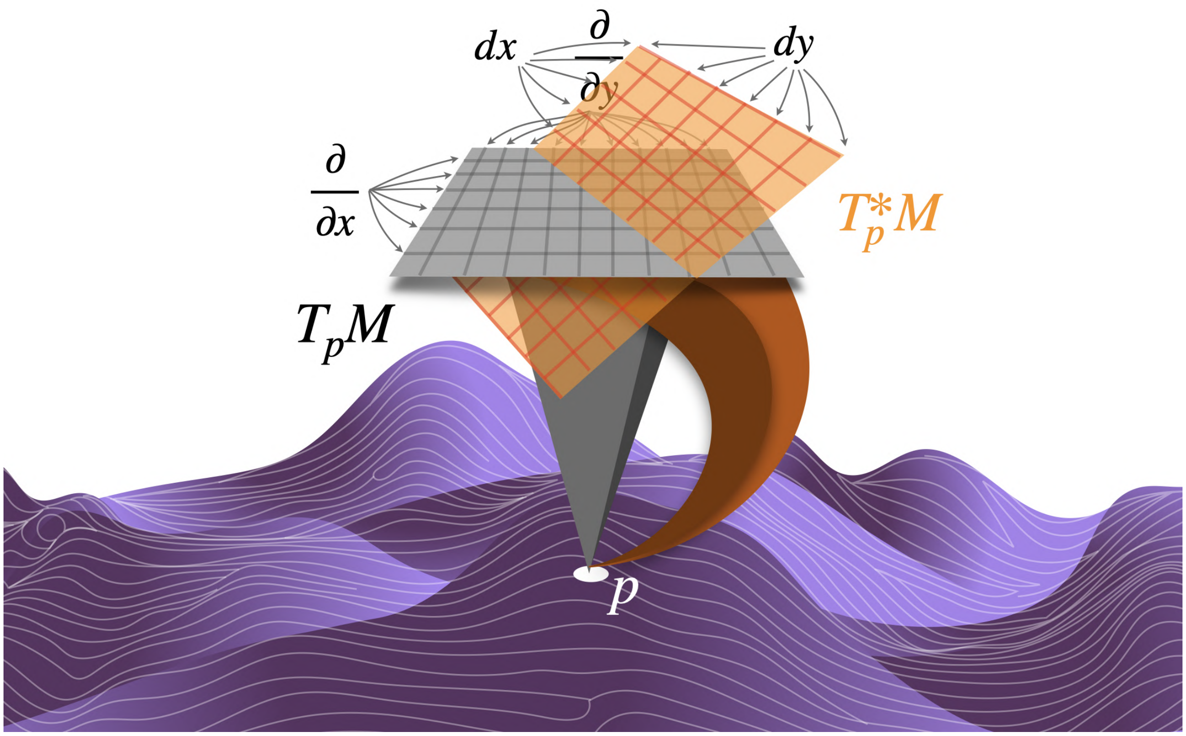
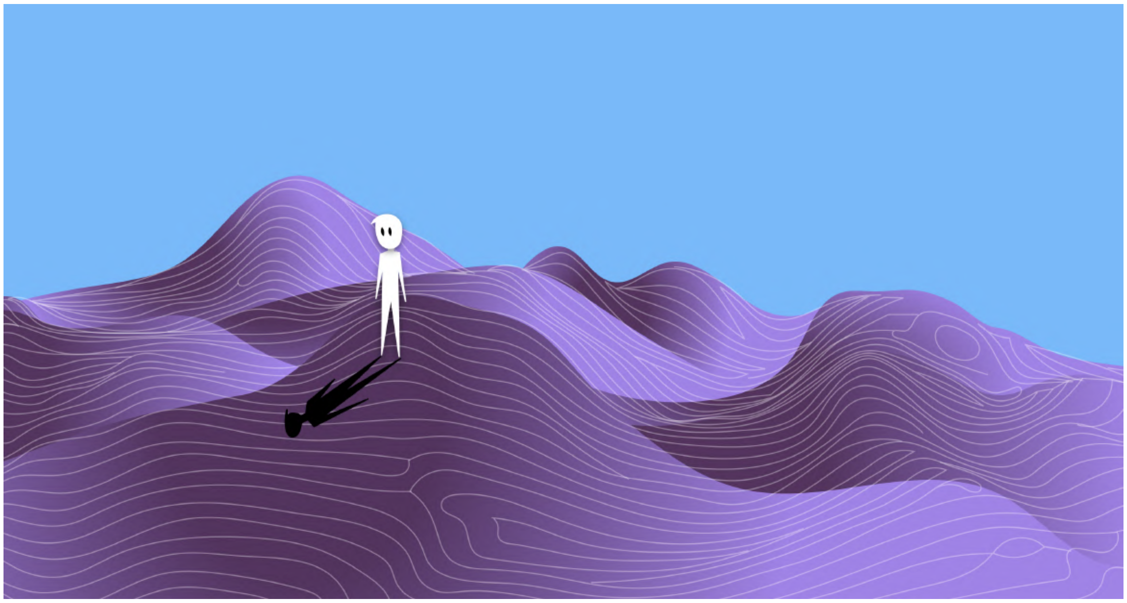
The Core of Differential Forms

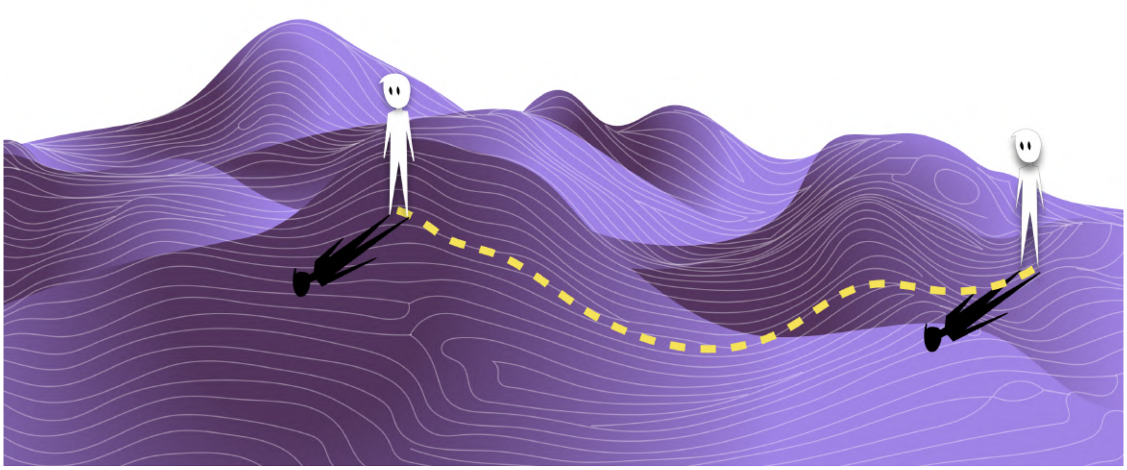
by DiBeos



Imagine you're hiking across a smooth hill landscape. This landscape is your space.



You can walk in any direction from any point, but let's forget about embedding it into a 3-dimensional space. Just focus on the ground under your feet, no up or down directions.

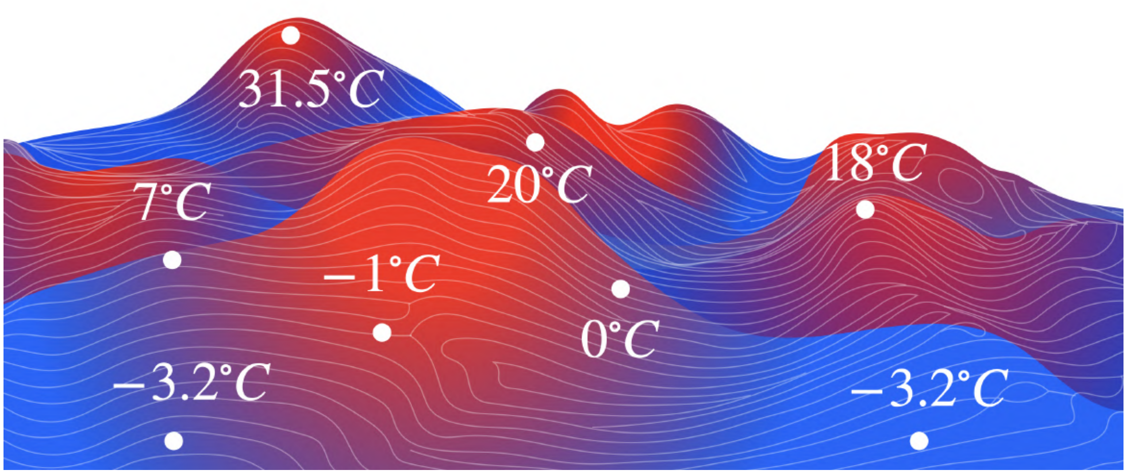


If this sounds weird, especially because there is clearly curvature involved here, you might want to watch this video on the channel and read the PDF file below, where we cover this notion of *embedding* in detail.

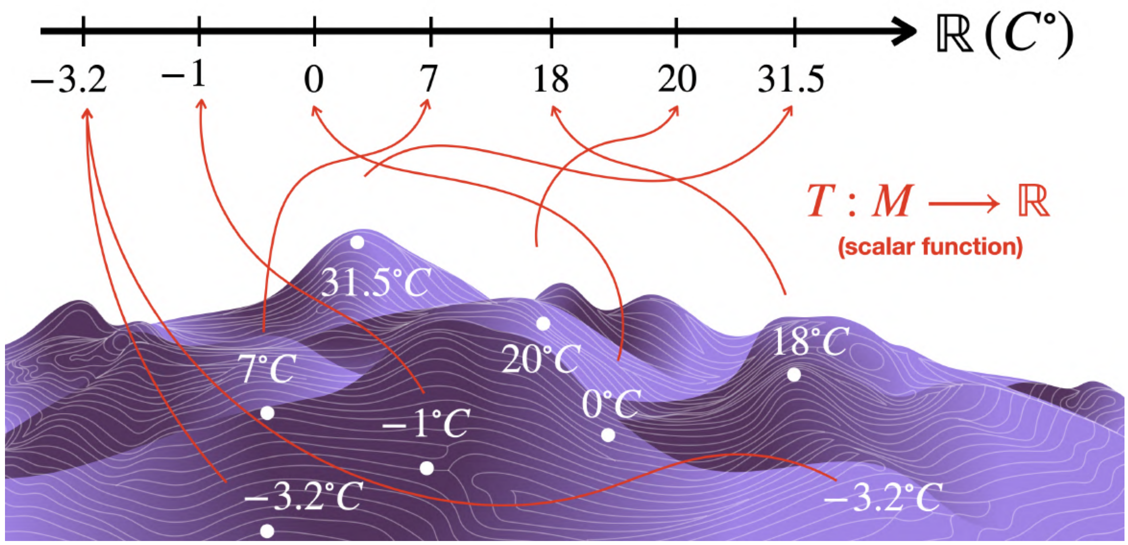
What is the Universe Expanding Into? (According to Pure Math)

PDF link: Eigenvalues & Eigenvectors

Now, suppose that at every point on this surface, there is a temperature assigned to it. The "higher up" you are on the hill, the hotter it gets.

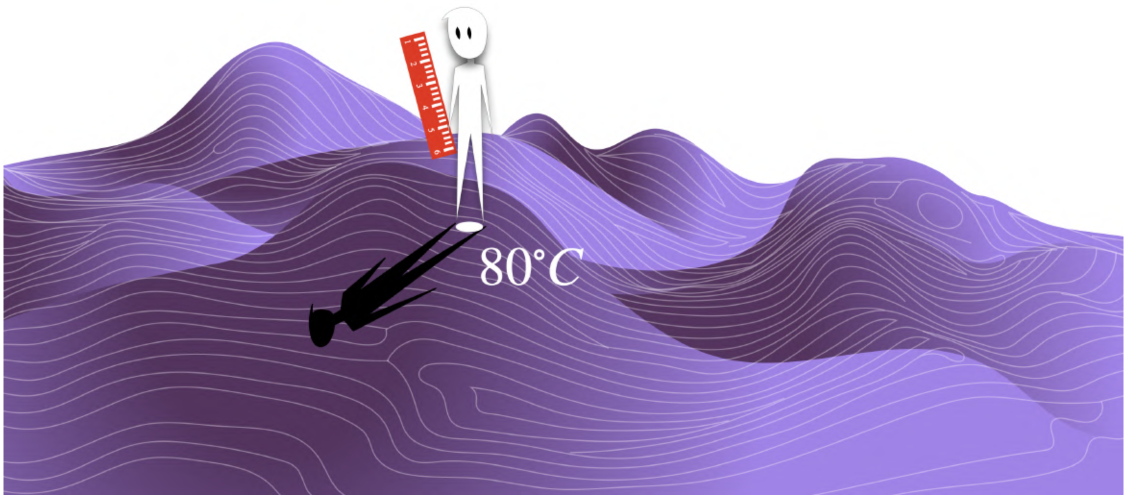


It's important to notice that temperature is just a real number, it doesn't have any direction associated with it.

