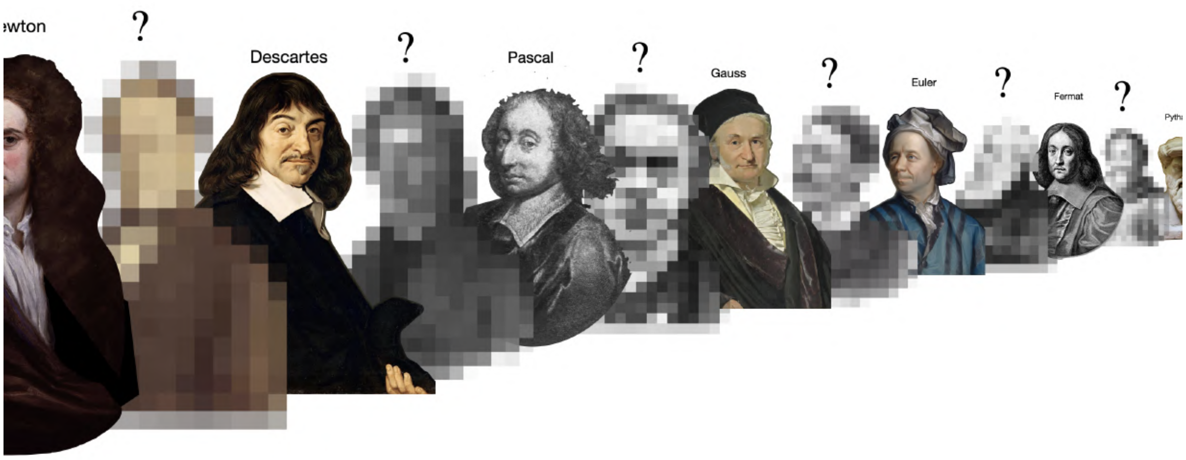
# Why Were These 6 Mathematicians Overshadowed?

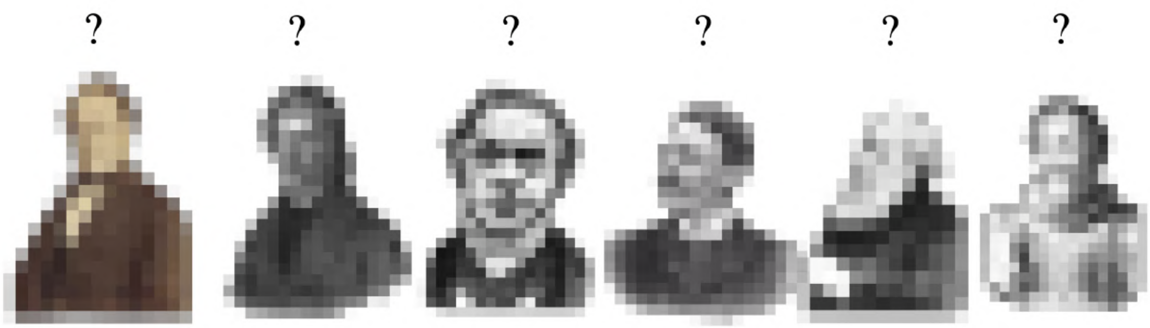
by DiBeos



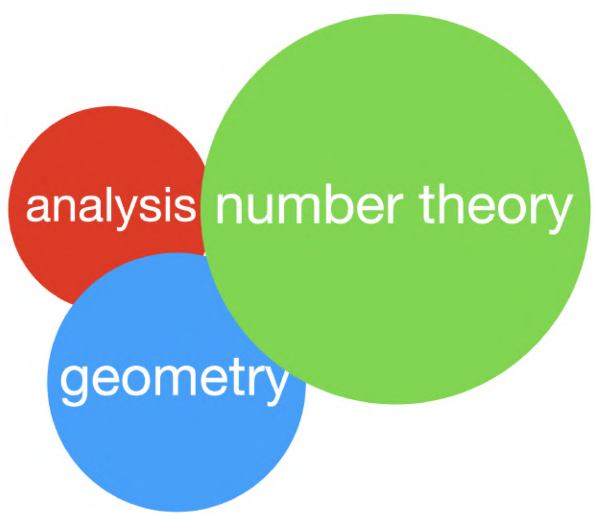

*"If I have seen further it is by standing on the shoulders of giants."*
**– Isaac Newton**

## Introduction

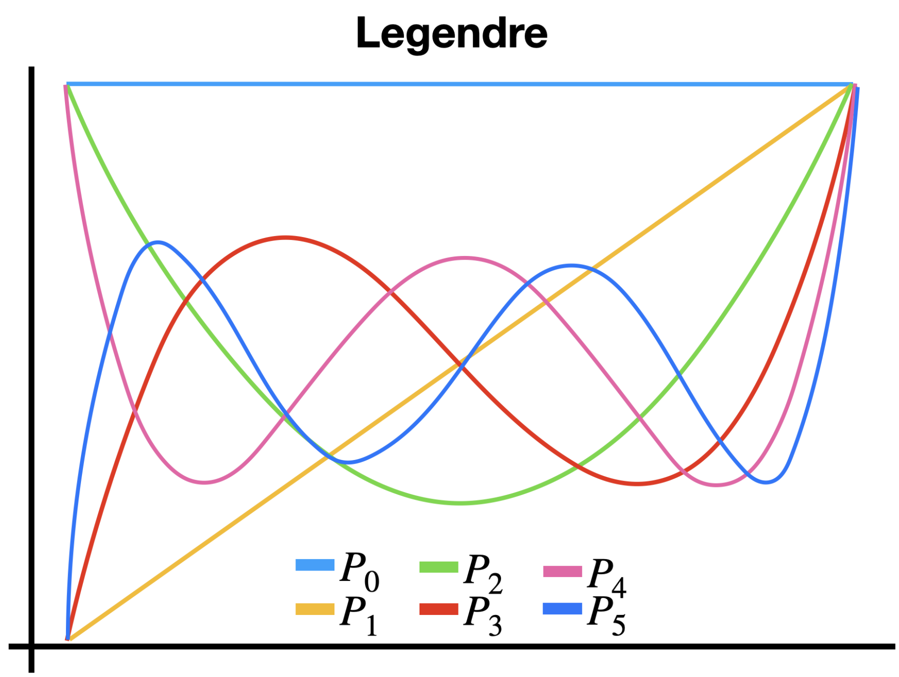
The greatest mathematicians often tend to get the credit for a lot of the discoveries done in their fields, but here are some mathematicians that didn't get as much credit as they deserve, and even got lost among other big names.




## 1 Adrien-Marie Legendre



Legendre worked across several fields, like Analysis, Geometry, and Number Theory. He studied the *special functions* that now bear his name: the **Legendre polynomials**.




And he produced extensive work on *elliptic integrals*, which was later refined and surpassed by mathematicians Abel and Jacobi.

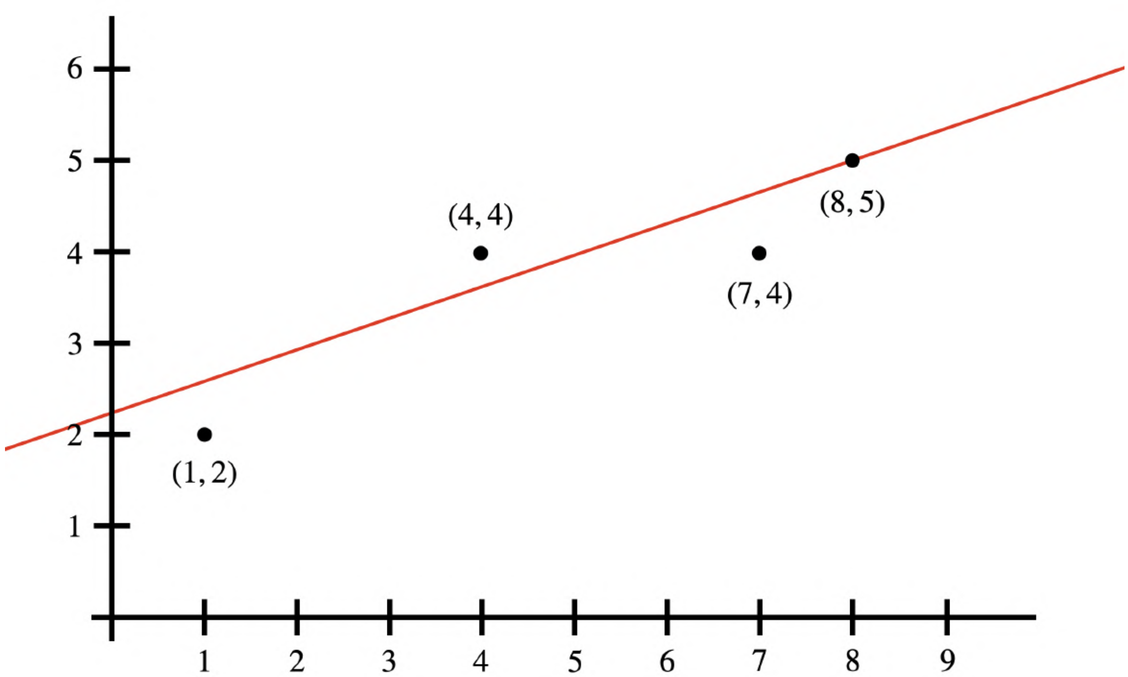
$$F(\phi \mid m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$$

