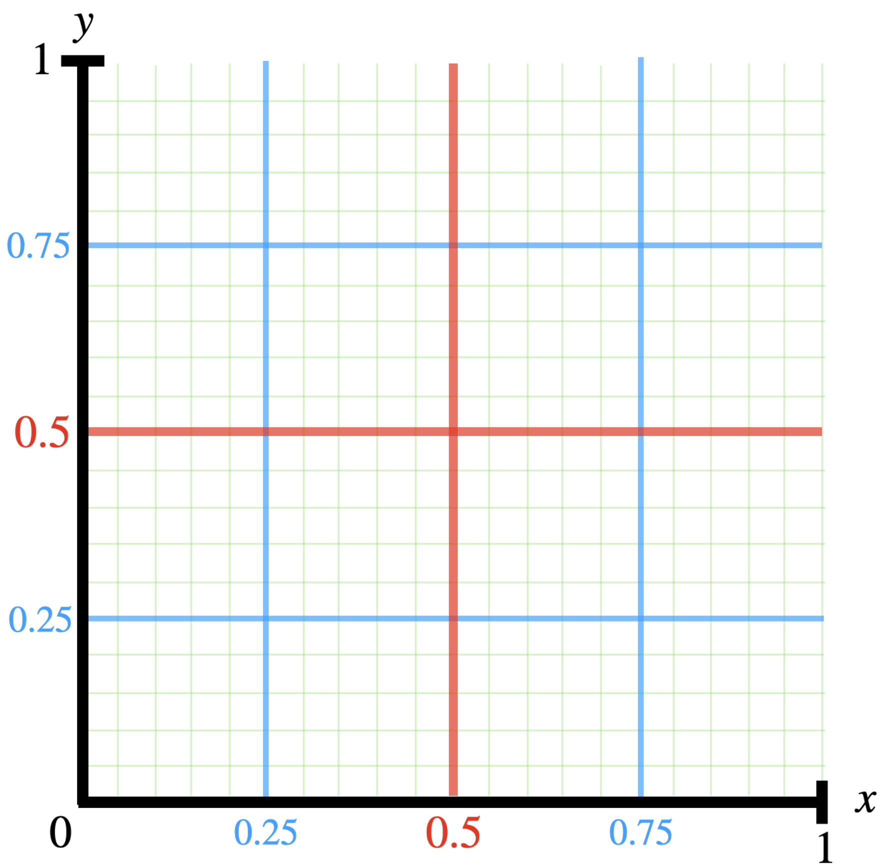
ρ is the *matter-energy density*, k is the *curvature parameter*, Λ is the *cosmological constant*, and a is the **scale factor**, which depends on time t: $a = a(t)$.

So, the scale factor changes over time. Its first derivative (or rate of change) $\frac{da}{dt} = \dot{a}$ is what we call the **expansion rate** of the universe.



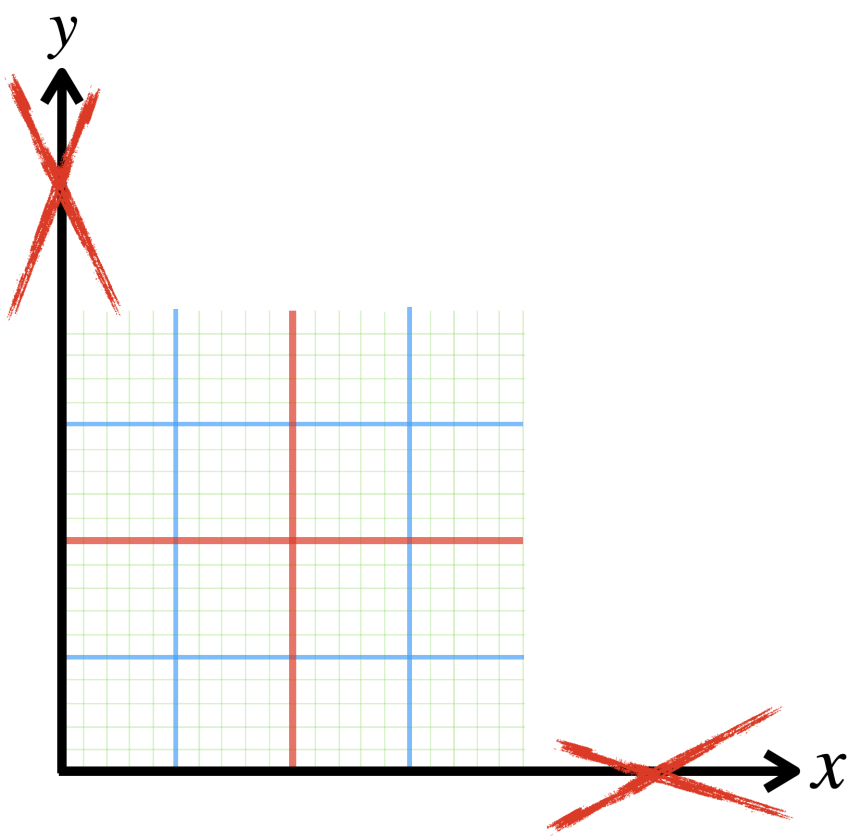
Let's see a simplified illustration of how the scale factor $a(t)$ works.