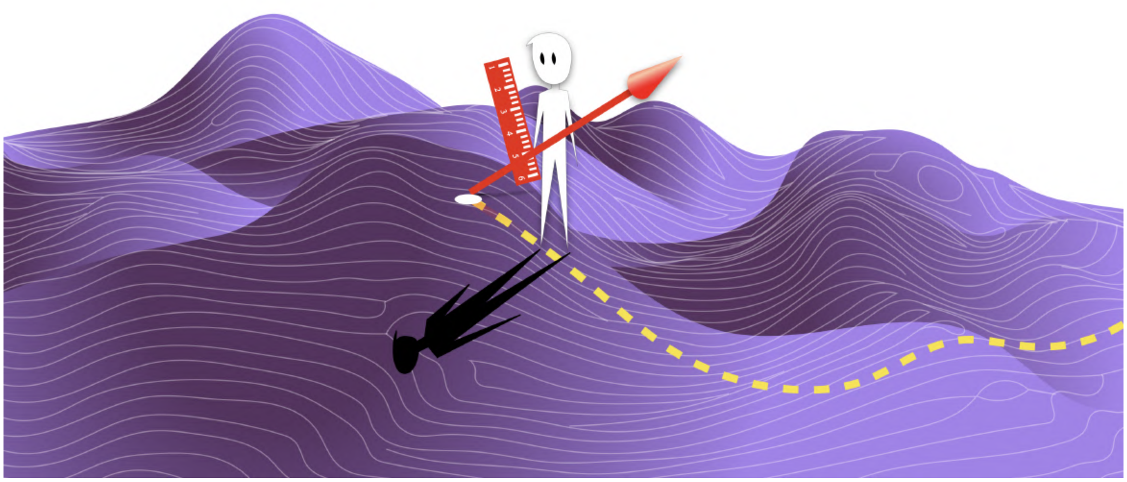


This is the concept of a *scalar function*. Denoted as "T" in this case, which goes from our hilled space called M to the real line.

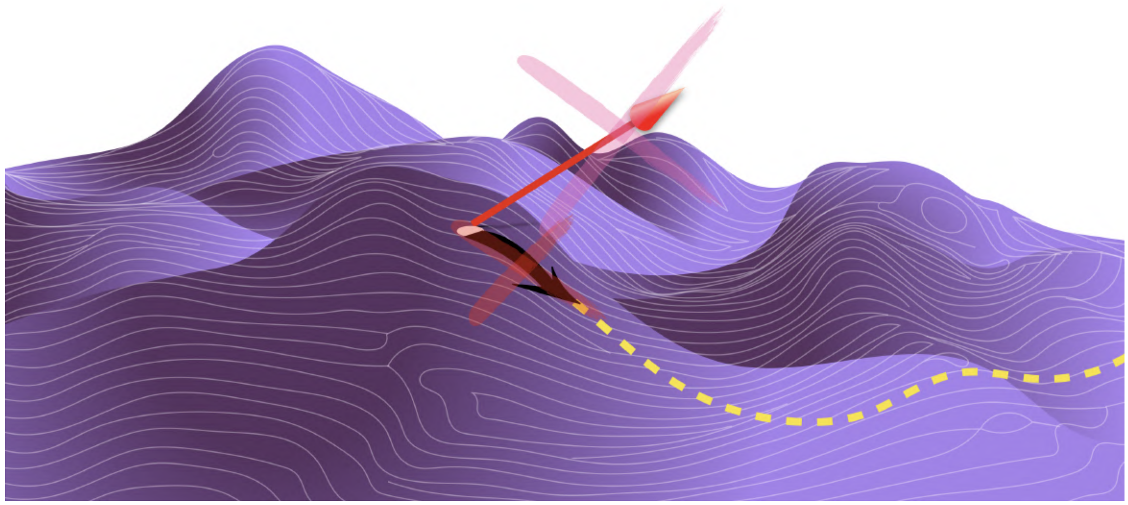
You stand on a point and look around. At that moment, you're standing on a spot that has a temperature value (say, 80°C). What can we measure?



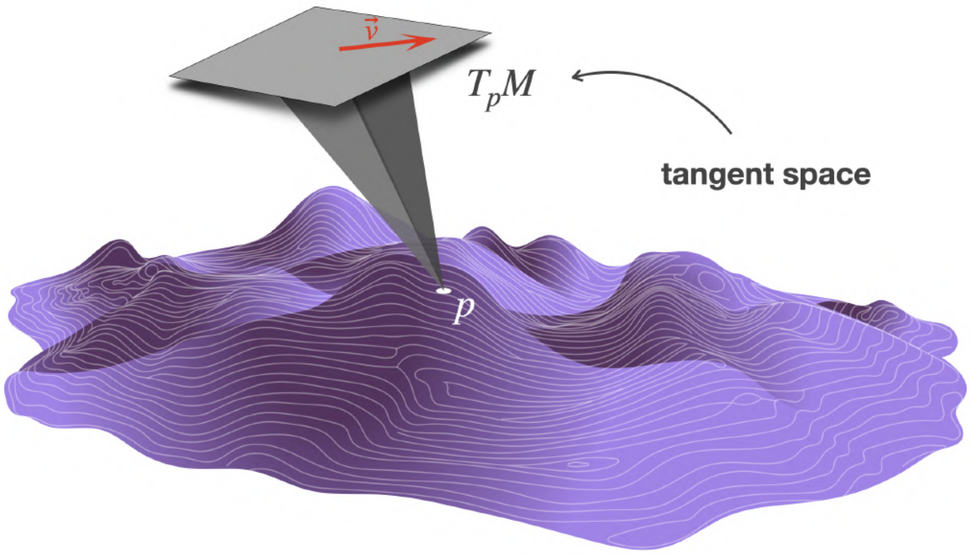
If a specific direction is given to you, you can take a tiny step along it. This little displacement can be measured with a vector. However, it says nothing about what's changing in the environment.



It's also important to note that the vector cannot be represented this way, "sticking out of the floor", since there is no external ambient space. We also can't just draw its sort of "shadow" on the floor because vectors do not bend like that.

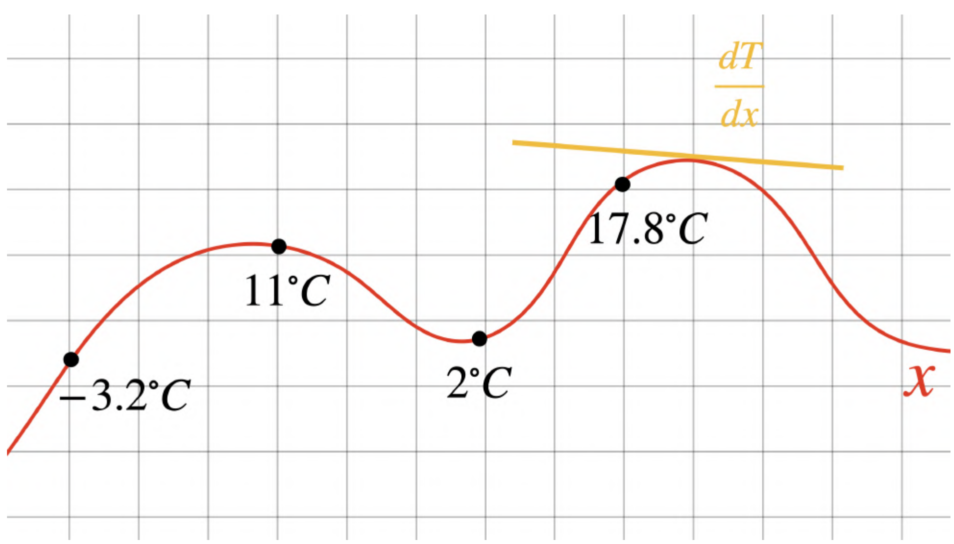


The vector actually lives in a different, separate space, called the tangent space. If the hill landscape is called M (which stands for *manifold*), then the vector $\vec{v}$, based on the point p, lives in the tangent space T_pM, which is specific for this point.



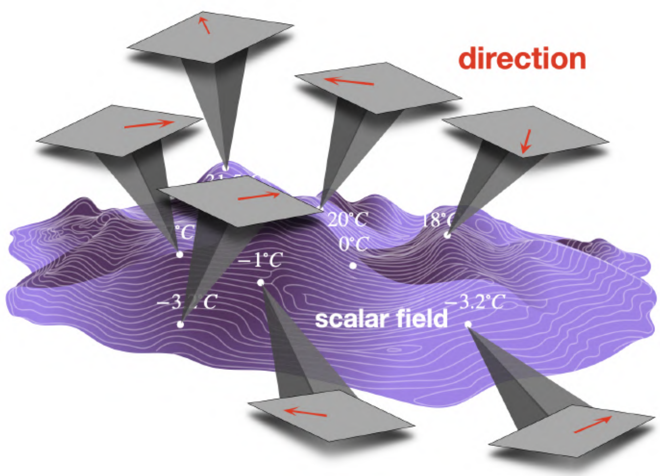
Now, think about what a *simple derivative* does:

It measures the rate of change of something (like temperature) with respect to something else (like distance), but only as you move along a straight line, usually the x-axis.



For example, if you move east, the derivative might tell you how fast the temperature changes per meter walked. That's great, but it does so only in one specific direction, not in general…

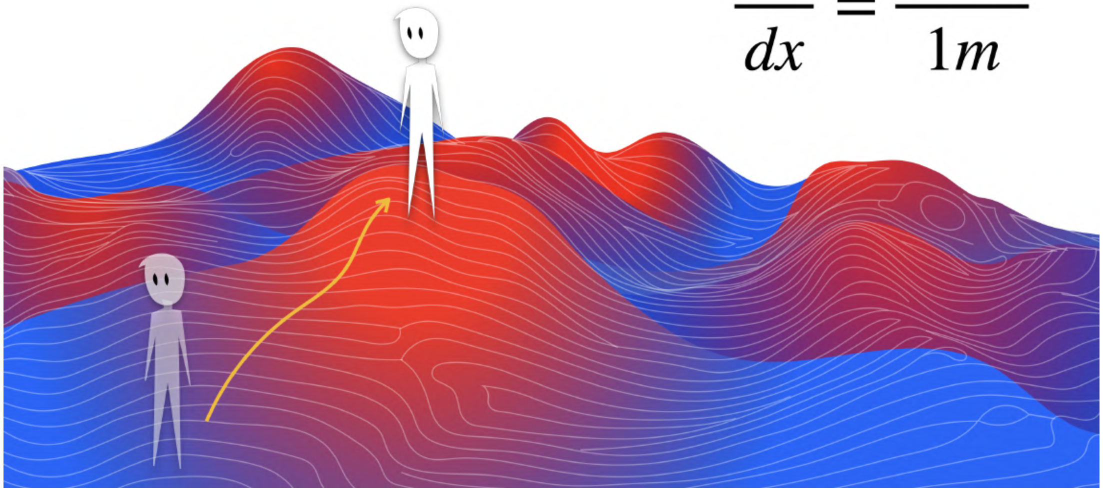
However, wouldn't it be nice if we could combine the best of both worlds? I.e., the *directionality* description of vectors and the capacity of tracking *rates of change* of scalar fields?



$$\Longrightarrow \; n \in \mathbb{R}$$

That's what a **directional derivative** does! It requires both a scalar field and a direction, and it gives you back a number. Like, "you'll feel the temperature increase at a rate of 30°C per meter in the direction you're heading".

$$\frac{dT}{dx} = \frac{30^\circ C}{1m}$$



A directional derivative is often expressed like this:

directional derivative

scalar function (ex.: temperature)