$$E(\phi \mid m) = \int_0^{\phi} \sqrt{1 - m \sin^2 \theta} \, d\theta$$

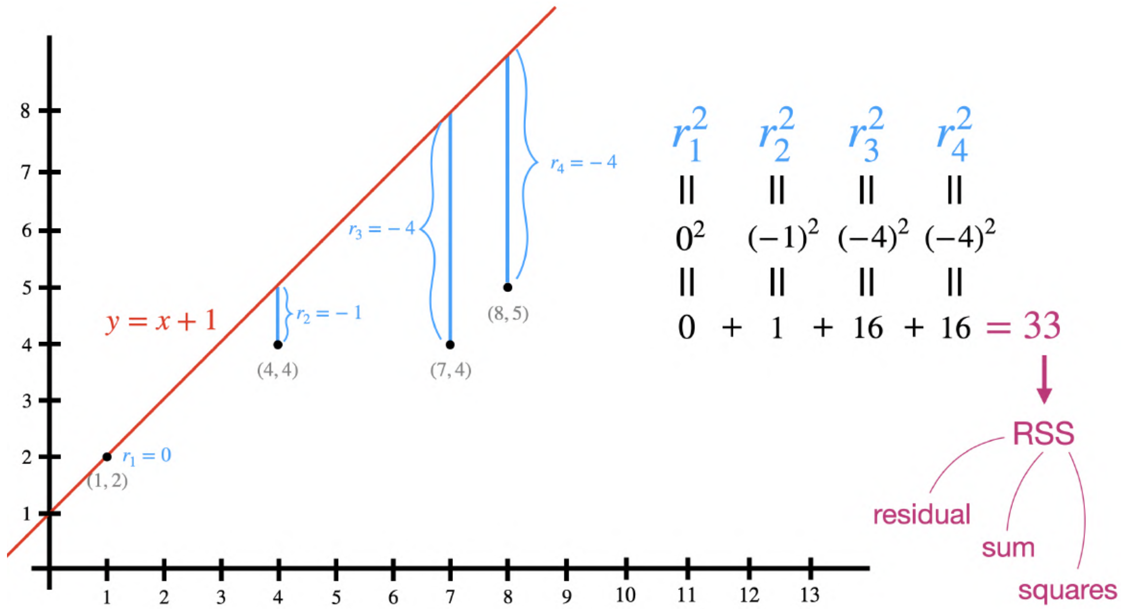
$$\Pi(n; \phi \mid m) = \int_0^{\phi} \frac{d\theta}{(1 - n \sin^2 \theta)\sqrt{1 - m \sin^2 \theta}}$$

Although he, of course, did plenty of work, his most celebrated contribution was in 1805 with the introduction of the *least-squares criterion for curve fitting*, which he developed in the context of calculating the orbits of comets.

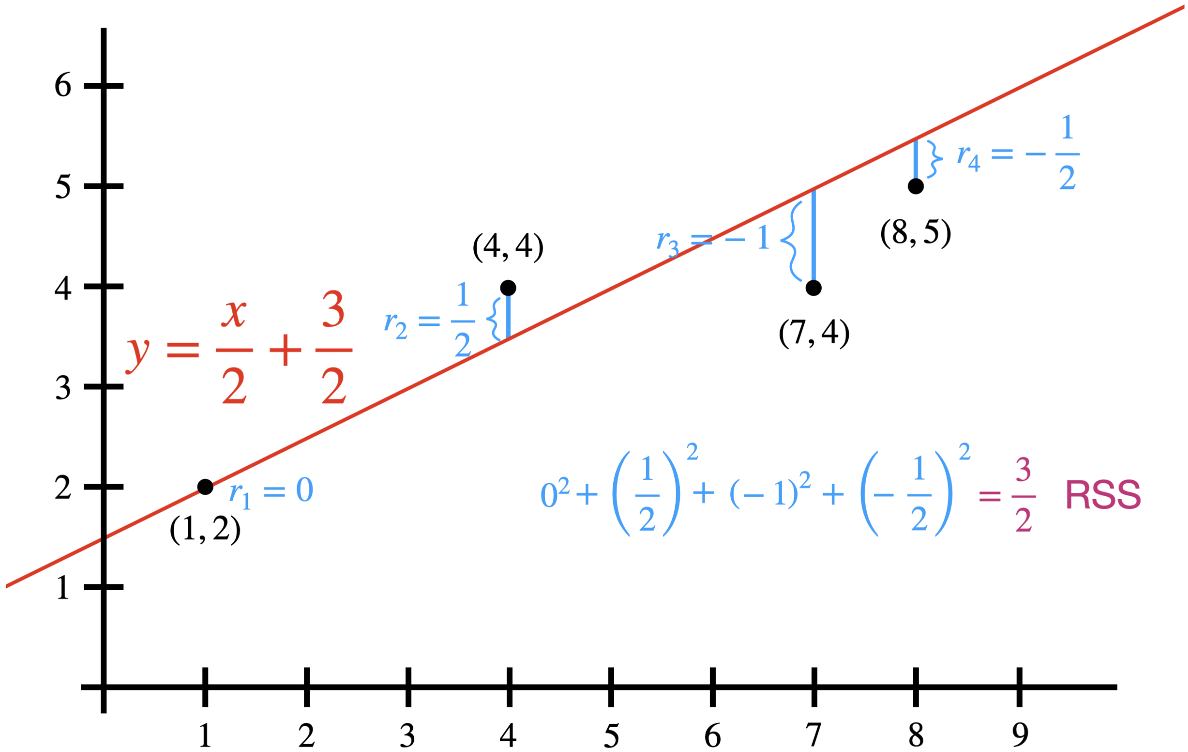
Suppose you have points $(1, 2)$, $(4, 4)$, $(7, 4)$, $(8, 5)$ and you want a straight line prediction.



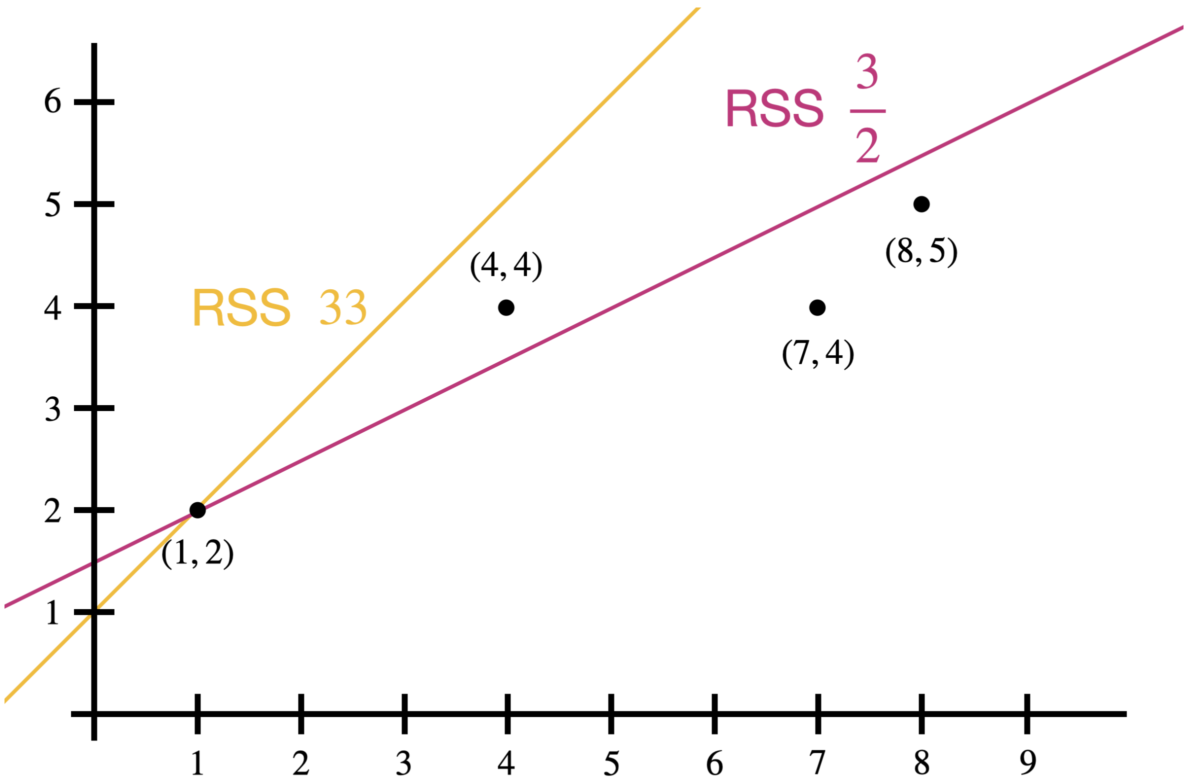
Try line $y = x + 1$. Residuals are 0, $-1$, $-4$ and $-4$. Square each one of these residuals, and then add them up. The result is 33. And this is the residual sum of squares ($RSS$).