We start by defining a $2D$ square manifold where each point is labeled by two intrinsic coordinates: $(x, y) \in [0, 1] \times [0, 1]$.

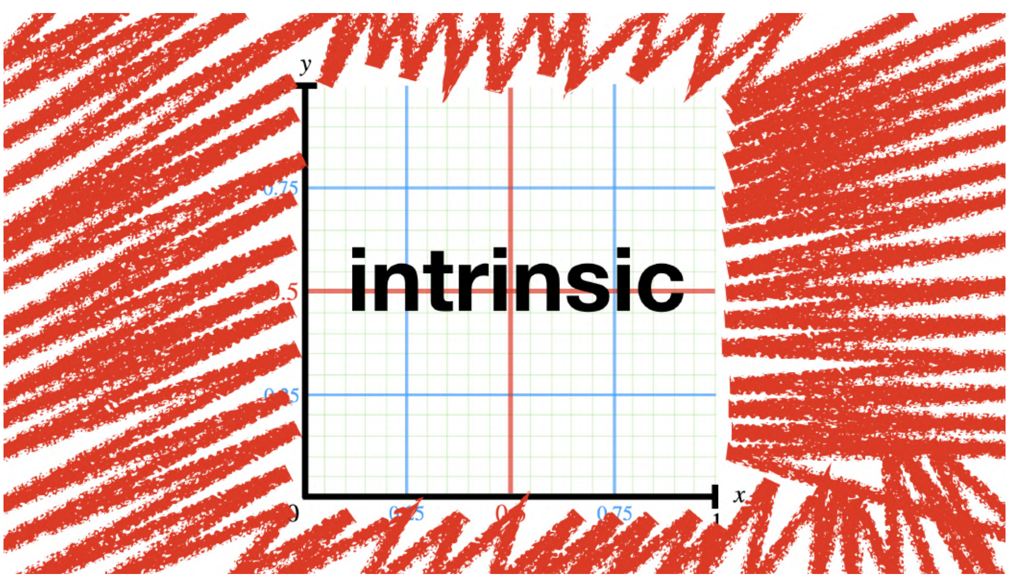


$$(x, y) \in [0, 1] \times [0, 1]$$

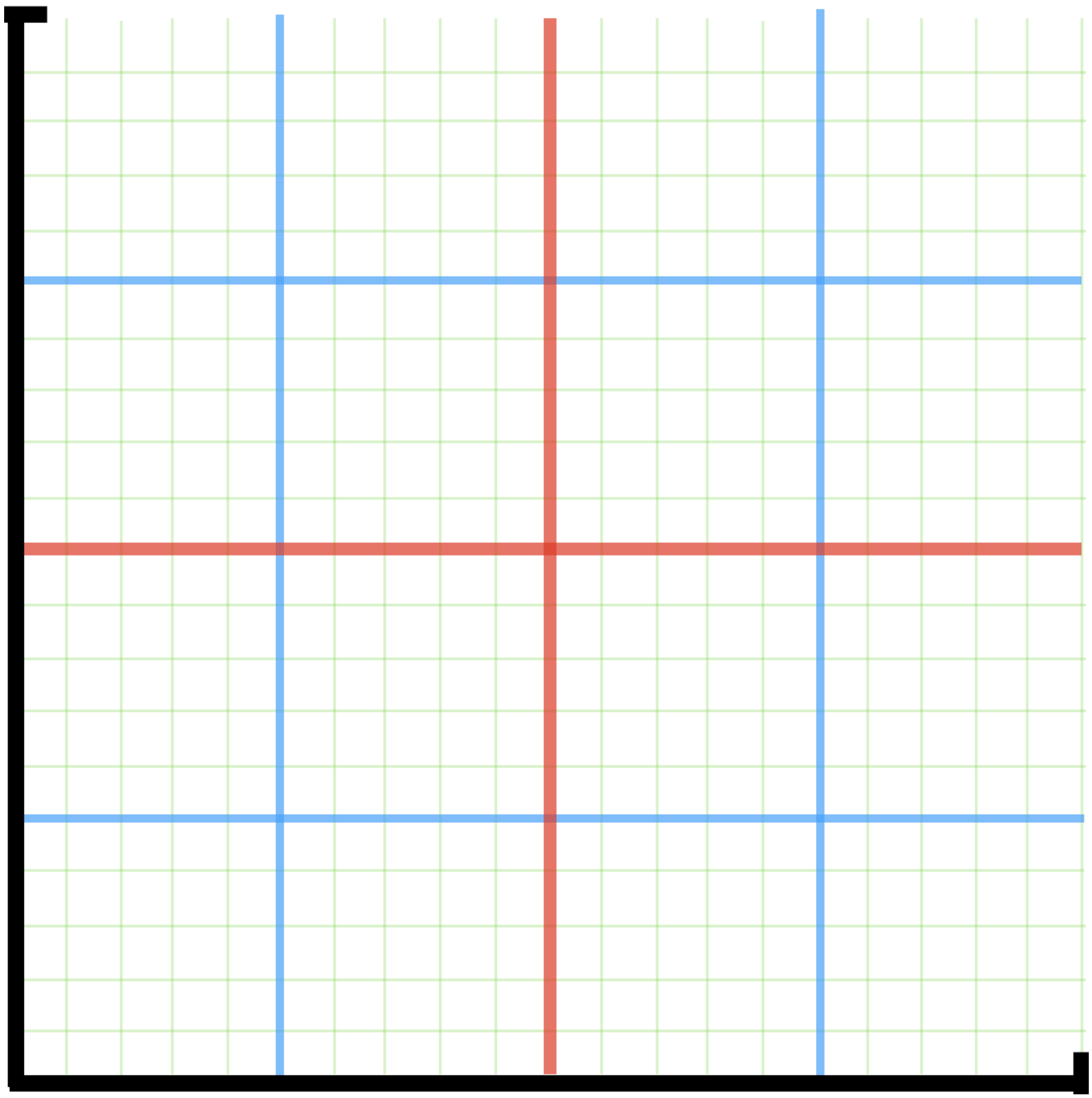
Notice how this coordinate system does not extend to points "outside" of the square, since it's not supposed to be embedded in a higher-dimensional space. And that's why these coordinates are *intrinsic*.



Actually, this representation itself is flawed, since there is no ambient space to begin with.

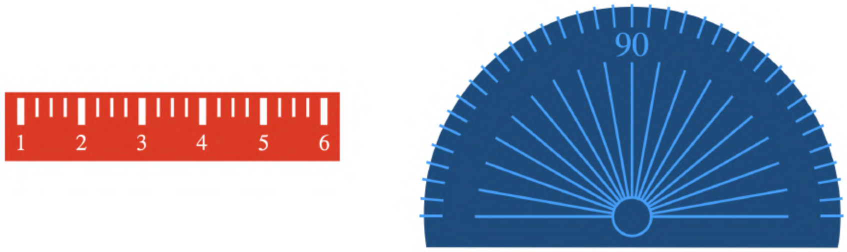


In order to really study only intrinsic properties, we should view this square covering the entire page, since it's all there is. Unfortunately, this is technically impossible because the page is rectangular... but you get the idea...