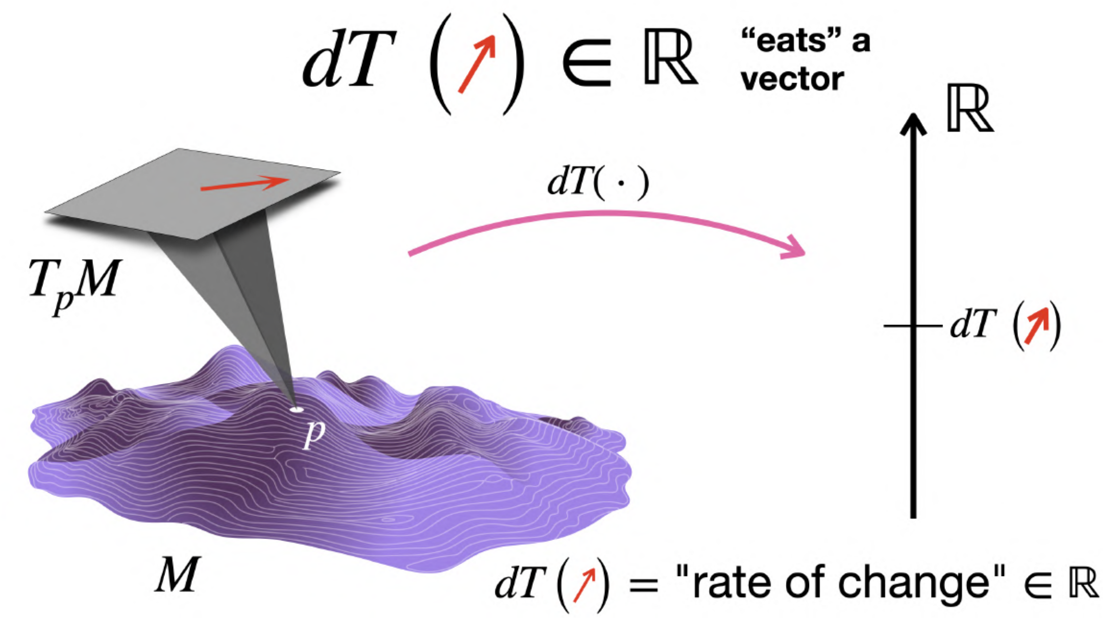
1-form

$$D_{\vec{v}} T = \boxed{dT}\,(\vec{v})$$

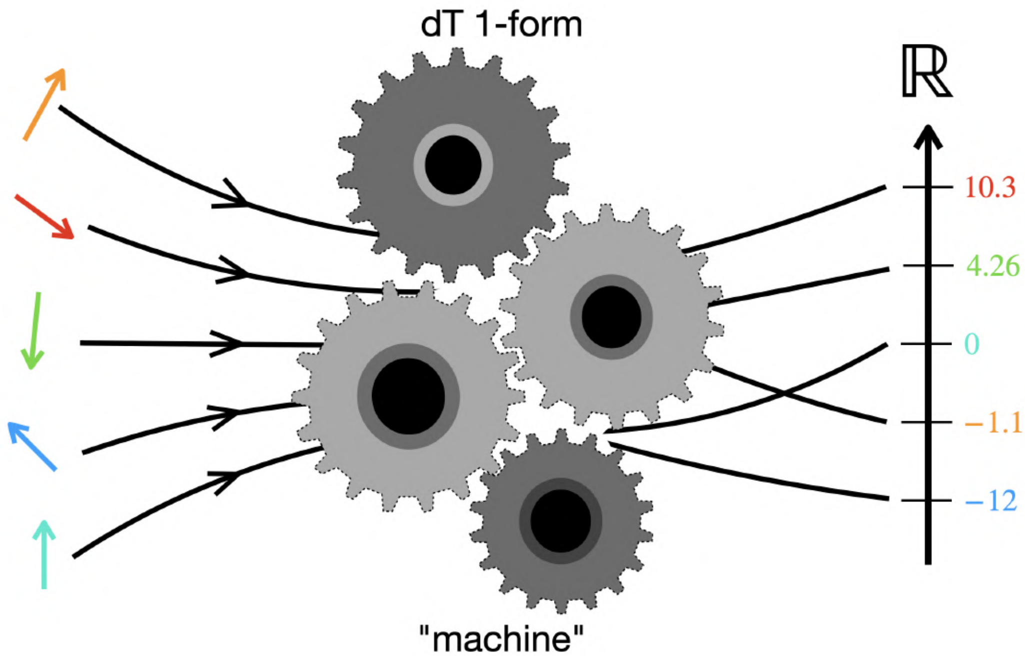
vectors

It reads as: "The rate at which the scalar function T changes as I move in the direction of $\vec{v}$ is given by feeding the vector $\vec{v}$ into the 1-*form* dT".

Ok... but what is a 1**-form**? A 1-form is a mathematical object that "eats" a vector (i.e. a direction) and returns a real number.

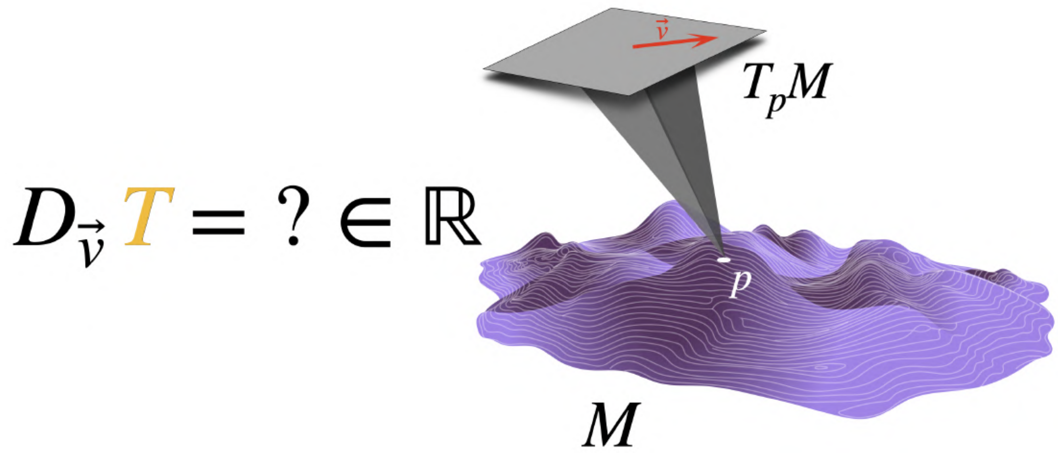


However, it's important to emphasize that dT on its own (with no vector attached to it) is <u>not</u> a real number. It's more like a mapping, or a "machine", that takes in a direction and tells you how the function T is changing along it. That's a 1-form!

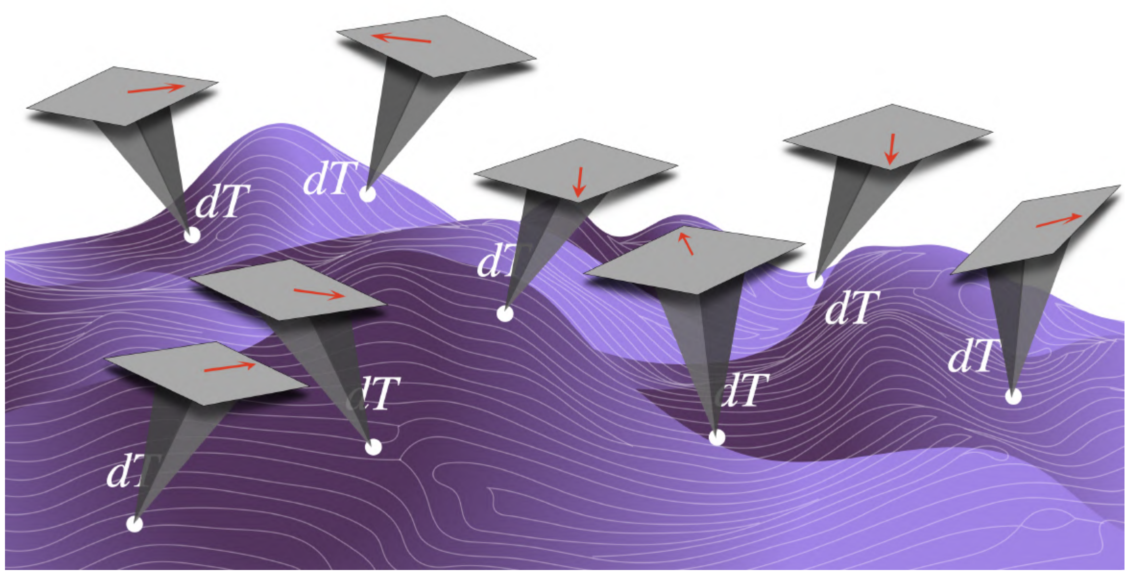


ATTENTION! That doesn't mean that 1-forms and directional derivatives are the same thing, but rather that they are closely related.

More specifically, a 1-form is a generalization of the directional derivative. The directional derivative $D_{\vec{v}}T$ is the specific result you get when you choose a scalar function T, choose a point p on the manifold, choose a direction $\vec{v}$, and as the question: "How does T change if I move in the direction $\vec{v}$ from p?". It's a number.

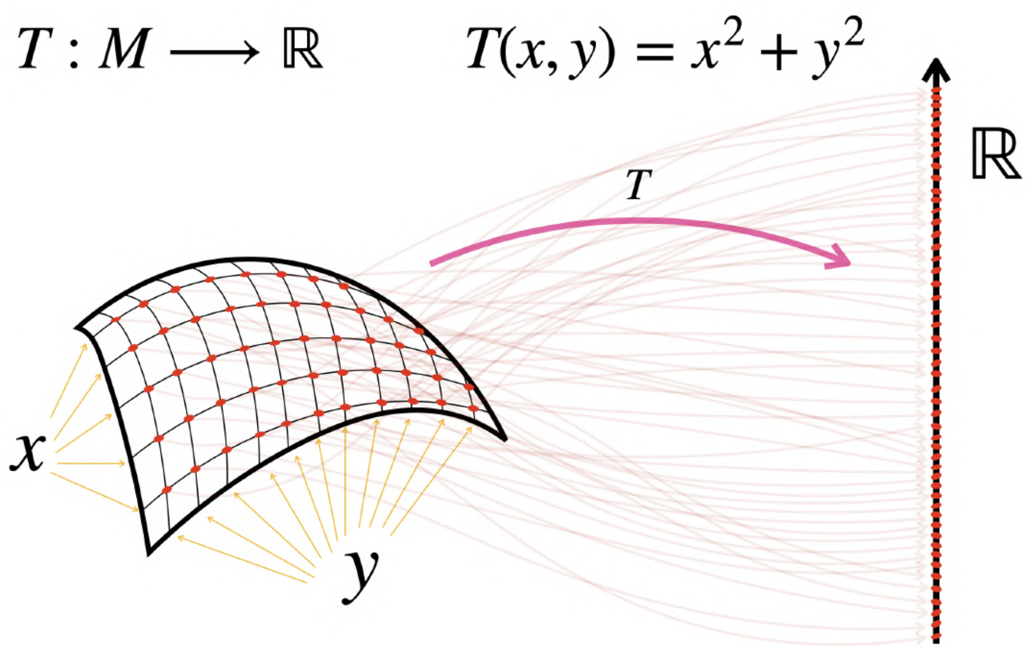


The 1-form dT, on the other hand, exists before we've chosen any direction. It lives at every point of the manifold. It takes any direction we give it, and tells how the function is changing in that direction.



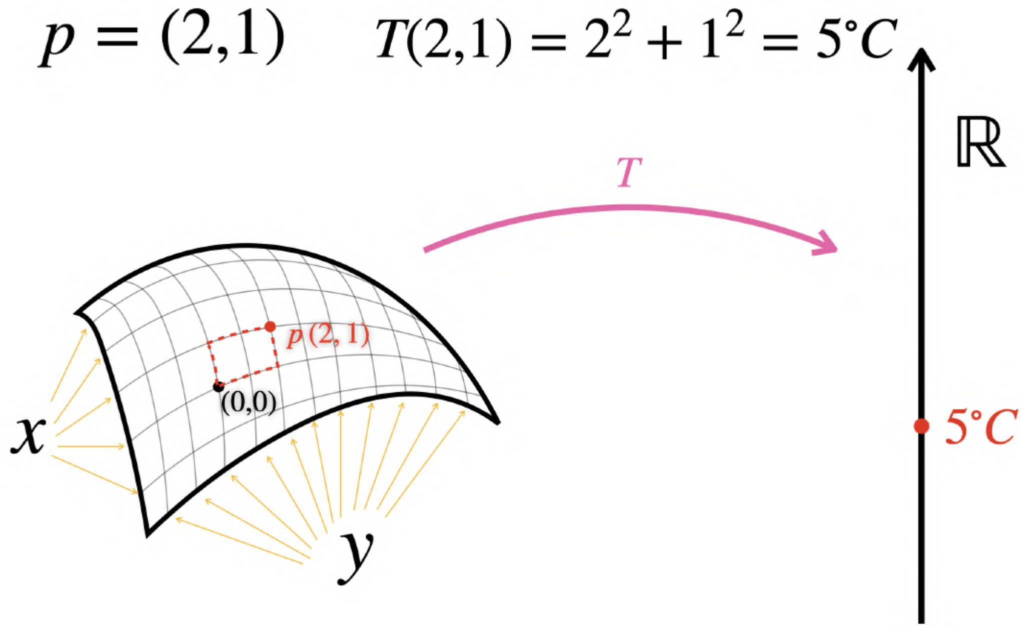
Let's see a concrete example:

A scalar field $T : M \to \mathbb{R}$ will be defined as $T(x, y) = x^2 + y^2$, where M is a 2-dimensional manifold (surface). This function gives the temperature at each point.