Try $y = \frac{x}{2} + \frac{3}{2}$. Residuals 0, $\frac{1}{2}$, $-1$, $-\frac{1}{2}$. The $RSS$ here is $\frac{3}{2}$.

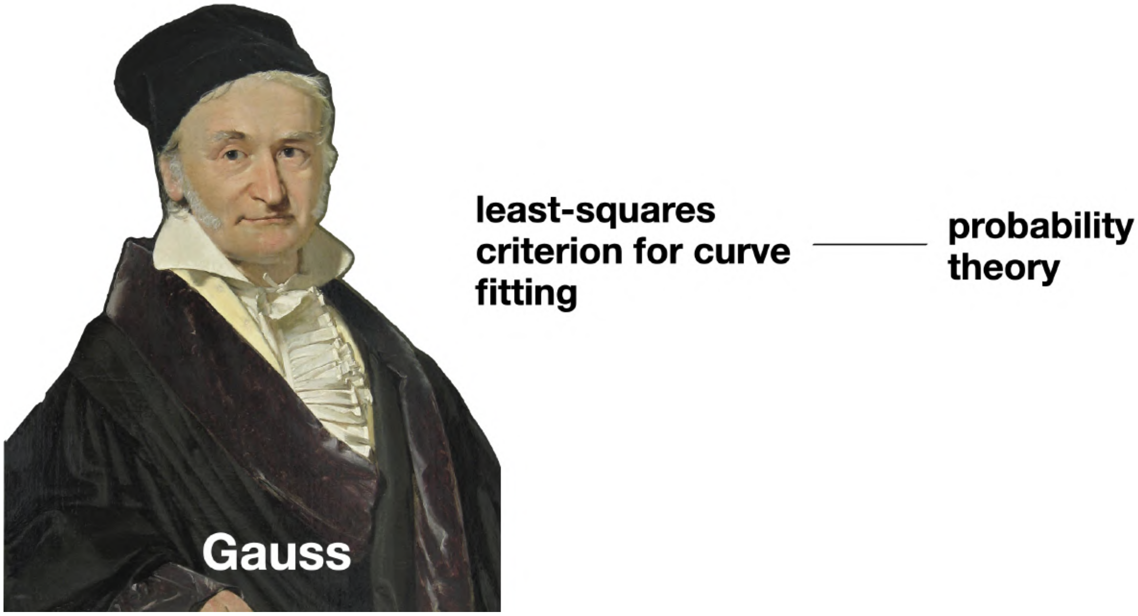



Because $\frac{3}{2} < 33$, the second line is better by *Legendre's rule*: you need to choose the model that makes the sum of squared errors smallest.

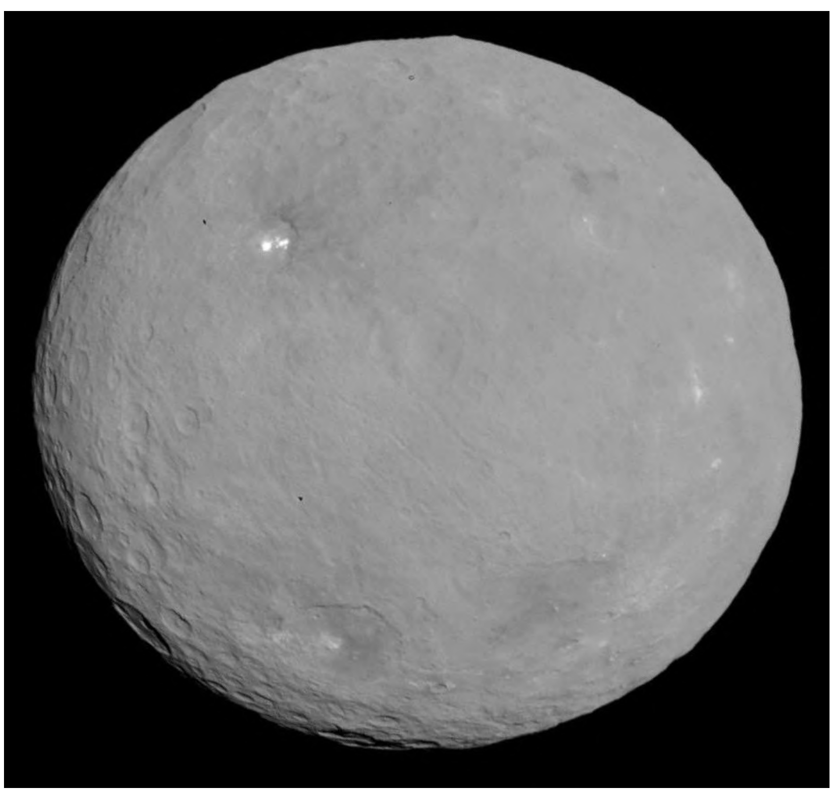


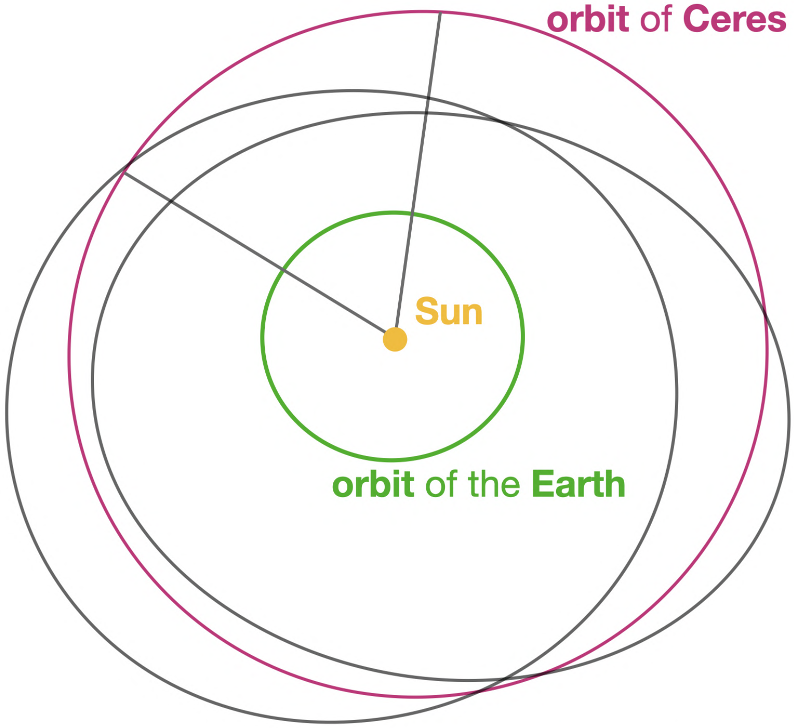

The "grasping least squares" example is just an illustration of the least squares principle, which shows numerically how Legendre's rule picks the best fit among many possible models. Of course, we used a linear model for simplicity, but you can easily imagine how to generalize it for *exponential*, *logarithmic* and *oscillatory* tendencies.

Legendre published the rule explicitly first. But Gauss later presented the same method a few years later, after having linked it to probability theory, since he used it earlier to compute the elliptical orbit of the famous dwarf planet *Ceres*.



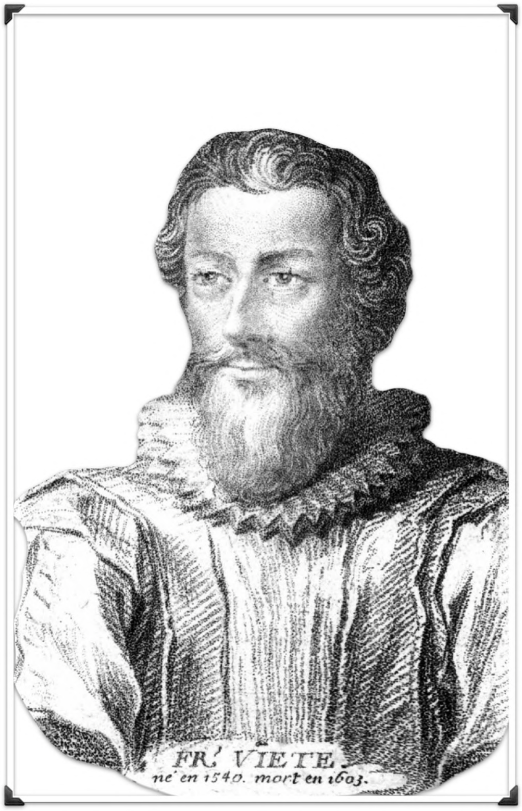

Because of that, Gauss tends to be given more credit.