Now, let's define a *metric* in it, i.e. a ruler to measure distances and angles.

$$g_{\mu\nu} = \begin{bmatrix} a(t) & 0 \\ 0 & a(t) \end{bmatrix}$$



This metric describes a flat space, but with a time-dependent scale factor $a(t)$ in it:

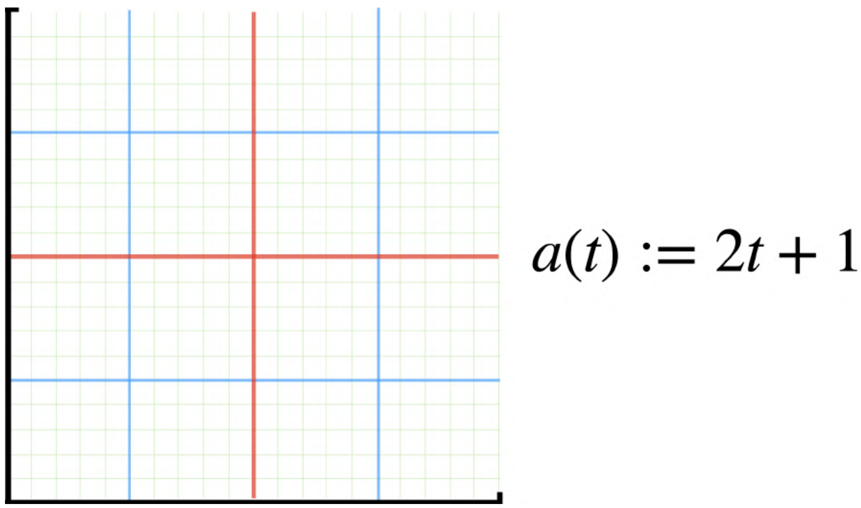
$$ds^2 = a(t)^2 \left(dx^2 + dy^2\right)$$

expanding geometry

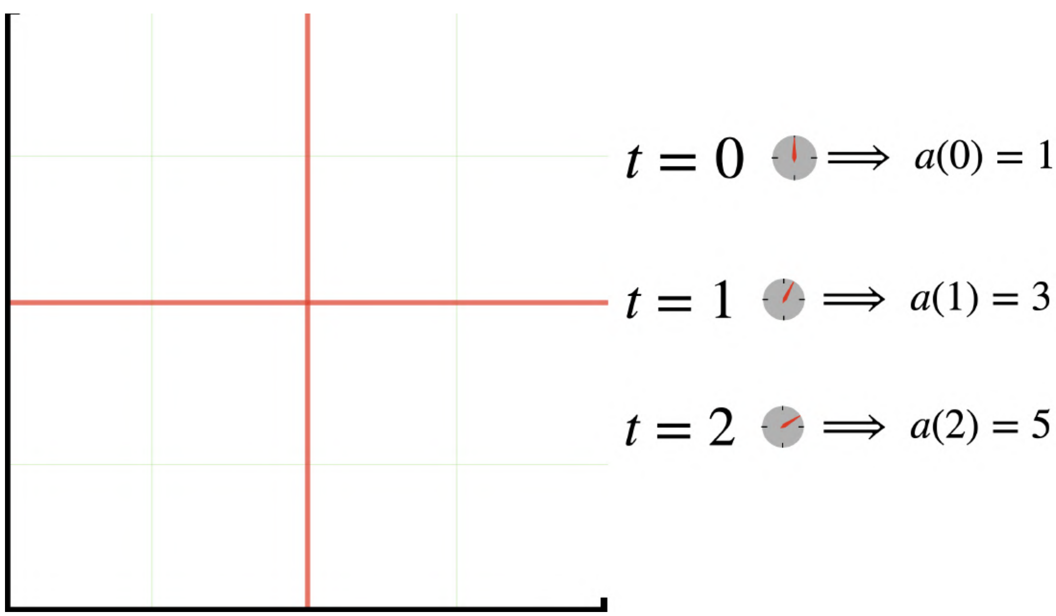
infinitesimal proper distance

Here, ds is the *infinitesimal proper distance* between neighboring points.

In order to make the example more concrete, let's define a *linear* scale factor explicitly:



This means that at time $t = 0$ the square has "side length" 1. At time $t = 1$, $a(1) = 3$, so all distances are tripled. At $t = 2$, $a(2) = 5$, and so all distances are $5\times$ longer than at the beginning. And so on...