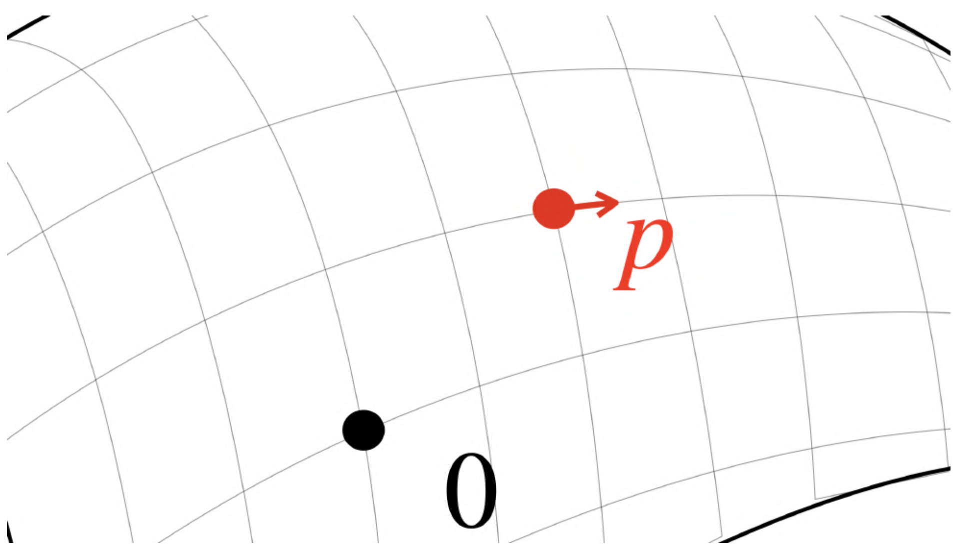


Pick a point $p = (2, 1)$. At this point the temperature is:

$$T(2, 1) = 2^2 + 1^2 = 5°C$$



What happens to the temperature if I move only in the x-direction?

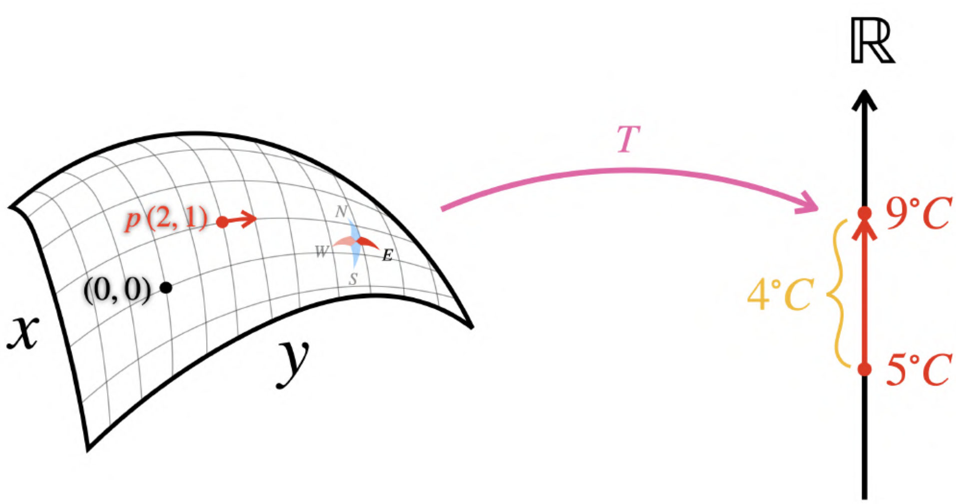


Well, let's see:

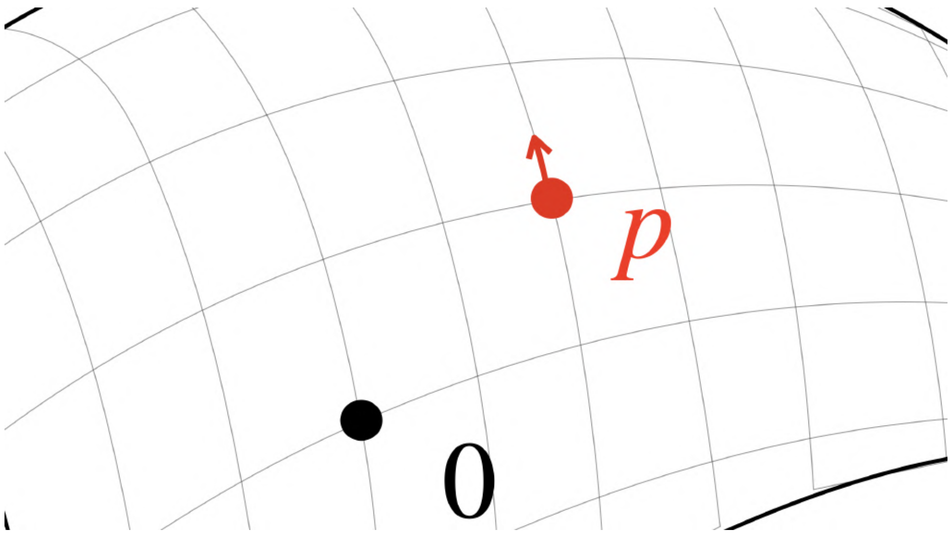
$$\frac{\partial T}{\partial x}(x, y) = 2x \implies$$

$$\implies \frac{\partial T}{\partial x}\overset{p}{(2,1)} = 2 \cdot 2 = 4°C$$

$\therefore$ For every unit you walk east, the temperature increases by $4°C$.



What happens if you move only in the y-direction?

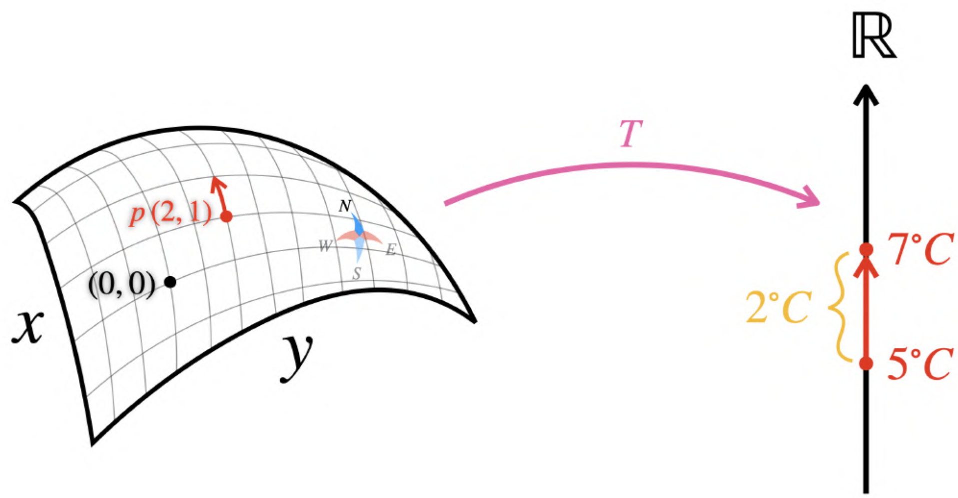


Using the same logic we find the following:

$$\frac{\partial T}{\partial y}(x, y) = 2y \implies$$

$$\implies \frac{\partial T}{\partial y}\overset{p}{(2,1)} = 2 \cdot 1 = 2^{\circ}C$$

$\therefore$ For every unit you walk north, the temperature increases by $2^{\circ}C$.



That's great, but what about moving *diagonally*? Let's say 45° north and 45° east.

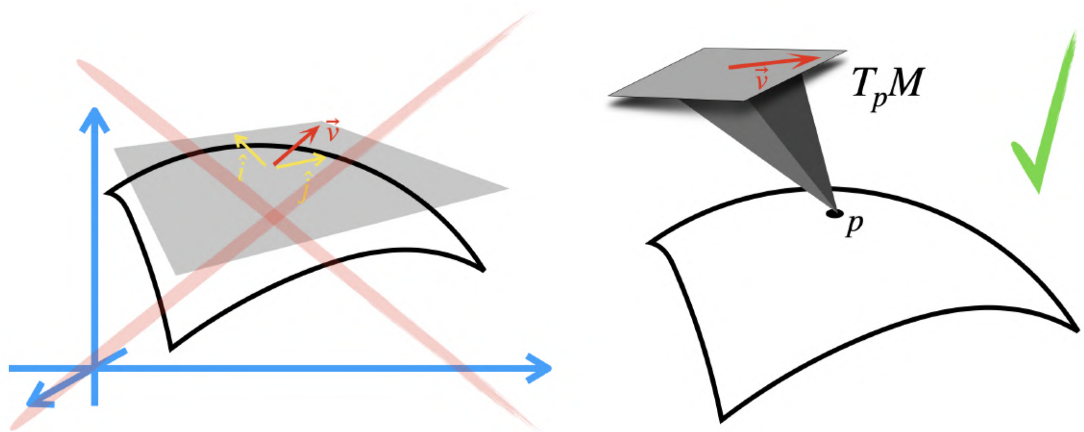
Now, we need to define a vector $\vec{v}$ that will encode this direction:

$$\vec{v} = \cos(45°)\,\hat{i} + \sin(45°)\,\hat{j}$$
$$\Downarrow$$
$$\vec{v} = \frac{\sqrt{2}}{2}\,\hat{i} + \frac{\sqrt{2}}{2}\,\hat{j}$$

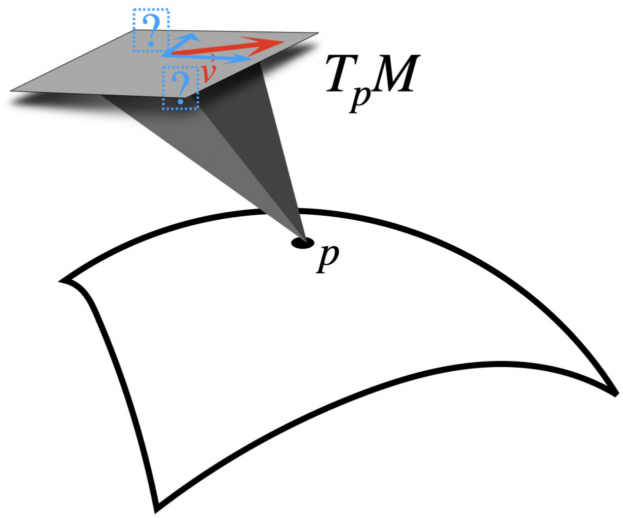
Actually, this expression makes **no sense at all** in this context, because we are assuming the manifold M to be abstract (i.e. possibly curved and not embedded into a higher-dimensional space).

$$\vec{v} = \frac{\sqrt{2}}{2}\cancel{\hat{i}} + \frac{\sqrt{2}}{2}\cancel{\hat{j}}$$

So, we cannot use the basis vectors $\hat{\imath}$ and $\hat{\jmath}$ of *Calculus* in order to write the vector $\vec{v}$ as a linear combination of them.



Visually, we're saying: the tangent vector $\vec{v}$ cannot stick out of the manifold. It needs to live inside of its own tangent space T_pM.
As a consequence, we need a new pair of basis vectors that are capable of spanning the entire tangent space at point p, such that any tangent vector $\vec{v} \in T_pM$ can be described as a linear combination of them:

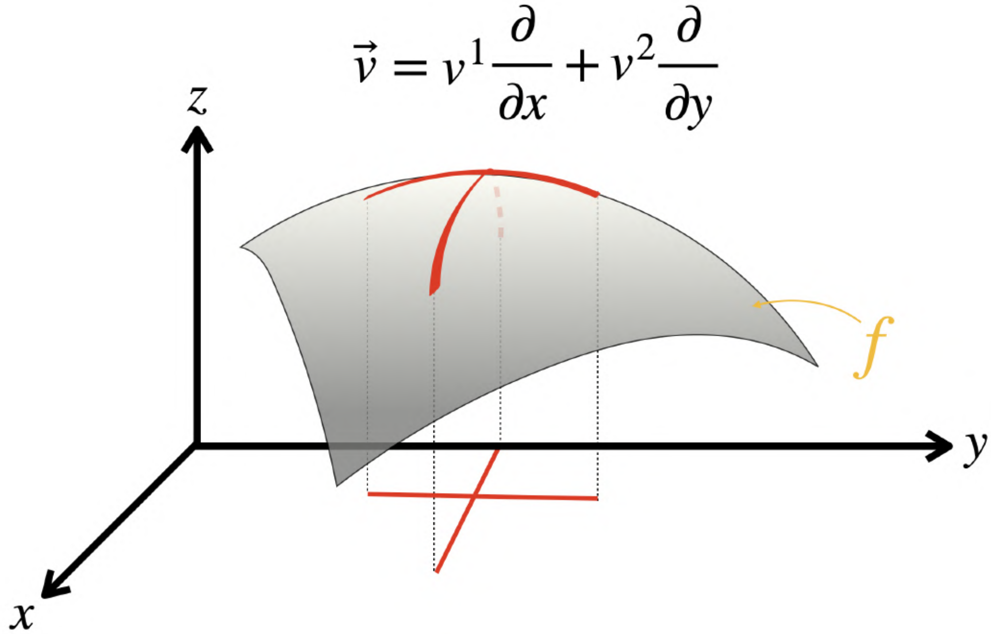


$$\vec{v} = \frac{\sqrt{2}}{2}\,\boxed{?} + \frac{\sqrt{2}}{2}\,\boxed{?} \in T_pM$$