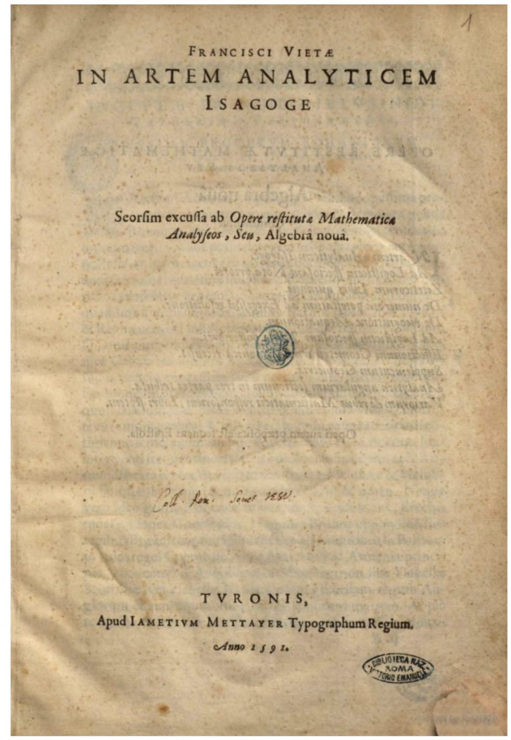





Based on Gauss' sketch of the orbit of Ceres

## 2 François Viète

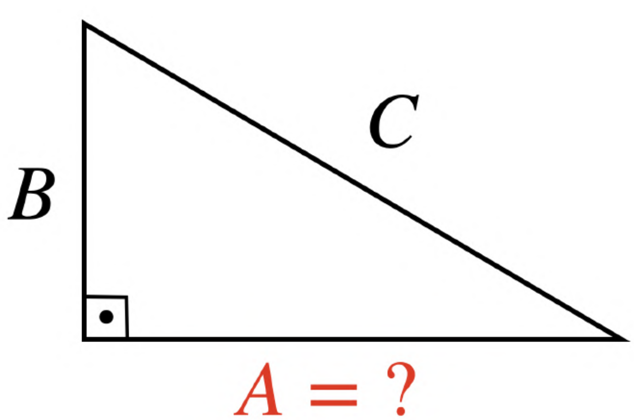




FRANCISCI VIETÆ
IN ARTEM ANALYTICEM
ISAGOGE

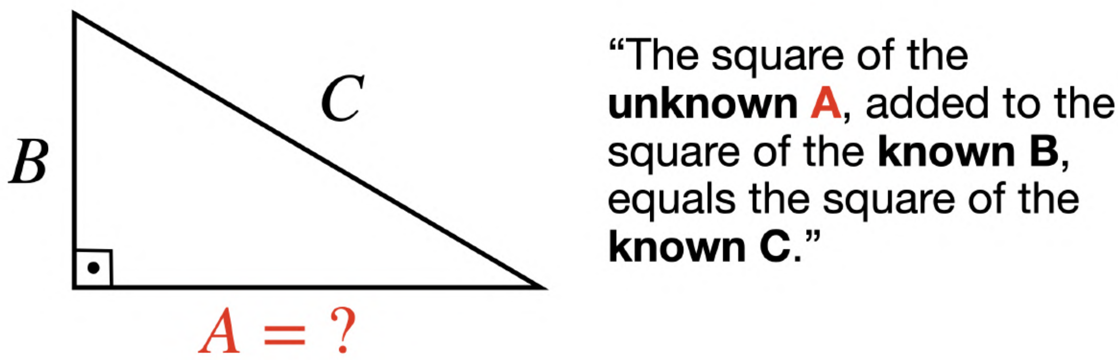
Seorsim excussa ab Opere restitutæ Mathematicæ Analyseos, Seu, Algebrâ nouâ.

TVRONIS,
Apud IAMETIVM METTAYER Typographum Regium.
Anno 1591.

The work for which Viète is best known is "Introduction to the analytic art" (*In Artem Analyticem Isagoge*) in 1591. There, Viète began to combine classical Greek geometry with algebraic methods that had originated from Islamic sources, and by doing so he laid the foundations for the algebraic approach to geometry.



Imagine a right triangle. Let's say you know the length of the hypotenuse and one of the legs. And don't know the value of the other leg.




Viète's innovation was to treat all given and unknowns with symbols, not numbers. Viète used consonants for knowns, vowels for unknowns.

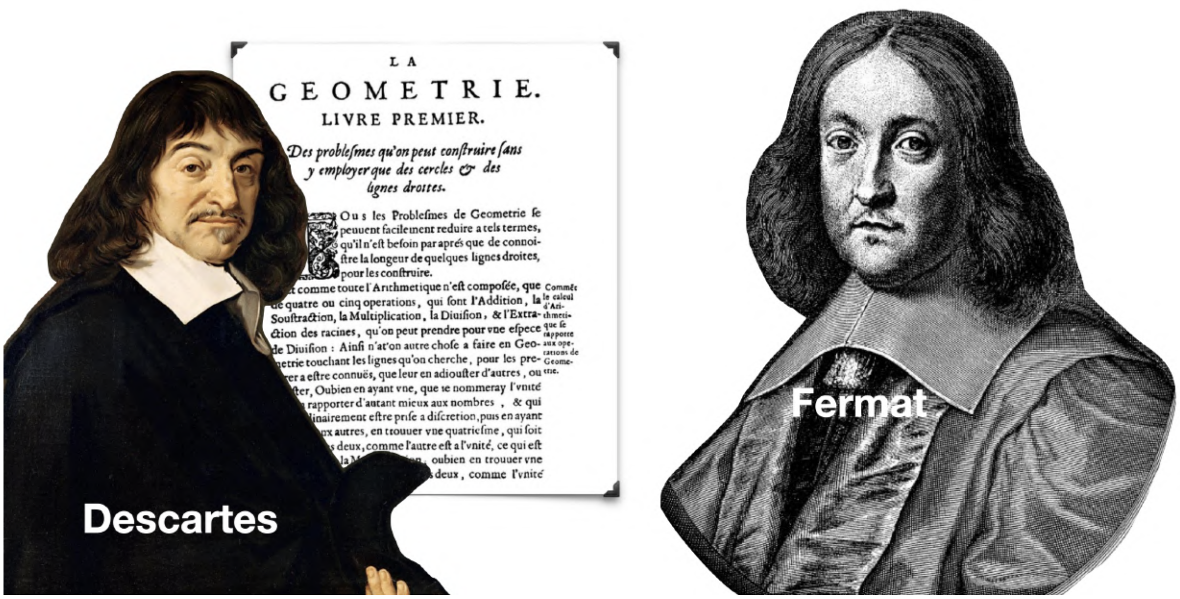
Known length (leg): **B**
Known length (hypotenuse): **C**
Unknown length (the other leg): **A**

So, instead of plugging in actual numbers immediately. Viète worked in general, i.e. abstractly.
Using the Pythagorean relationship, he would write something like:

*"The square of the unknown A, added to the square of the known B, equals the square of the known C."*

In symbols, the expression can be translated to this: $A^2 + B^2 = C^2 \implies A^2 = C^2 - B^2 \implies A = \sqrt{C^2 - B^2}$.

This might obviously seem easy to you, but it was a big deal in Viète's time to have an algebraic method to express the answer generally in terms of givens. He then wouldn't stop at Algebra, but would show how to construct the length $A$ with ruler-and-compass, and would turn the result from Algebra into Geometry.