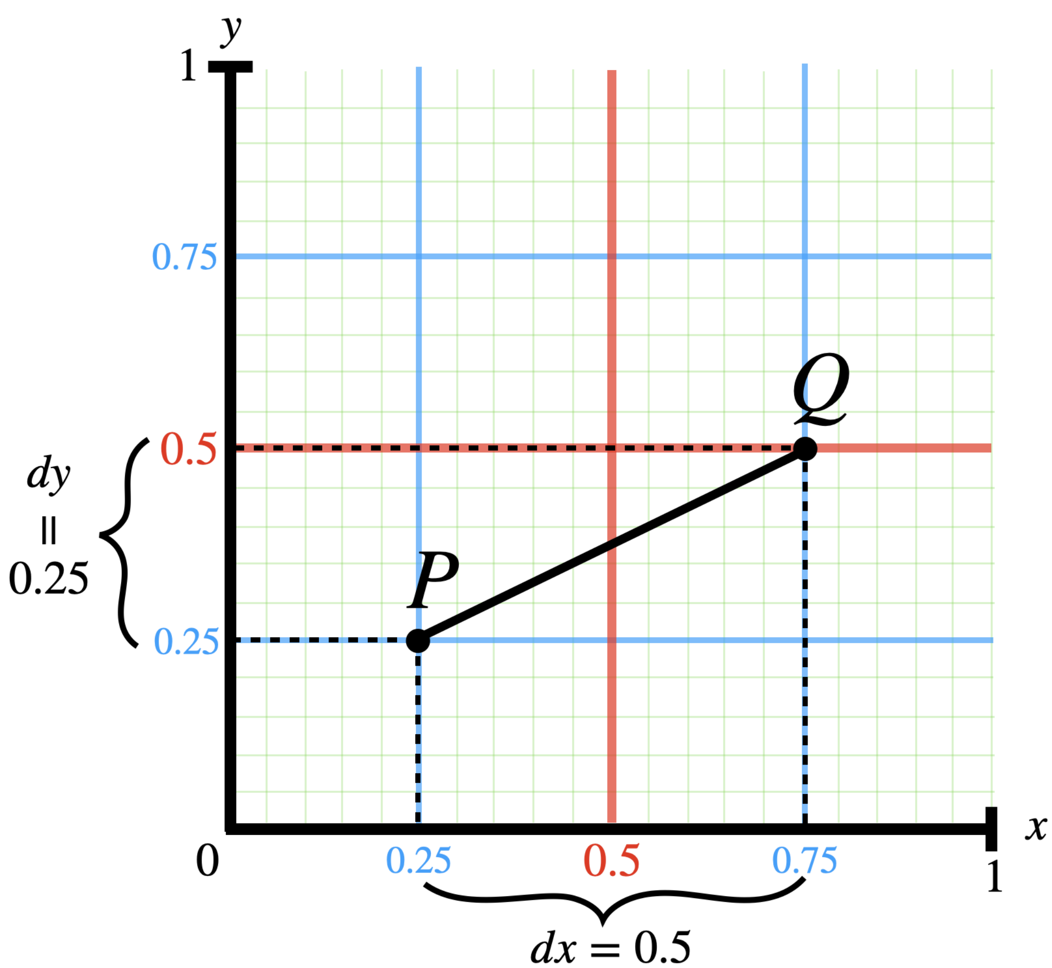
This universe is expanding at *constant speed*.

By the way, our actual universe is not expanding at a constant rate, but rather at an increasing rate! This means that the expansion is speeding up. The distances between galaxies are growing faster and faster.

In mathematical terms, the second derivative of the scale factor $a(t)$ is positive. Actually, in some really distant parts of the universe, galaxies are moving away from us faster than the speed of light, not because they're speeding through space, but because space itself is stretching!

Back to our toy model of the universe, let's take two points:

$$P(0.25\,,\,0.25) \quad \text{and} \quad Q(0.75\,,\,0.5)$$



Their coordinate separations are: $dx = 0.5$ and $dy = 0.25$.