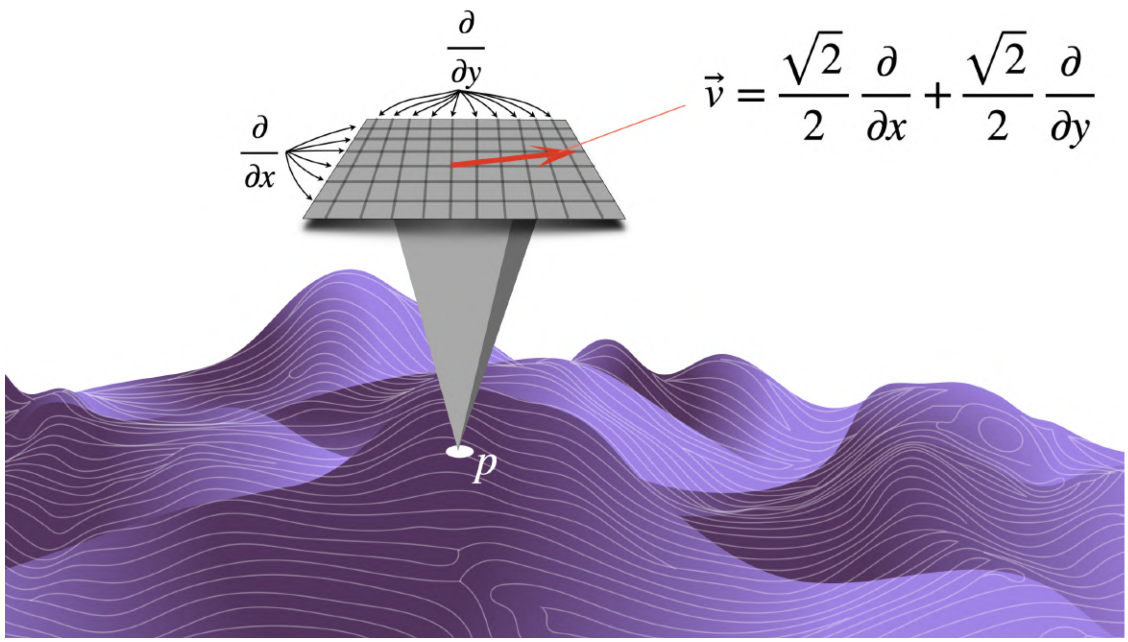
In *Differential Geometry*, we define tangent vectors as *derivations* because we don't have a global ambient space. This is a hard concept to grasp, and usually one of the greatest sources of confusion when people are trying to learn **differential forms**. So, pay attention!

Instead of defining a vector by its <u>direction</u> in space, we define it by what it <u>does</u>:

A tangent vector tells us how a function changes at a point. And this is exactly what a derivation does. The goal is to describe it only in terms of **intrinsic properties**.



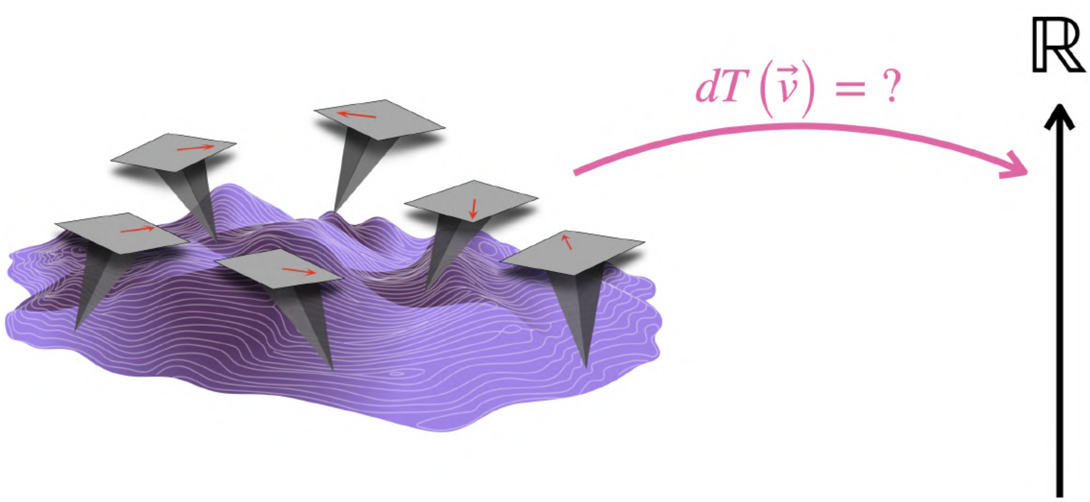
The differential operator $\frac{\partial}{\partial x}$ does exactly that. It takes a function f and returns the rate of change in the x-direction. Hence, the vector $\vec{v}$ is described as a linear combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, which are not only considered differential operators (in this context), but also vectors themselves. More precisely, the pair $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$ is the basis of the tangent space T_pM. And, therefore, the appropriate picture you should have in mind is this:



Hence, vectors themselves are considered differential operators when dealing with abstract, curved spaces!

Back to our example: we can apply the 1-form dT to this diagonal direction given by the vector $\vec{v} = \frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}$. The coefficients $\frac{\sqrt{2}}{2}$ come from the fact that the $\sin(45°)$ and $\cos(45°)$ are both $\frac{\sqrt{2}}{2}$.

Remember, a 1-form is a mapping that takes vectors and returns real numbers.



Since the vector $\vec{v}(\,\cdot\,)$ is an operator, we can act it on the scalar function T,which is literally the definition of directional derivative!

$$\vec{v}(T) = \left(\frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\right)(T)$$

$$\Downarrow$$

$$\vec{v}(T) = \frac{\sqrt{2}}{2}\frac{\partial T}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial T}{\partial y} = dT(\vec{v})$$

(directional derivative)

Thus, we can also write it in terms of the result of acting the 1-form on the vector $\vec{v}$.

So, the question is: what does dT look like before acting on the vector $\vec{v}$?

$$dT(\quad) = ?$$

Well, what about this:

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy \quad ?$$

Let's see why this expression for the 1-form dT works perfectly:

$$\begin{aligned}
dT(\vec{v}) &= dT\left(\frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\right) \\
&= \left(\frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy\right)\left(\frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\right) \\
&= \frac{\sqrt{2}}{2}\frac{\partial T}{\partial x} \overbrace{dx \frac{\partial}{\partial x}}^{=1} + \frac{\sqrt{2}}{2}\frac{\partial T}{\partial x} \cancel{dx \frac{\partial}{\partial y}} \\
&\quad + \frac{\sqrt{2}}{2}\frac{\partial T}{\partial y} \cancel{dy \frac{\partial}{\partial x}} + \frac{\sqrt{2}}{2}\frac{\partial T}{\partial y} \overbrace{dy \frac{\partial}{\partial y}}^{=1} = \\
&= \frac{\sqrt{2}}{2}\frac{\partial T}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial T}{\partial y}
\end{aligned}$$

Therefore, when we consider $dx\frac{\partial}{\partial x} = dy\frac{\partial}{\partial y} = 1$ and $dx\frac{\partial}{\partial y} = dy\frac{\partial}{\partial x} = 0$ (in the last step) we get the exact same expression we wanted!

Ok, but the question now becomes: how come?

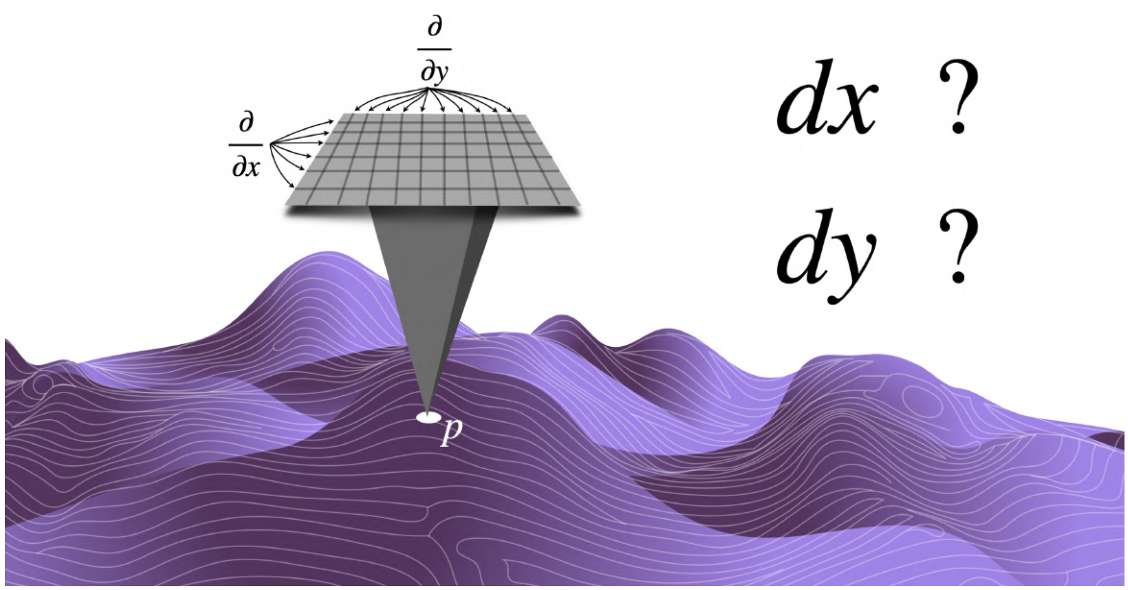
$$dx\frac{\partial}{\partial x} = dy\frac{\partial}{\partial y} = 1$$

$$\wedge$$

$$dx\frac{\partial}{\partial y} = dy\frac{\partial}{\partial x} = 0$$

How come?

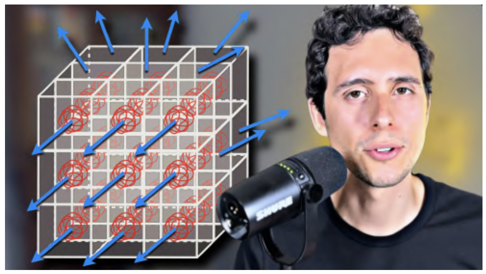
Well, we've seen that $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ is the basis for the tangent space T_pM, for a point $p \in M$. But what about dx and dy?



dx and *dy* are *covectors* and also 1-forms. Indeed, **covector** $=$ 1**-form**.

When we refer to it as a covector though, we are usually in the context of *Linear Algebra*.

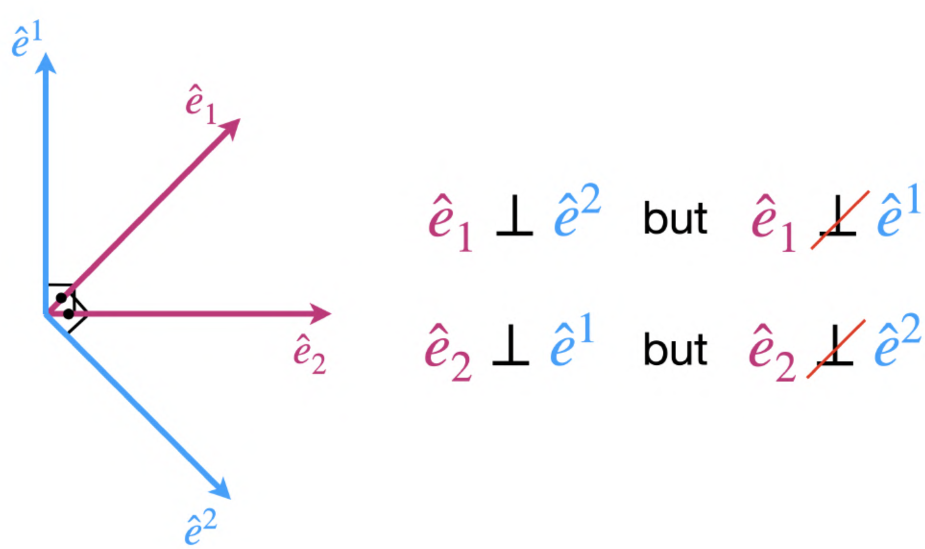
If you want to learn about it in detail, check out the following PDF and video in the channel:



The Core of Tensor Calculus

PDF link: Tensor Calculus

But just to give you guys a taste of it, when you have basis vectors $\hat{e}_1$ and $\hat{e}_2$ (no matter if they are orthogonal or not), you can find mathematical objects (denoted as **covectors** $\hat{e}^1$ and $\hat{e}^2$) such that $\hat{e}_1 \perp \hat{e}^2$ and $\hat{e}_2 \perp \hat{e}^1$, but not to their respective counterparts:



This fact can be expressed in a more compact way as:

$$\hat{e}_i\, \hat{e}^j = \delta_i^j \quad \begin{cases} \hat{e}_1\, \hat{e}^1 = 1 \\ \hat{e}_1\, \hat{e}^2 = 0 \\ \hat{e}_2\, \hat{e}^1 = 0 \\ \hat{e}_2\, \hat{e}^2 = 1 \end{cases} \qquad i,j \in \{1,2\}$$