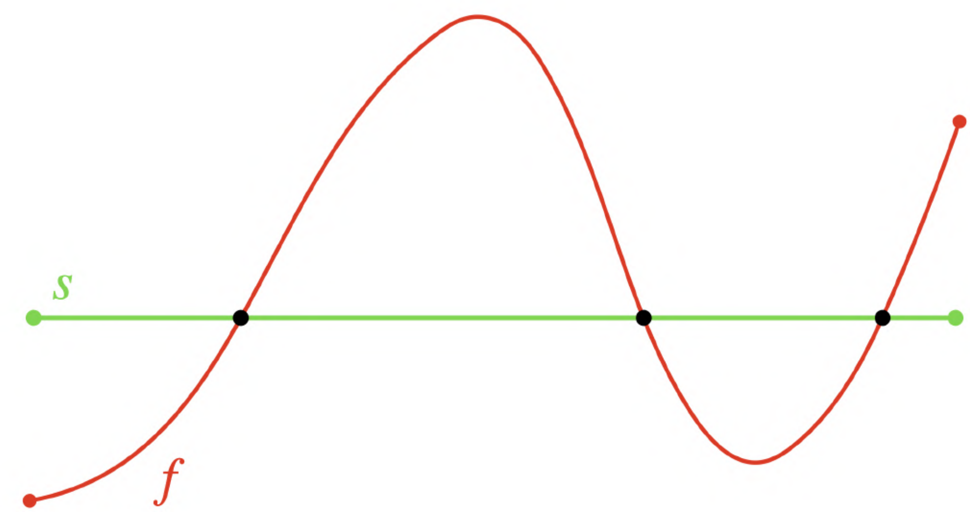



A generation later, Descartes' *La Géométrie* made this program dominant by introducing coordinate geometry and cleaner symbolic notation (exponents like $x^2$, variables $x$, $y$, constants $a$, $b$). Fermat independently developed Analytic Geometry too. Because their notation and coordinate method became the standard, historians credit Descartes and Fermat as founders of Analytic Geometry, and Viète is considered to be more of a precursor.

# 3 Bernard Bolzano

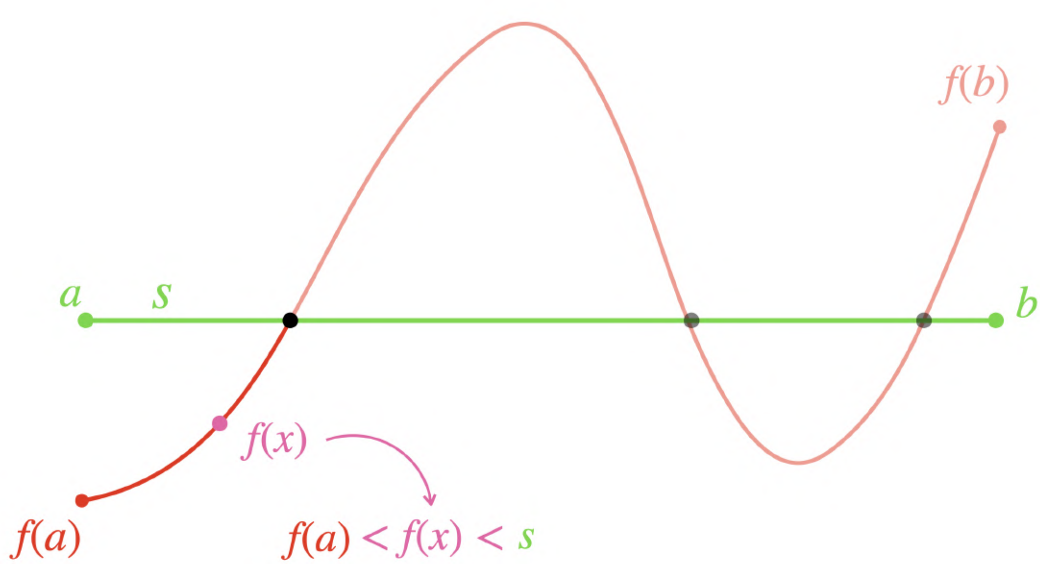
Bolzano was one of the first to push mathematics toward real **rigor** in Analysis.

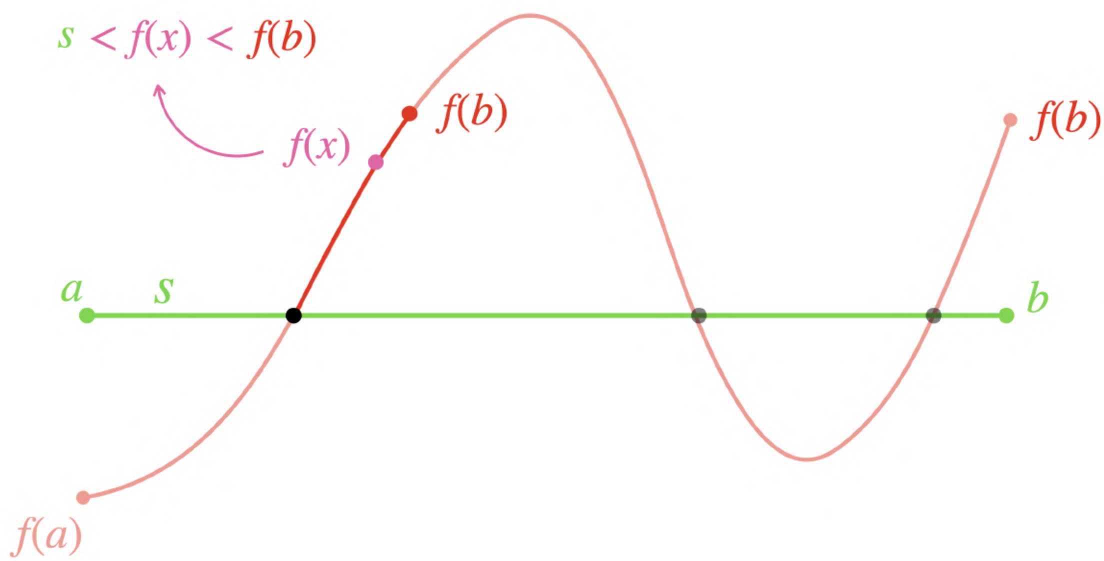
In 1817, he proved an early version of the *intermediate value theorem*.




Say we have some continuous function $f(x)$, and we choose a value $y = s$. The points mark where the curve intersects that horizontal line, i.e. where $f(x) = s$.

If $f$ is continuous and at one point $f(x)$ is **below** $s$ while at another point it is **above** $s$, then the function must cross the line $f(x) = s$ somewhere in between.



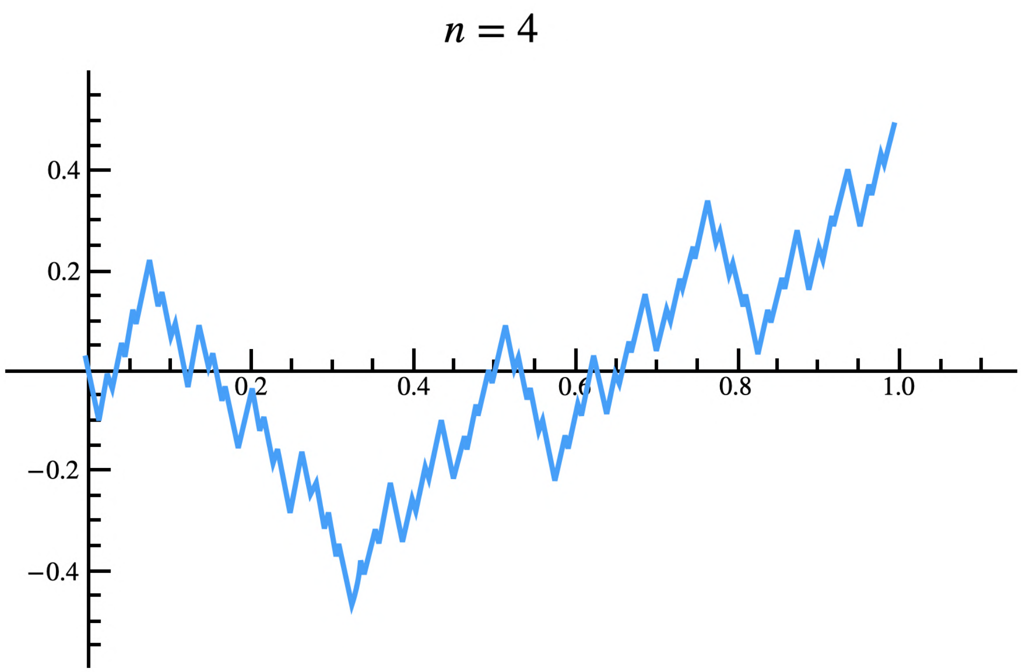



Formally the theorem is expressed as this:

**Intermediate Value Theorem:** Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $s \in \mathbb{R}$ satisfy $f(a) < s < f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = s$.

At the time mathematicians widely believed that a continuous function must be **differentiable** except at some isolated points. Intuitively speaking, differentiable means that if you zoom in very closely near that point, the function begins to look like a "straight line". But, Bolzano actually constructed a counterexample to that assumption.