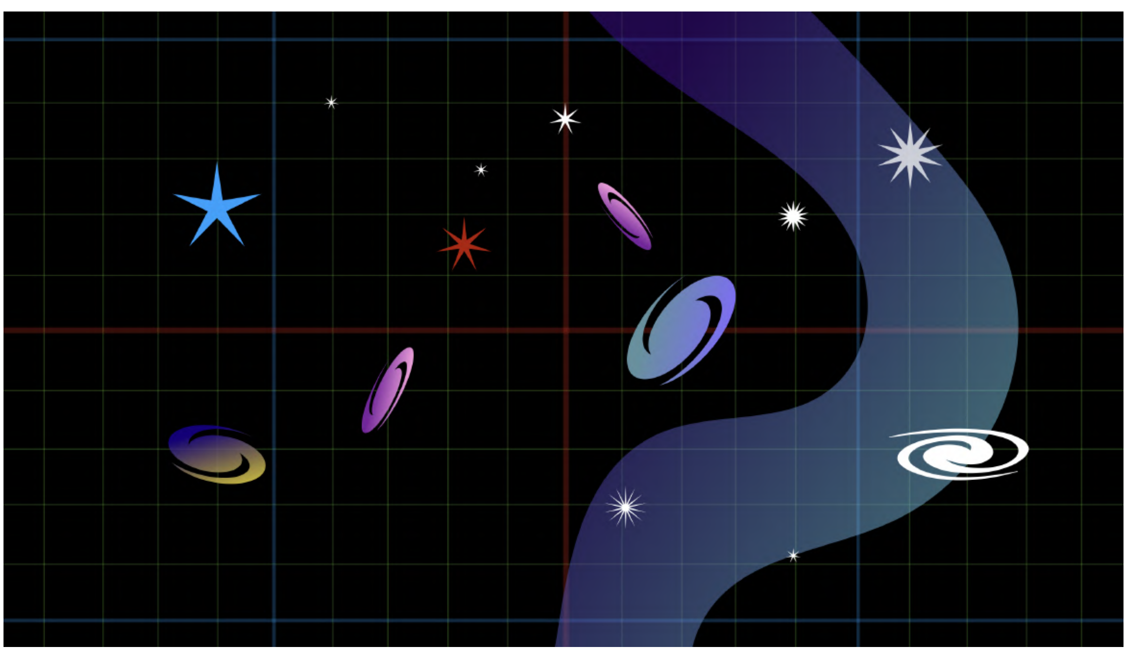
Therefore, the so-called *proper distance* between points P and Q, when using the metric, is:

$$
\begin{aligned}
D(t) = ds = a(t)\,\sqrt{dx^2 + dy^2}&\\
&= (2t+1)\,\sqrt{(0.5)^2 + (0.25)^2}\\
&= (2t+1)\,\sqrt{0.25 + 0.0625}\\
&= (2t+1)\,\sqrt{0.3125}\\
&\simeq (2t+1)\cdot 0.559\\
&\Rightarrow \boxed{D(t) \simeq 1.118\,t + 0.559} \quad \text{(proper distance)}
\end{aligned}
$$

As mentioned earlier, the expansion rate is calculated with the derivative of the scale factor with respect to time:

$$\dot{a}(t) = \frac{da}{dt}(t) = \frac{d}{dt}(2t+1) = 2$$

So, for each second, the distance between all points increases by a factor of 2 per unit distance.



Notice how we calculated the expansion rate without ever referring to any ambient space, just in terms of intrinsic properties.

In conclusion, the expansion of our universe does not depend on the existence of a sort of "larger ambient space" around it. We know the universe is expanding because we can experimentally observe that the space between galaxies is increasing (through the physical phenomenon of redshift). And importantly, this expansion is determined entirely by measurements of intrinsic properties, not by any external or higher-dimensional embedding.

Dive Deeper

$$\left(\frac{\dot{a}}{a}\right) = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$k = -1 \quad \Longrightarrow \quad \text{hyperbolic (open) universe.}$$

$$k = 0 \quad \Longrightarrow \quad \text{flat (open) universe.}$$

$$k = +1 \quad \Longrightarrow \quad \text{spherical (closed) universe.}$$

In order to simplify the problem, and thus find solutions that will allow us to derive approximate insights about the physical predictions of the nature of our universe, let us make some assumptions:

Assume that the universe can be modeled as a *dust*, like a fluid with no internal pressure. This is a decent approximation in a **matter-dominated** universe.

When modeling the universe on large scales, its contents are usually classified based on how their *energy density* (i.e. the amount of energy packed into a given volume of space) evolves/changes over time as the universe expands. There are two conventional elements to be considered: **matter** (i.e. dark matter and ordinary matter) and **radiation** (i.e. photons and neutrinos). The famous equation $E = mc^2$ is the one that allows us to treat both mass and radiation as forms of energy.