Kronecker delta

$dim(M)$

And that's precisely what's going on here (in the context of differential forms):

$$\frac{\partial}{\partial x} \perp dy \quad \text{but} \quad \frac{\partial}{\partial x} \not\perp dx$$

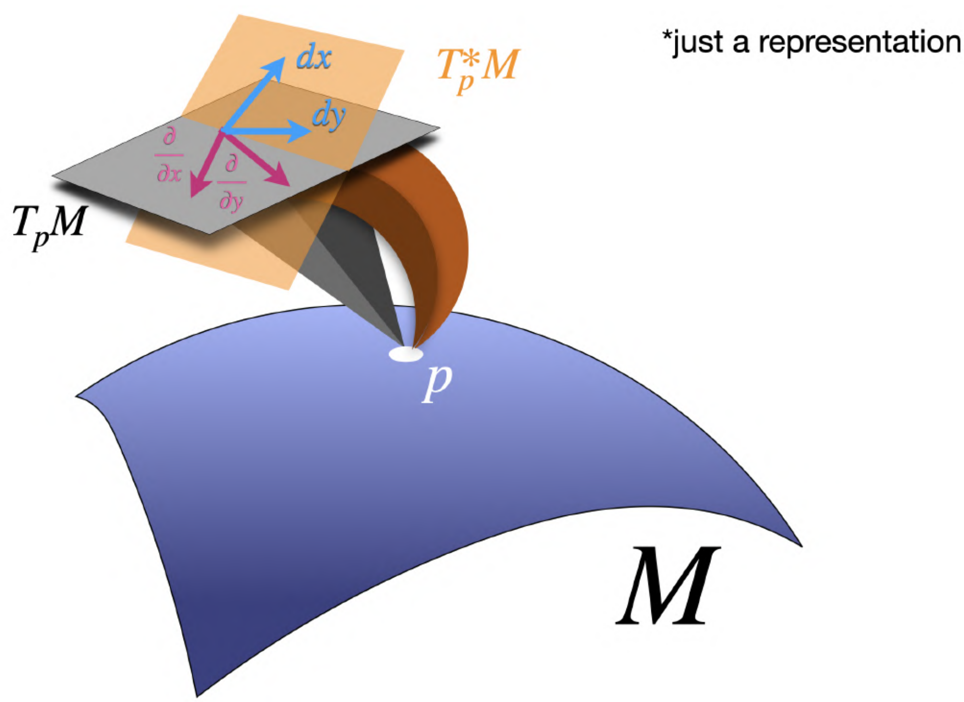
$$\frac{\partial}{\partial y} \perp dx \quad \text{but} \quad \frac{\partial}{\partial y} \not\perp dy$$

Basis vectors **Basis covectors**

These relations are valid (strictly) in the sense that:

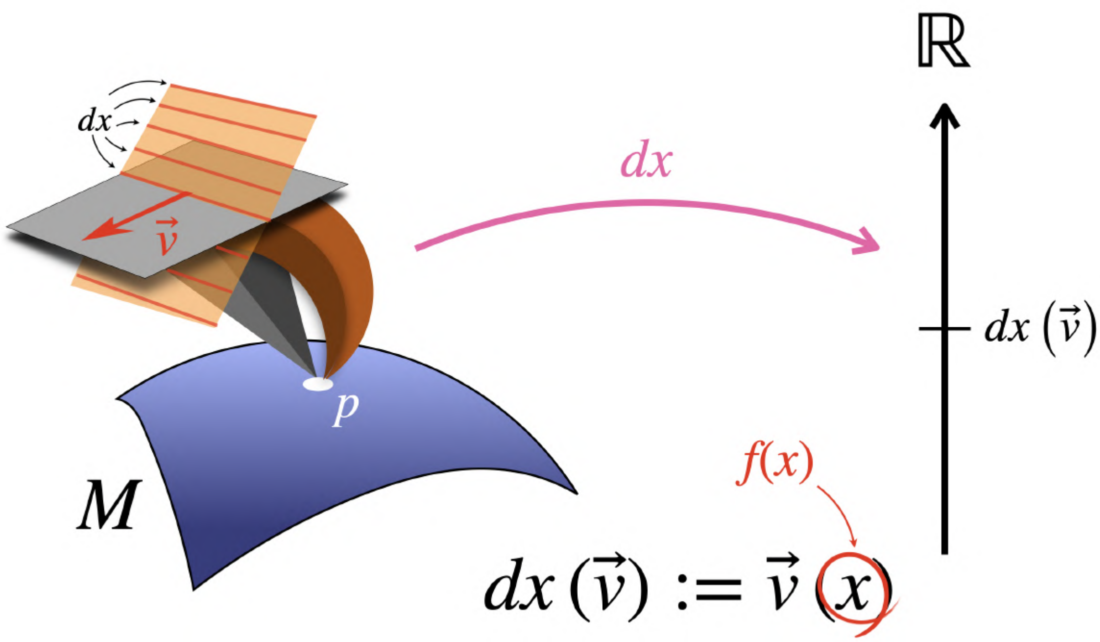
$dx\,\dfrac{\partial}{\partial x} = 1$	$dx\,\dfrac{\partial}{\partial y} = 0$
$dy\,\dfrac{\partial}{\partial x} = 0$	$dy\,\dfrac{\partial}{\partial y} = 1$

So, the covectors dx and dy form a basis $\{dx\,,\,dy\}$ for a space that's dual to the tangent space T_pM. This dual space is called **cotangent space** T_p^*M:

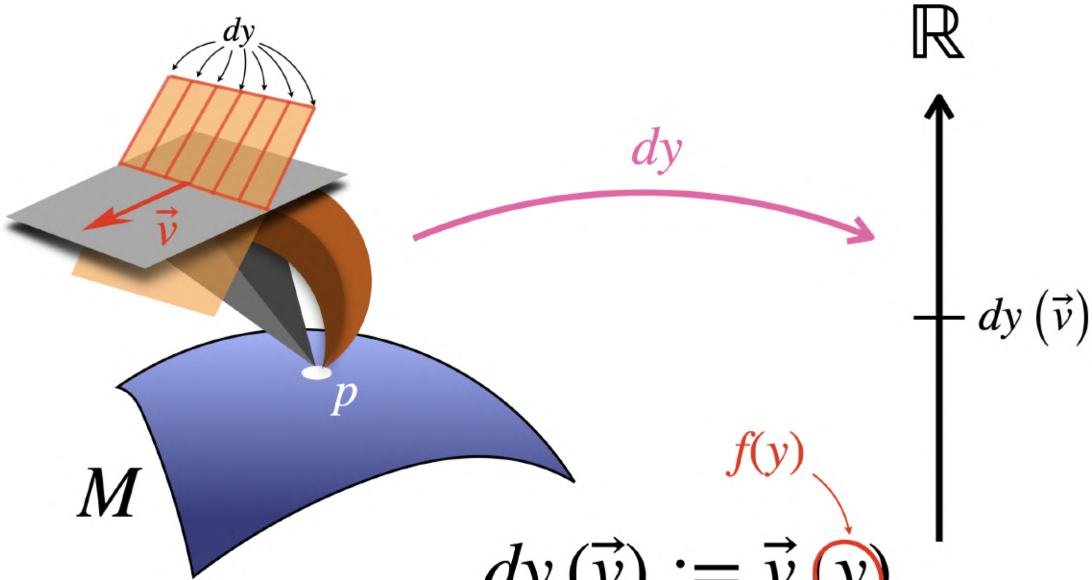


(This is not actually what it looks like geometrically. It's just a visual representation of the abstract Linear Algebra facts that we've seen earlier.)

In Differential Geometry, though, dx and dy are referred to as 1-forms, and therefore they act on vectors $\vec{v}$ in order to return scalars. For dx, this scalar (or number) is exactly the value of the directional derivative of the function x (or $f(x) = x$) in the direction of the vector $\vec{v}$.



A similar result is valid for dy.



$$dy\,(\vec{v}) := \vec{v}\,(\overset{f(y)}{y})$$

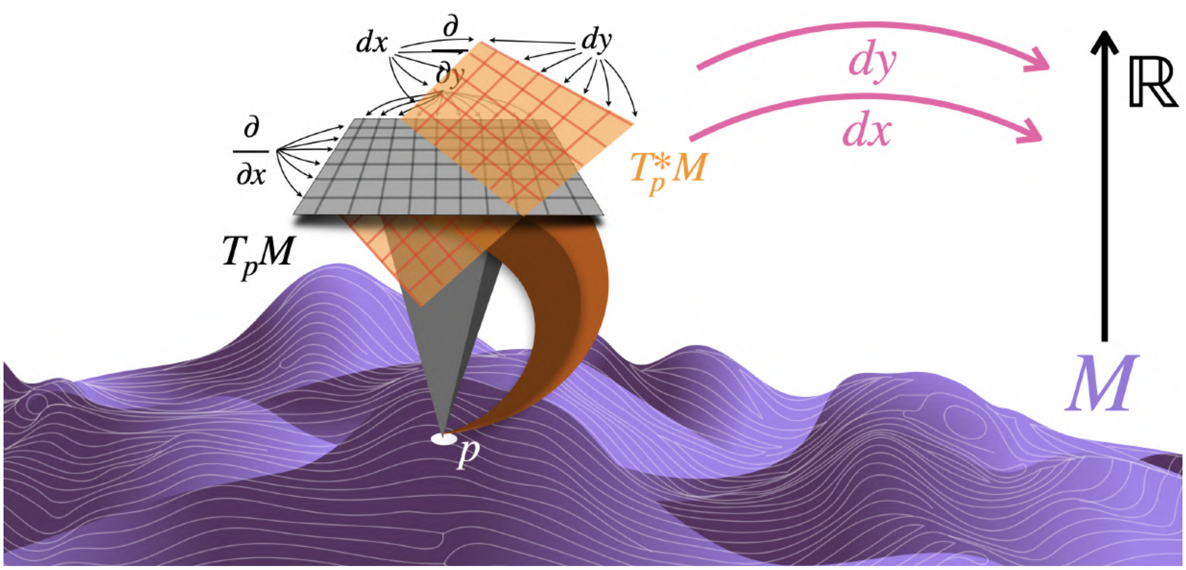
So, the 1-forms dx and dy are sort of "filters" that extract the x- and y-components of a vector, respectively.

When dx acts on $\vec{v}$, for example, we get the magnitude of the vector's projection onto the x-axis. And similarly for dy.

$$dx\,(\vec{v}) = \vec{v}\,(x) = \left(\frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}\right)(x) =$$

$$= \frac{\sqrt{2}}{2}\overset{=1}{\frac{\partial x}{\partial x}} + \cancelto{0}{\frac{\sqrt{2}}{2}\frac{\partial x}{\partial y}} = \frac{\sqrt{2}}{2}$$

Differential forms are especially useful in abstract, curved spaces, because they don't rely on an ambient space, or coordinates, or even a precise notion of distance. Instead of describing geometric objects as, like, "arrows in space", forms describe how scalar functions change. They act on vectors and extract quantities like rates of change, flux, and orientation. And all of it in a way that's 100% intrinsic to the space. That's why they're perfect for studying manifolds, even if some local regions are curved, twisted, or stretched, and even if global coordinates don't exist.



So, to recap:

The symbols $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ represent the basis vectors of the tangent space T_pM at a point p on a manifold. They describe directions of motion along the x- and y-coordinates that are intrinsic to the manifold. Their duals, dx and dy form a basis for the cotangent space T_p^*M, which consists of linear functionals (also known as 1-forms) that act on tangent vectors.

This dual pairing is what allows us to write and compute geometric quantities: a vector $\vec{v}$ is expressed as a linear combination of the basis vectors, and a 1-form (like dT) is expressed as a linear combination of dx and dy.

$$\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\} \longleftrightarrow \{dx, dy\}$$

$$\text{vector:} \quad \vec{v} = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}$$

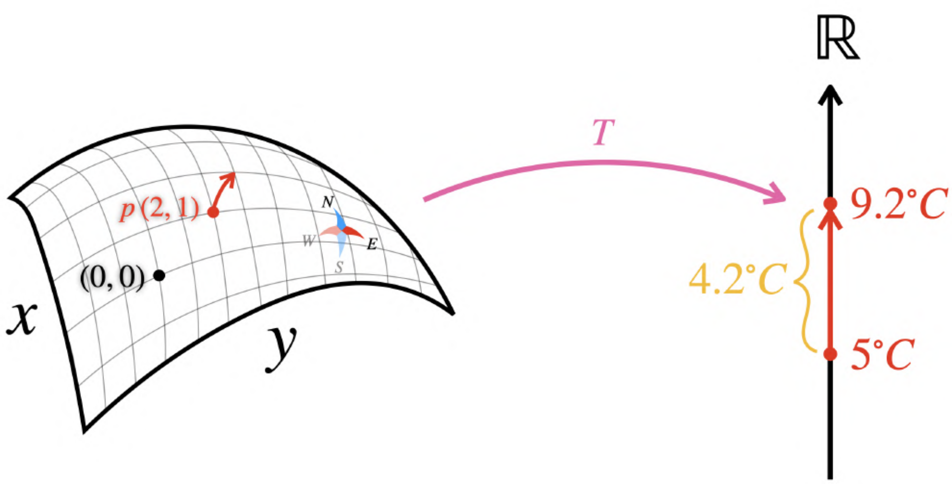
$$\text{1-form connector:} \quad dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy$$

When we apply the 1-form to a vector, the result is a scalar (so, a measurable quantity, like a rate of change), which reflects how much the vector "points in the directions" given by the 1-forms.