## Bolzano’s continuous but nowhere differentiable function

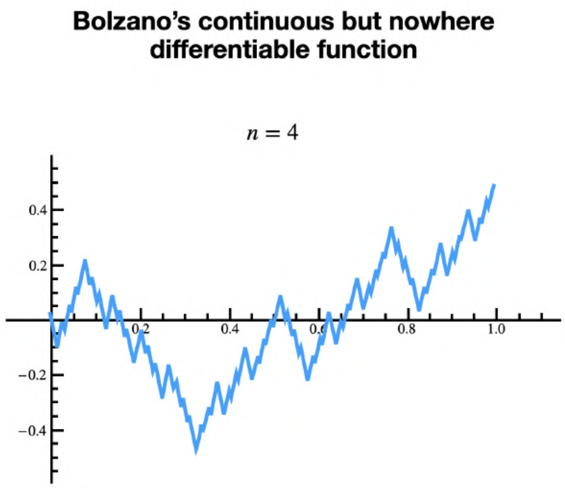


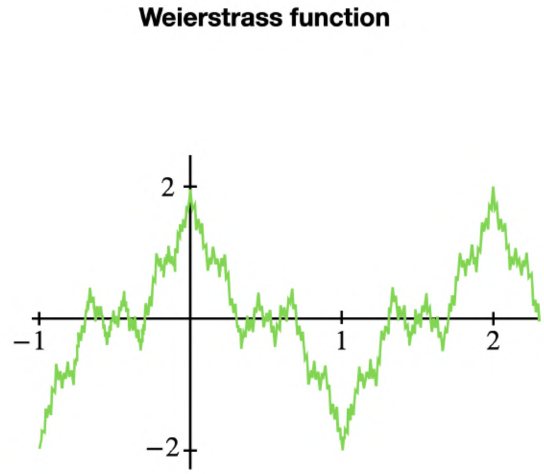

The function Bolzano built in the early 1830s is usually called *Bolzano’s nowhere-differentiable function*.

*(If you want to explore and play with this function, check out the link below!)*

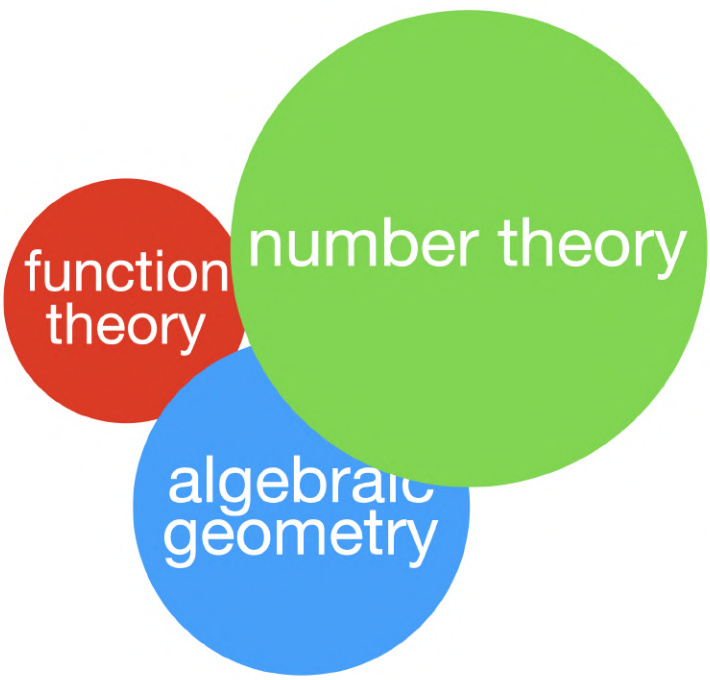
Bolzano’s Continuous But Nowhere Differentiable Function

Unfortunately, he never published it, so no one knew about it until historians later reconstructed it from his manuscripts, and it’s now recognized as the first known example of a function that is continuous everywhere but differentiable nowhere. All of this, more than 30 years before Weierstrass’s famous 1872 example.




# 4 Ernst Eduard Kummer




He made important contributions in several fields, like Function Theory and Algebraic Geometry, but his greatest discovery was in Number Theory, namely the creation of the theory of *ideal prime factors* in the 1840s.

Kummer originally wanted to generalize the law of **quadratic reciprocity** (which was first proved by Gauss) to higher powers.

## law of quadratic reciprocity

Let $p$ and $q$ be distinct odd prime numbers, and define the Legendre symbol as

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } n^2 \equiv q \mod p, \text{for some integer } n \\ -1 & \text{otherwise}. \end{cases}$$

then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)}{2}\cdot\frac{(q-1)}{2}}$$

As a concrete example of this law, let's pick $p = 13$ and $q = 3$.

**Step 1:** **Compute the Legendre symbols separately**

$\left(\frac{3}{13}\right)$: check if 3 is a square mod 13.

$$
\begin{aligned}
1^2 &\equiv 1 \pmod{13}\\
2^2 &\equiv 4 \pmod{13}\\
3^2 &\equiv 9 \pmod{13}\\
4^2 &= 16 \equiv 3 \pmod{13}\\
6^2 &= 36 \equiv 10 \pmod{13}
\end{aligned}
$$

Since $4^2 = 16 \equiv 3 \pmod{13}$, we have $\left(\frac{3}{13}\right) = 1$.

$\left(\frac{13}{3}\right)$: check if 13 is a square mod 3.

$$
\begin{aligned}
1^2 &\equiv 1 \pmod{3}\\
2^2 &= 4 \equiv 1 \pmod{3}\\
3^2 &= 9 \equiv 0 \pmod{3}\\
&\vdots\\
13^2 &= 169 \equiv 1 \pmod{3}
\end{aligned}
$$

Thus $n^2 \equiv 1 \pmod{3}$ has solutions ($n \equiv 1, 2$), so $\left(\dfrac{13}{3}\right) = 1$.

**Step 2:** **Multiply them**

$$
\left(\frac{3}{13}\right)\left(\frac{13}{3}\right) = 1 \cdot 1 = 1
$$

**Step 3:** **Quadratic Reciprocity**

For odd primes $p, q$:

$$
\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}
$$

With $p = 13,\ q = 3$:

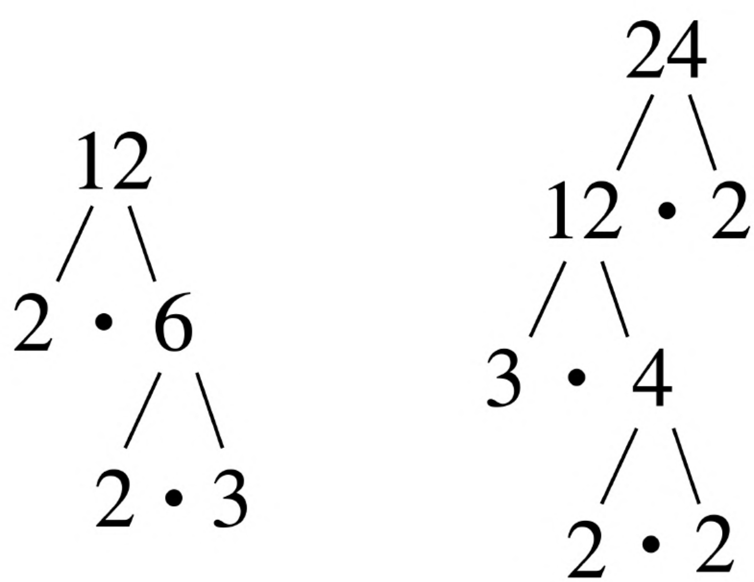
$$(-1)^{\frac{(13-1)(3-1)}{4}} = (-1)^{\frac{12\cdot 2}{4}} = (-1)^6 = 1$$

$$\boxed{\left(\frac{3}{13}\right)\left(\frac{13}{3}\right) = 1 = (-1)^6}$$

Therefore, the Law of Quadratic Reciprocity holds for $p = 13,\ q = 3$.

This is not a proof, but it allows us to verify that the law holds at least for this particular example.

Kummer wanted to generalize this law for higher powers. But before seeing how, let's understand something first.




We can factorize any integer into prime factors, and this factorization is unique (up to reordering the factors, and multiplying them by $-1$). But this idea can be extended.

For example, we can consider the Gaussian integers, which is the set of complex numbers of the form $a + b\,i$, where $a$ and $b$ are integers.

In this number system, 5 is not considered a prime, because you can factorize it into $(1+2i)(1-2i) = 1^2 - (2i)^2 = 1+4 = 5$. However, 3 is still prime, because the only way to factorize it would be into $(\sqrt{2}+i)(\sqrt{2}-i) = (\sqrt{2})^2 - i^2 = 2-(-1) = 3$, but the problem is that $\sqrt{2}$ is not an integer, and therefore these factors are not Gaussian integers, by definition.

What is cool about that is that Gaussian integers also have unique factorizations. Any Gaussian integer can be written as a product of *irreducibles* (i.e. elements that cannot be further factored into non-units).

Is this always true? I mean, is it true that for any number you can think of, there is always a unique way of factorizing it? Unfortunately not...