Back to our dusty, pressureless, matter-dominated universe, galaxies are modeled as non-interacting particles and random velocities are negligible. It's also assumed to be a *perfect fluid*, i.e. isotropic, with no viscosity and with no heat conduction.

Isotropic means that it's the same in all directions (no privileged direction). *No viscosity* means there isn't internal friction. And *no heat conduction* means that there is no flow of thermal energy (heat) from one part of the fluid to another.

All of these **physical** assumptions can be boiled down to one **mathematical** expression:

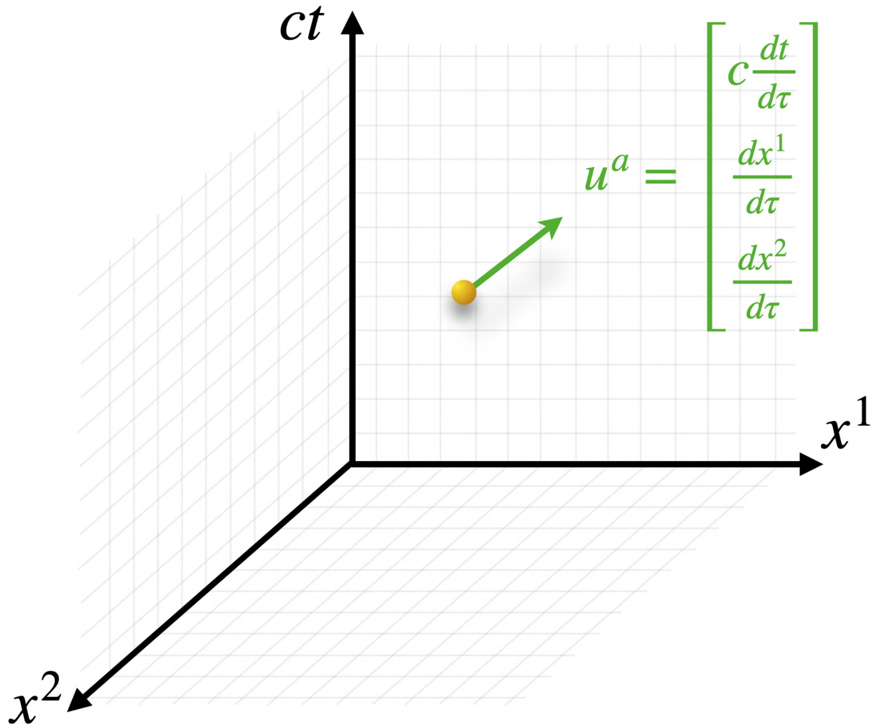
$$T_{ab} = \rho \, u_a u_b$$

This is the *stress-energy tensor* of a perfect fluid with zero pressure. This will be our right hand-side (RHS) of Einstein's equations ($G \equiv 1$):

$$R_{ab} - \frac{1}{2} R \, g_{ab} = 8\pi \, T_{ab}$$

(no dark energy considered: $\Lambda = 0$)

u^a is the 4-velocity of a particle (or fluid element). A 4-velocity is a velocity vector with 4 components (1 for c-time and 3 for space) in the spacetime diagram.



(This is a 3D representation, since the 4D one is impossible to draw...)

$$u^a = \begin{bmatrix} c\frac{dx^0}{d\tau} \\ \frac{dx^1}{d\tau} \\ \frac{dx^2}{d\tau} \\ \frac{dx^3}{d\tau} \end{bmatrix} = \begin{bmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{bmatrix}$$

τ is *proper time* (i.e. the time measured by the particle's "internal clock", in its own reference frame).

u_a is the *lowered-index version* of u^a, which is defined using the metric g_{ab}:

$$u_a = g_{ab}\, u^b$$

This is called "lowering the index". u^a is a **contravariant** vector, and u_a is a **covariant** version (or covector).

Covectors can be thought of, intuitively, as mathematical objects that "act on" vectors to produce scalars (like taking a directional measurement or projection):

$$u_a\, v^a = \text{scalar}$$

For a vector v^a.