Let's finish this topic by applying the 1-form dT (with $T(x,y) = x^2 + y^2$) to the (diagonal) vector $\vec{v} = \frac{\sqrt{2}}{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial y}$ at point $p = (2, 1)$:

$$
\begin{aligned}
dT\left(\vec{v}_p\right) &= \left(\frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy\right)_p \left(\frac{\sqrt{2}}{2}\,\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\,\frac{\partial}{\partial y}\right) = \\
&= \left(\frac{\partial}{\partial x}\,\left(x^2+y^2\right)\,dx + \frac{\partial}{\partial y}\,\left(x^2+y^2\right)\,dy\right)_p \left(\frac{\sqrt{2}}{2}\,\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\,\frac{\partial}{\partial y}\right) = \\
&= \left(2x\,dx + 2y\,dy\right)_p \left(\frac{\sqrt{2}}{2}\,\frac{\partial}{\partial x} + \frac{\sqrt{2}}{2}\,\frac{\partial}{\partial y}\right) = \\
&= \left(2x\,\frac{\sqrt{2}}{2}\,dx\frac{\partial}{\partial x} + 2x\,\frac{\sqrt{2}}{2}\,dx\frac{\partial}{\partial y} + 2y\,\frac{\sqrt{2}}{2}\,dy\frac{\partial}{\partial x} + 2y\,\frac{\sqrt{2}}{2}\,dy\frac{\partial}{\partial y}\right)_{p=(2,1)} = \\
&= \left(\cancel{2}x\,\frac{\sqrt{2}}{\cancel{2}}\,\overset{=1}{dx\frac{\partial}{\partial x}} + \cancel{2}x\,\frac{\sqrt{2}}{\cancel{2}}\,\overset{=0}{\cancel{dx\frac{\partial}{\partial y}}} + \cancel{2}y\,\frac{\sqrt{2}}{\cancel{2}}\,\overset{=0}{\cancel{dy\frac{\partial}{\partial x}}} + \cancel{2}y\,\frac{\sqrt{2}}{\cancel{2}}\,\overset{=1}{dy\frac{\partial}{\partial y}}\right)_{p=(2,1)} = \\
&= \left(\sqrt{2}\,x + \sqrt{2}\,y\right)_{p=(2,1)} = \\
&= \sqrt{2}\cdot 2 + \sqrt{2}\cdot 1 = 3\sqrt{2} \approx 4.2 \in \mathbb{R}
\end{aligned}
$$

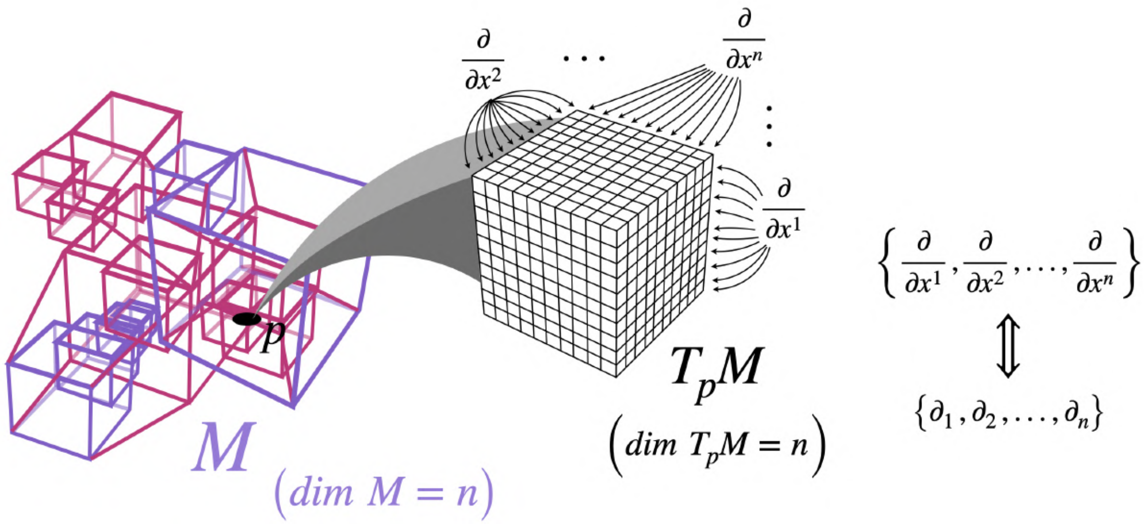
$\therefore$ for every unit you walk northeast, the temperature increases by 4.2°C.



Great!

Before moving on to **2-forms**, let's generalize the definition of a 1-form for manifolds M of any dimension n:

The dimension of the tangent space T_pM is the same as the dimension of M. Let's say "n".



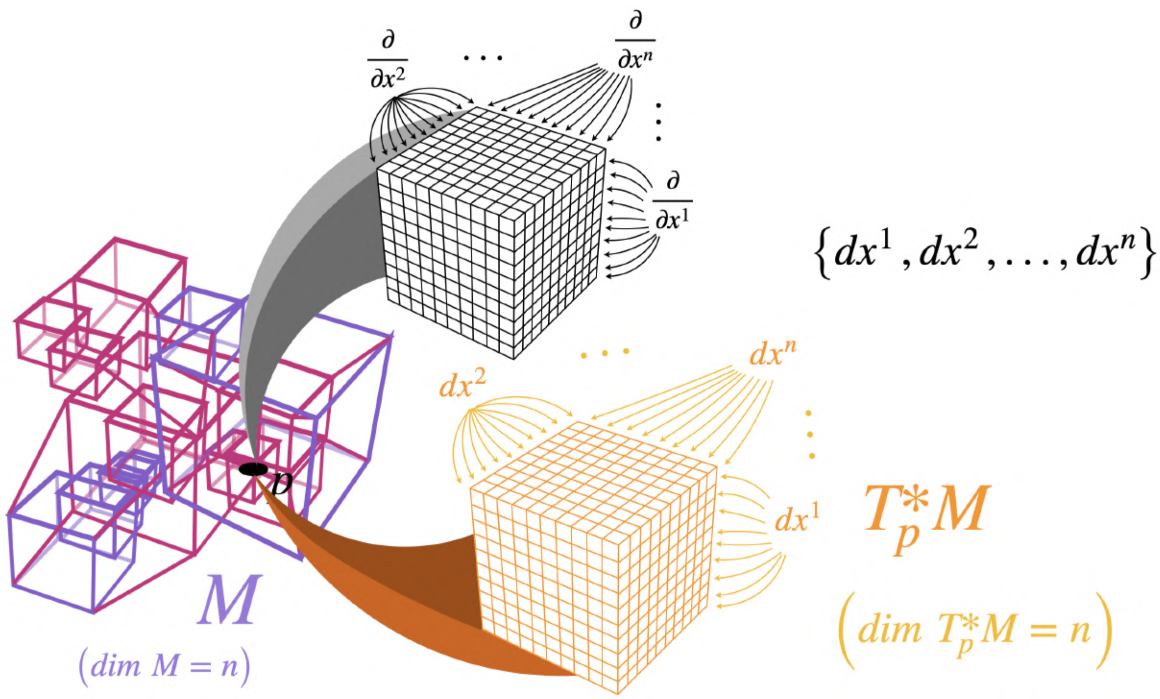
The basis of the tangent space T_pM is:

$$\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$$

Or simply,

$$\{\partial_1, \partial_2, \ldots, \partial_n\}$$

The cotangent space T_p^*M (so, its dual) is the space of 1-forms at the point p.



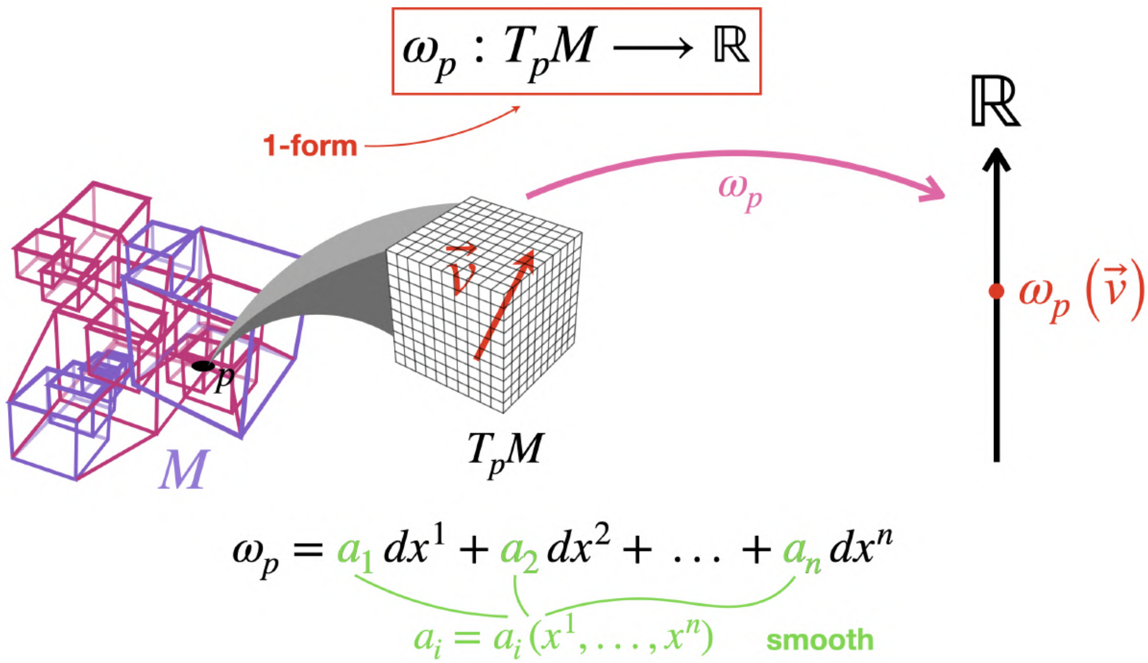
Its dimension is also n and its basis is:

$$\{dx^1, dx^2, \ldots, dx^n\}$$

From the definition of dual basis, this relation must always be satisfied:

$$dx^i \partial_j = \delta_j^i \quad , \quad \text{for } i, j \in \{1, \ldots, n\}$$

Thus, what is a 1-form on this abstract manifold M?



A 1-form (commonly denoted as ω_p) is a mapping that assigns to each point $p \in M$ a linear map:

$$\omega_p : T_pM \longrightarrow \mathbb{R}$$

In coordinates, any 1-form can be written as

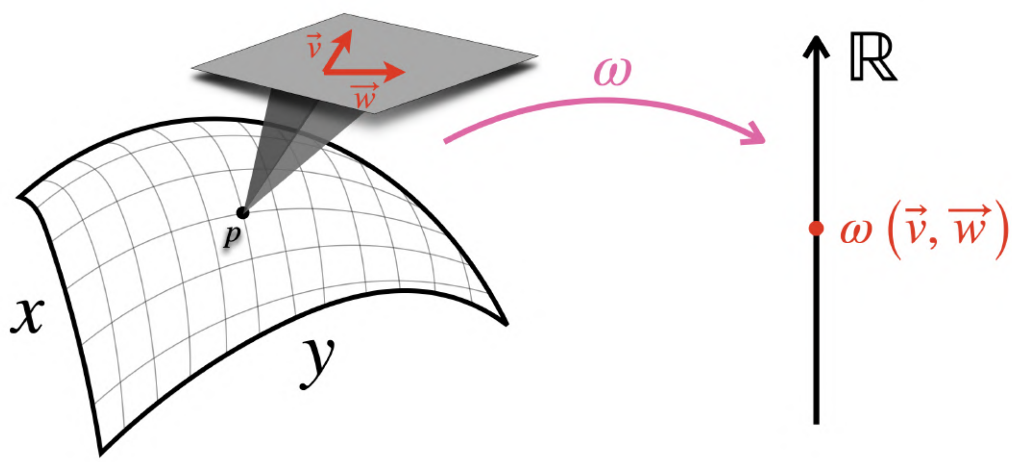
$$\omega_p = a_1\, dx^1 + a_2\, dx^2 + \ldots + a_n\, dx^n$$

where the coefficients $a_i = a_i(x^1, \ldots, x^n)$ are smooth functions on M.

Awesome! We are ready to move on to 2-forms.