The Gaussian integers are an example of a ring in mathematics. There are many others, but if you don't know what it is, just think of it as a set of elements that possesses addition and multiplication, both of which are *associative*, *commutative*, and where the *distributive* property is valid.

There are some rings that do not present this uniqueness in factorization. One example is the ring $\mathbb{Z}[\sqrt{-5}]$ which are the numbers of the form $a + b\sqrt{-5}$, $a, b \in \mathbb{Z}$.

Take $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. We just factorized it into two different elements of the same ring $\mathbb{Z}[\sqrt{-5}]$.

This was a problem, because one of the early attempts to prove Fermat's Last Theorem was to factorize the equation over some extension of the integers (which is also called a ring of integers), and then use unique prime factorization to show that this failed. But, if we can't have unique prime factorization, then it doesn't make any sense to use this approach.

So Kummer had a bright idea. He said that, though it's true that for a general ring of integers, prime factorization was impossible, one could come up with a concept of an "ideal number", and you could uniquely factor an element from a ring of integers into such ideal numbers.

In the example of $\mathbb{Z}[\sqrt{-5}]$ you would replace the existing primes with the "ideal numbers".

$$\mathbb{Z}[\sqrt{-5}]$$

**Ideal numbers:**

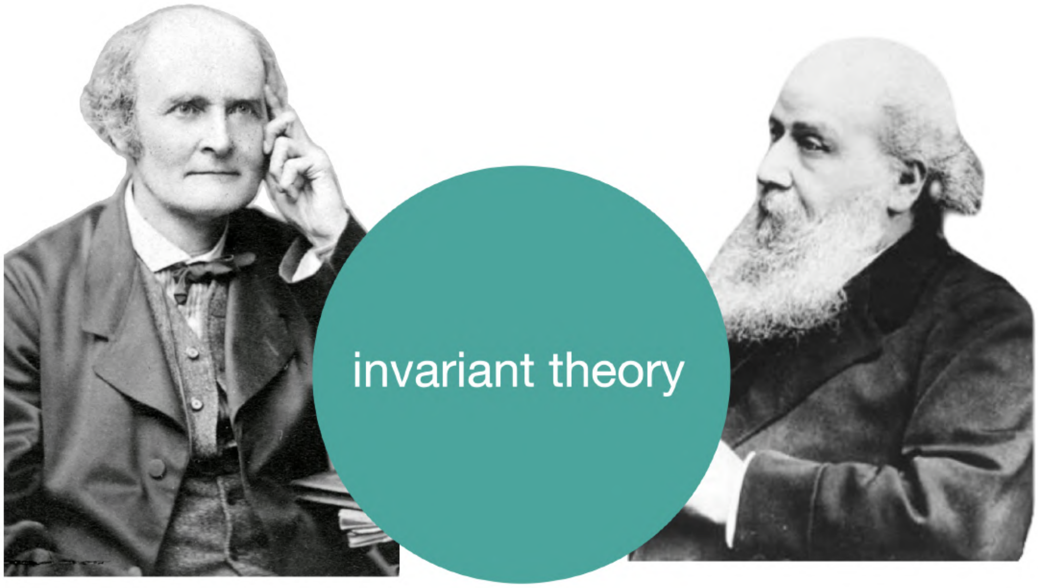
$$\mathfrak{p}_1 = (2, 1+\sqrt{-5}) \qquad \mathfrak{p}_2 = (3, 1-\sqrt{-5})$$

$$(6) = \mathfrak{p}_1 \cdot \overline{\mathfrak{p}_1} \cdot \mathfrak{p}_2 \cdot \overline{\mathfrak{p}_2}$$

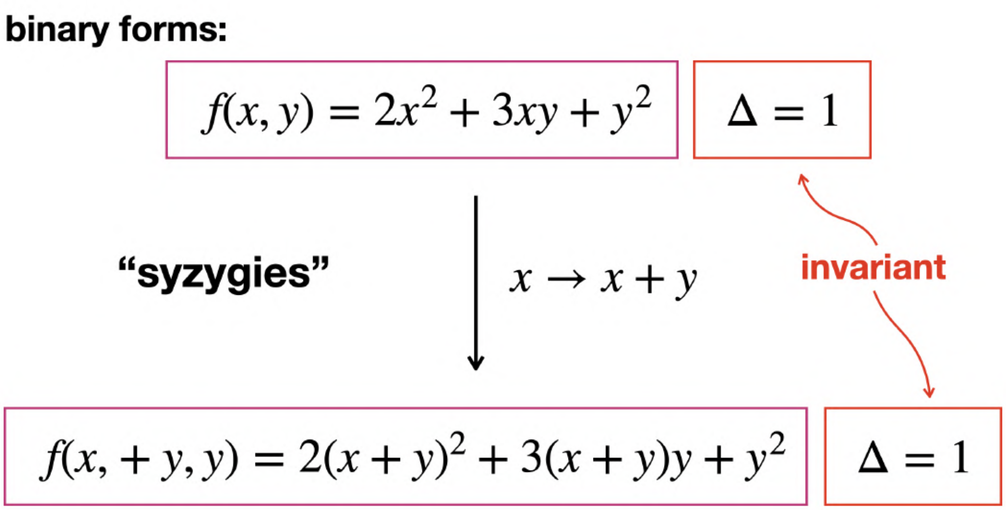
Decades later, Richard Dedekind took Kummer's idea and generalized it by introducing ideals in ring theory, and proved their unique factorization. Rings with this property are now called **Dedekind domains**. They're basically conceptual successors of Kummer's "ideal primes".

# 5 James Joseph Sylvester

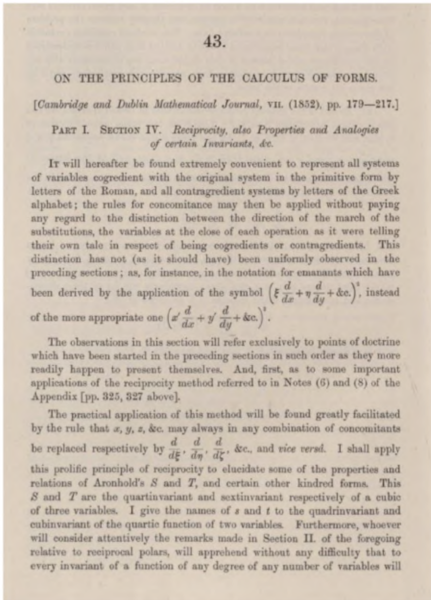
James Joseph Sylvester was a 19th-century algebraist who made various significant contributions, but his most famous was helping to create the modern field of invariant theory, and the related notion of *covariants*, alongside the more famous mathematician Arthur Cayley.




More concretely speaking, they focused on devising techniques for explicitly finding invariants and covariants of *binary forms* and determining their algebraic relations, otherwise known as *"syzygies"*.




Sylvester focused on resolving these questions in two important papers, and in this one, he proved, among other results, *Sylvester's law of inertia*.



43.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

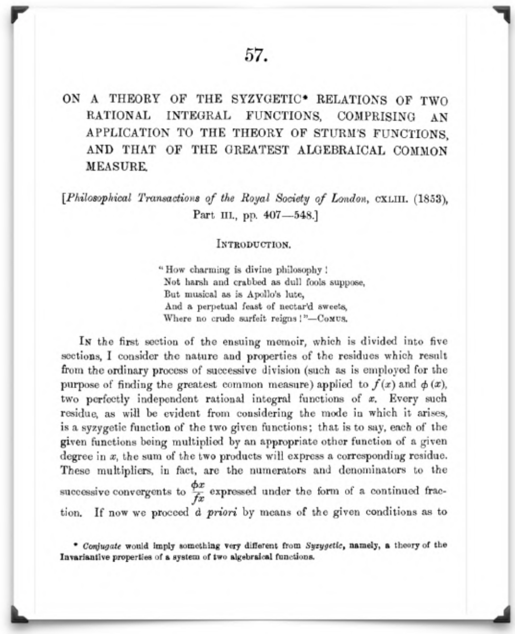
[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 179—217.]

PART I. SECTION IV. *Reciprocity, also Properties and Analogies of certain Invariants, &c.*

IT will hereafter be found extremely convenient to represent all systems of variables cogredient with the original system in the primitive form by letters of the Roman, and all contragredient systems by letters of the Greek alphabet; the rules for concomitance may then be applied without paying any regard to the distinction between the direction of the march of the substitutions, the variables at the close of each operation as it were telling their own tale in respect of being cogredients or contragredients. This distinction has not (as it should have) been uniformly observed in the preceding sections; as, for instance, in the notation for emanants which have been derived by the application of the symbol $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\&c.\right)^i$, instead of the more appropriate one $\left(x'\frac{d}{dx}+y'\frac{d}{dy}+\&c.\right)^i$.