In the *rest frame of the fluid*, its 4-velocity looks like that:

$$u_a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the dust is at rest in this reference frame.

In order to calculate u_a explicitly, we need to lower the index of u^a using the appropriate metric, which in this case will be the following:

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right)$$

$$g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{bmatrix}$$

This is called the **flat Friedmann-Lamaître-Robertson-Walker (FLRW) metric**, and it considers the universe (on large scales) to be flat ($k = 0$). So far, experimental observations have indicated that the macroscale of the universe is flat, but we can't really know for sure yet for the same reason the Earth looks flat to us, even though it's just because its much bigger than us. Anyway, this approximation will be enough here.

$$u_a = g_{ab}\, u^b = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$T_{ab} = \rho\, u_a u_b = \rho \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix} = \rho \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(The operation above is called the ***outer product****)*

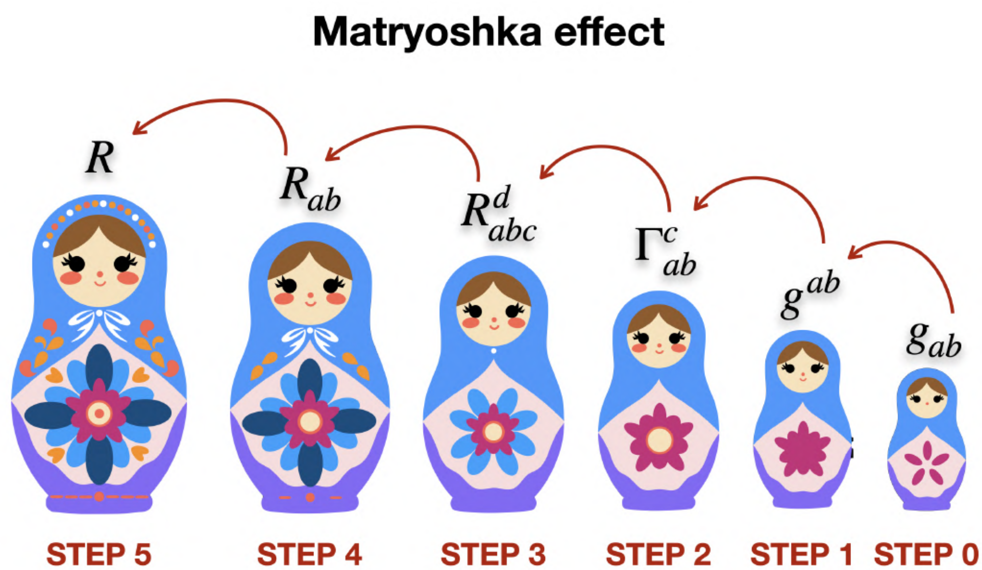
So, the form of the stress-energy tensor is the same as in Minkowski (flat) space.

The next logical step is to plug this tensor into the RHS of Einstein's equations, and use the FLRW metric to calculate its left hand-side (LHS):

$$R_{ab} - \frac{1}{2} R \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{bmatrix} = 8\pi\rho \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ok, we need to calculate the **Ricci tensor** R_{ab} and the **scalar curvature** R. If we compute the first we get the second.

Let's check our "Matryoshka doll effect" in order to understand what order we must follow:



Thus, we have 4 steps to go from the metric g_{ab} to the Ricci tensor R_{ab}, and then another one to get to the scalar curvature R.

We won't show all these huge calculations here, since it would be way too long. But the final results, for the *temporal* and *spatial* components (respectively), are the following two equations:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho} \quad \text{(expansion rate)}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho} \quad \text{(acceleration)}$$

Both of these equations are telling us that, since the energy density ρ is not zero (in fact, $\rho > 0$, otherwise there would be absolutely no matter or radiation in the universe!), then the rate of change of the scale factor ($\dot{a}(t)$) must always be $\neq 0$ (LHS of the first equation), and its acceleration ($\ddot{a}(t)$) must always be negative (LHS of the second equation).

In other words, the universe must always be expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$), and must always be decelerating ($\ddot{a} < 0$), as long as the universe is not empty ($\rho > 0$). It can never be completely static! A universe at rest is simply not possible under these conditions. At least not without adding something else to Einstein's equations... like **dark energy** Λ (in which case ($\ddot{a} > 0$))!

But this is a story for another time...

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