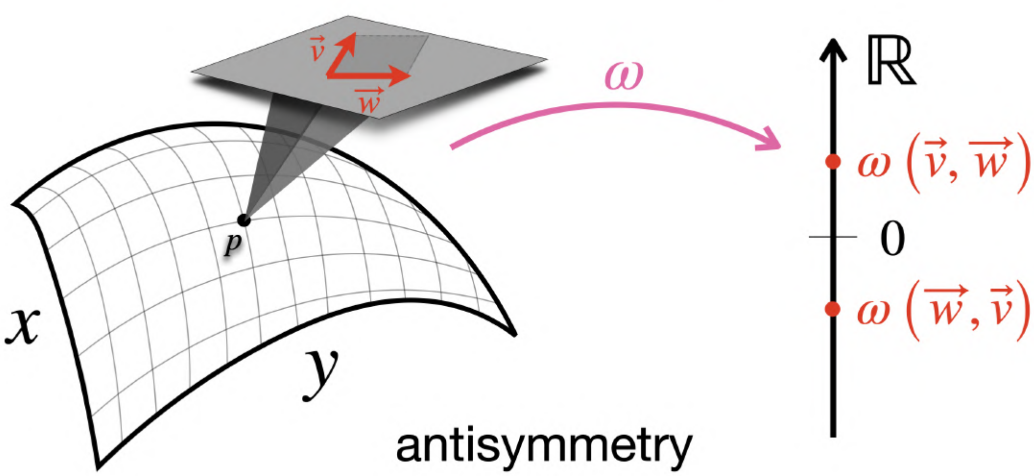
A 2-form is a function that takes two vectors and returns a number.

$$\omega : T_pM \times T_pM \longrightarrow \mathbb{R}$$

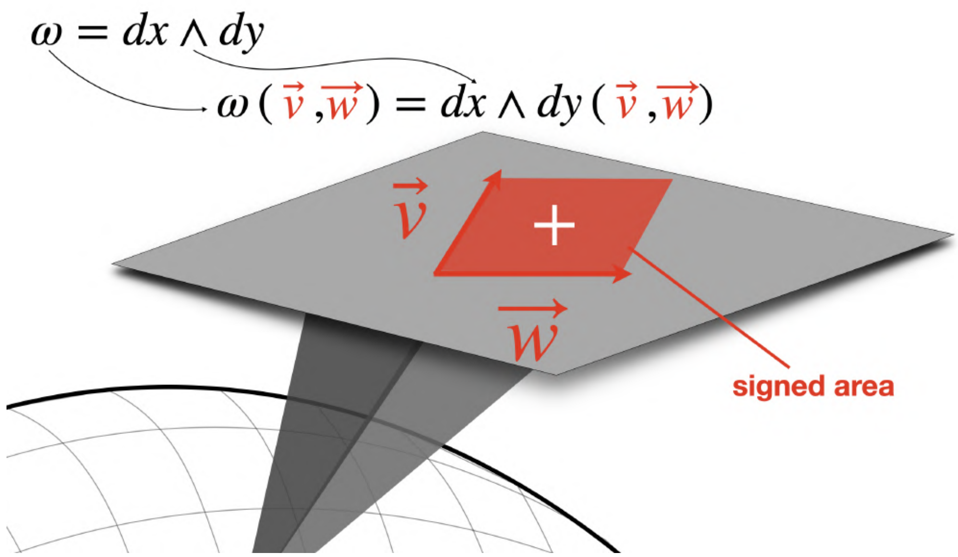


But, there is a little caveat here: the order of the vectors matter.

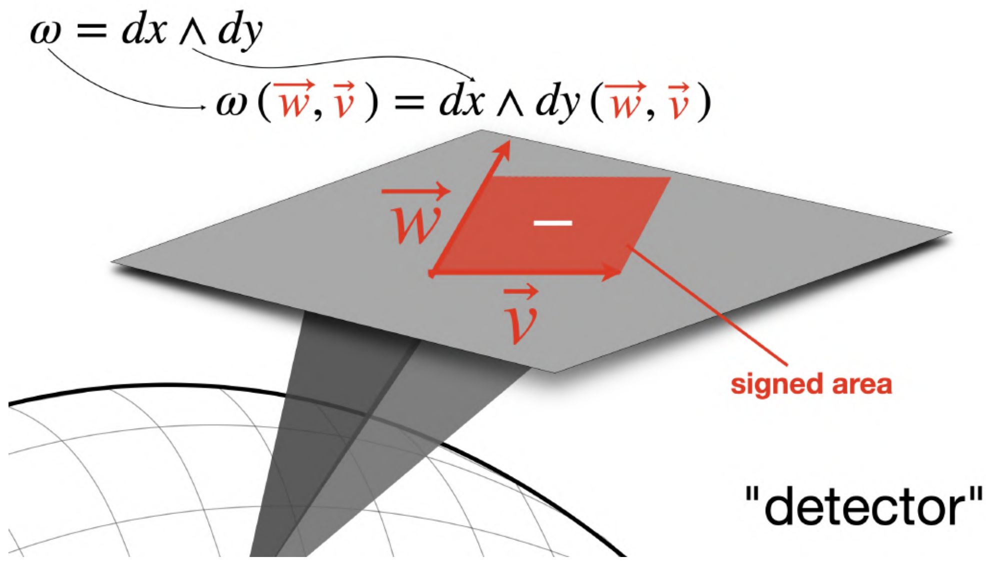
$$\omega\left(\vec{v}, \vec{w}\right) = -\,\omega\left(\vec{w}, \vec{u}\right)$$



This property is called **antisymmetry**.



In order to deeply understand antisymmetry, think of vectors $\vec{v}$ and $\vec{w}$ at the point p forming a small parallelogram. When a 2-form ω (which can be written in coordinates as $dx \wedge dy$) acts on those vectors, it measures the signed areas of that parallelogram. So, not just how much area it spans, but which way it's oriented (positively or negatively).



Swapping the input vectors flips the sign of the result, producing a reversal in orientation according to the convention you pick. This property is super useful because it gives us a sort of "detector" of local orientation on the manifold.

Now, you must be wondering: what is this symbol ($\wedge$)? This is called the **wedge product**, and it's what allows us to construct higher-order differential forms (like 2-forms) using 1-forms as building blocks.

$$dx \wedge dy$$

wedge product

$$dx \wedge dy \wedge dz \wedge \ldots \wedge d\omega$$

Think of it like this: just as adding vectors gives a new vector, wedging two 1-forms gives a new mathematical object that knows how to handle two vectors instead of one. However, just as we said before, in an antisymmetric way.

$$dx \wedge dy = -\,dy \wedge dx$$

Since 2-forms measure oriented areas, guess what area the wedge product of a basis covector with itself measures?

$$dx \wedge dx = 0$$

Let's see what happens when the 2-form $dx \wedge dy$ acts on the vectors $\vec{v}$ and $\vec{w}$, as the basis vectors of the tangent space T_pM:

$$dx \wedge dy \longrightarrow \boxed{\begin{array}{c} \vec{v} = \dfrac{\partial}{\partial x} \\ \hline \vec{w} = \dfrac{\partial}{\partial y} \end{array}}$$

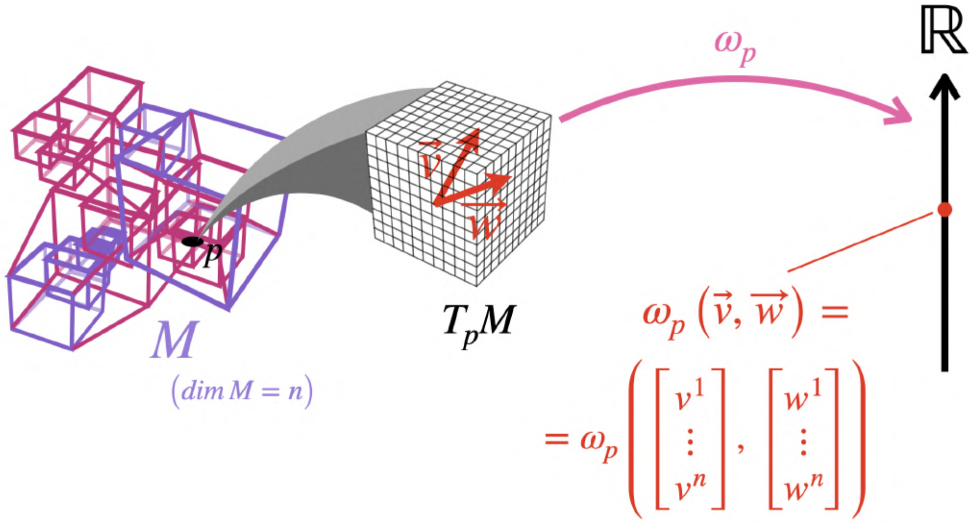
$$dx \wedge dy \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \underbrace{dx\frac{\partial}{\partial x}}_{1} \cdot \underbrace{dy\frac{\partial}{\partial y}}_{1} - \underbrace{dx\frac{\partial}{\partial y}}_{0} \cdot \underbrace{dy\frac{\partial}{\partial x}}_{0} = +1$$

The result is $+1$ because the orientation agrees with the order of the wedge: first dx, then dy. If you flip the order (i.e. $dy \wedge dx$) the result is

$-1.$

Now we generalize 2-forms to n-dimensions:

$$\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$



This is an antisymmetric, bilinear map that takes two tangent vectors at a point p and returns a number.

In local intrinsic coordinates $(x^1, x^2, \ldots, x^n)$, the basis for 2-forms is built using wedge products of 1-forms.

$$\boxed{\text{intrinsic coordinates:} \quad (x^1, x^2, \ldots, x^n)}$$

$$\overset{\text{1-form}}{dx^i} \wedge \overset{\text{1-form}}{dx^j}$$

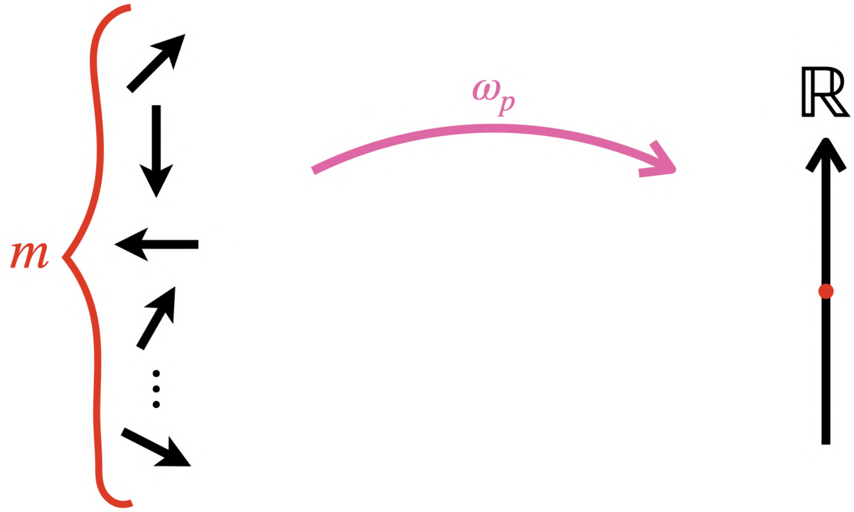
There are $\binom{n}{2} = \frac{n!}{2!(n-2)!}$ such basis elements in total. A general 2-form looks like this:

$$\omega = \sum_{i<j} f_{ij}(x)\, dx^i \wedge dx^j$$

We could also think about generalizing it to m-forms in n-dimensions.

$$\omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{m \text{ times}} \longrightarrow \mathbb{R}$$

This is a totally antisymmetric, multilinear map that takes m vectors and returns a number.



$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_m}$$

$$i_1 < i_2 < \ldots < i_m$$

Each m-form is written in terms of wedge products of m distinct 1-forms. So, the basis for the space of m-forms at p is:

$$dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_m} \quad , \quad i_1 < i_2 < ... < i_m$$

There are $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ such independent m-forms.

To recap and conclude it:

0-form	Scalar function	$f(x^1, \ldots, x^n)$
1-form	Linear map (covector)	$\alpha = f_1\, dx^1 + \ \ldots \ + f_n\, dx^n$
2-form	Oriented area element (takes 2 vectors)	$\omega = \sum_{i<j} f_{ij}\, dx^i \wedge dx^j$
3-form ⋮	Oriented volume element (takes 3 vectors)	$\theta = \sum_{i<j} f_{ijk}\, dx^i \wedge dx^j \wedge dx^k$
m-form ⋮	Oriented m-volume element	$\eta = \sum_{i_1<\ldots<i_m} f_{i_1\ldots i_m}\, dx^i_1 \wedge \ldots \wedge dx^i_m$
n-form	Top-form (full volume element on M)	$\mu = f\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$

The last one in the table above (i.e. **top-form**) is the ultimate *volume-measuring tool* on an n-dimensional manifold. It's a single object that "eats" n tangent vectors at a point and returns the oriented volume they span. In coordinates, it's written like $f(x)dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, and it's exactly what is integrated over the whole manifold in order to calculate things like total mass, charge and flux.