The observations in this section will refer exclusively to points of doctrine which have been started in the preceding sections in such order as they more readily happen to present themselves. And, first, as to some important applications of the reciprocity method referred to in Notes (6) and (8) of the Appendix [pp. 325, 327 above].

The practical application of this method will be found greatly facilitated by the rule that $x, y, z$, &c. may always in any combination of concomitants be replaced respectively by $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this prolific principle of reciprocity to elucidate some of the properties and relations of Aronhold's $S$ and $T$, and certain other kindred forms. This $S$ and $T$ are the quartinvariant and sextinvariant respectively of a cubic of three variables. I give the names of $s$ and $t$ to the quadrinvariant and cubinvariant of the quartic function of two variables. Furthermore, whoever will consider attentively the remarks made in Section II. of the foregoing relative to reciprocal polars, will apprehend without any difficulty that to every invariant of a function of any degree of any number of variables will



57.

ON A THEORY OF THE SYZYGETIC* RELATIONS OF TWO RATIONAL INTEGRAL FUNCTIONS, COMPRISING AN APPLICATION TO THE THEORY OF STURM'S FUNCTIONS, AND THAT OF THE GREATEST ALGEBRAICAL COMMON MEASURE.

[*Philosophical Transactions of the Royal Society of London*, CXLIII. (1853), Part III., pp. 407—548.]

INTRODUCTION.

"How charming is divine philosophy!
Not harsh and crabbed as dull fools suppose,
But musical as is Apollo's lute,
And a perpetual feast of nectar'd sweets,
Where no crude surfeit reigns!"—COMUS.

IN the first section of the ensuing memoir, which is divided into five sections, I consider the nature and properties of the residues which result from the ordinary process of successive division (such as is employed for the purpose of finding the greatest common measure) applied to $f(x)$ and $\phi(x)$, two perfectly independent rational integral functions of $x$. Every such residue, as will be evident from considering the mode in which it arises, is a syzygetic function of the two given functions; that is to say, each of the given functions being multiplied by an appropriate other function of a given degree in $x$, the sum of the two products will express a corresponding residue. These multipliers, in fact, are the numerators and denominators to the successive convergents to $\frac{\phi x}{fx}$ expressed under the form of a continued fraction. If now we proceed *à priori* by means of the given conditions as to

* *Conjugate* would imply something very different from *Syzygetic*, namely, a theory of the Invariantive properties of a system of two algebraical functions.

The gist of it is that any real quadratic expression can be turned, by a change of coordinates, into a sum of positive squares minus a sum of negative squares (and possibly some zeros). The numbers of "**+**" and "**–**" squares you get are fixed no matter how you change coordinates. That fixed pair is the **inertia**.

**Real Quadratic Form**

$$Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x}$$

$(p, q)$ = inertia

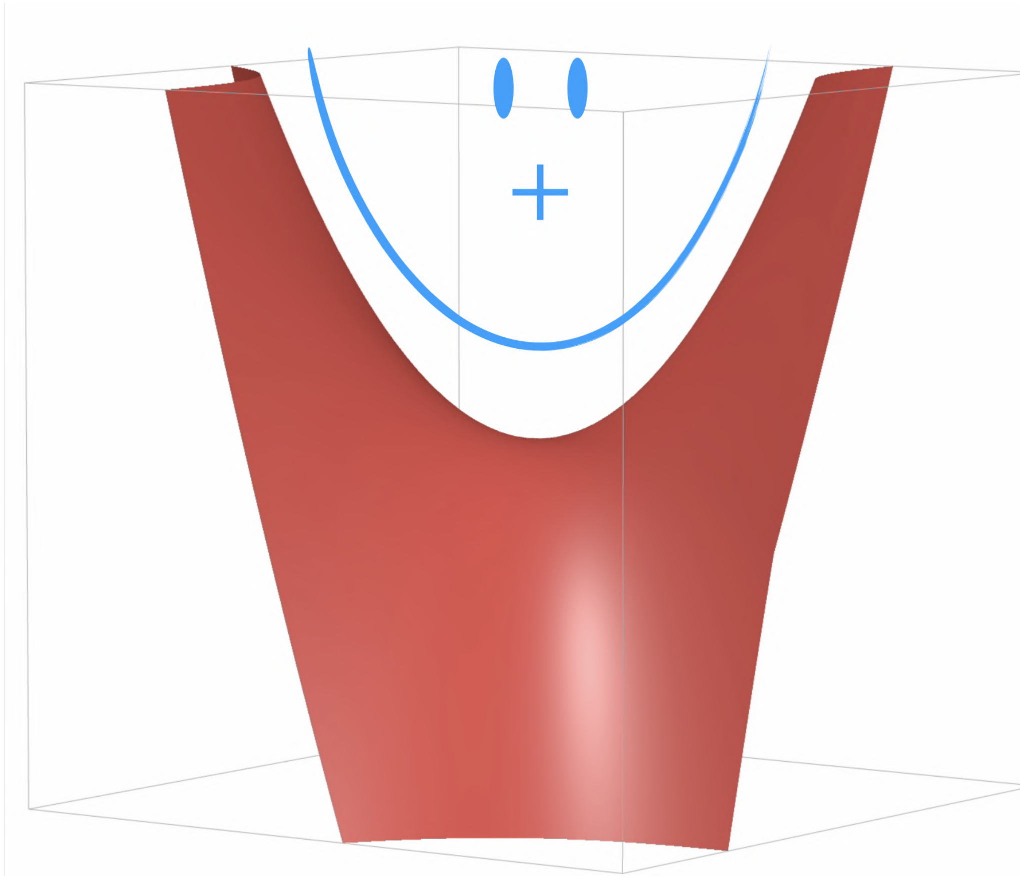
fixed

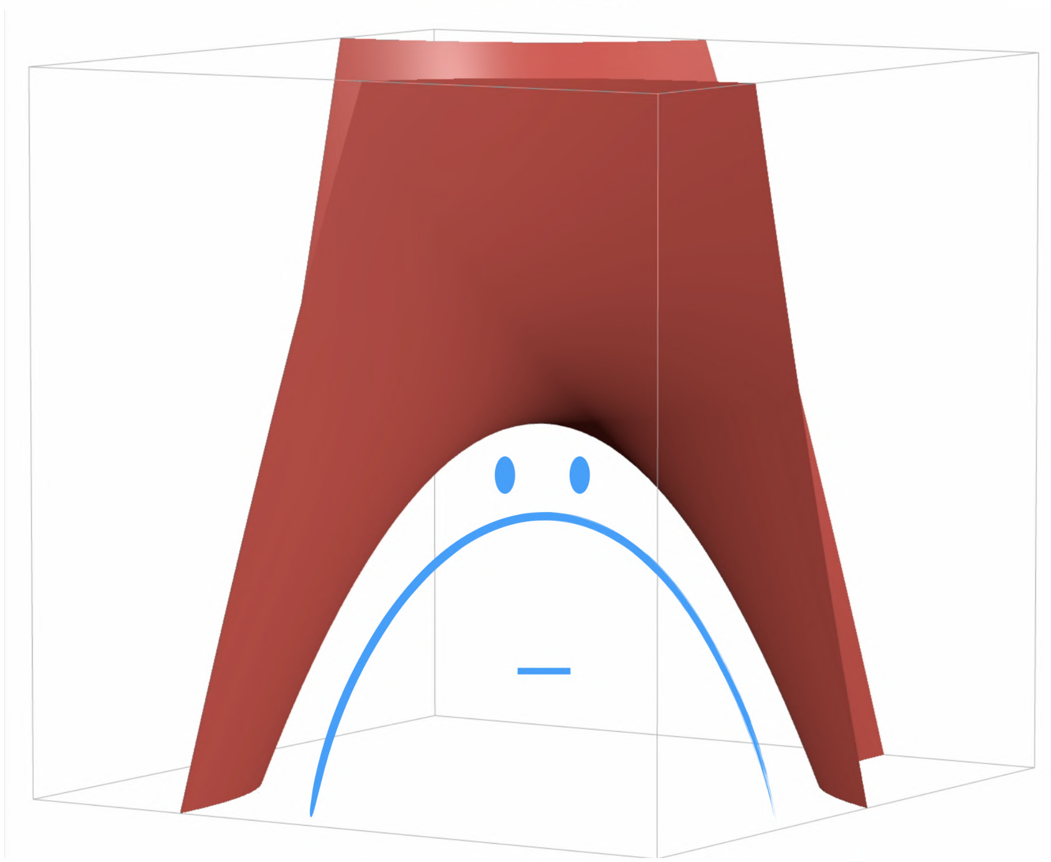
$$Q(y) = \sum_{i=1}^{p} y_i^2 - \sum_{j=1}^{q} y_{p+j}^2$$

Take this example:

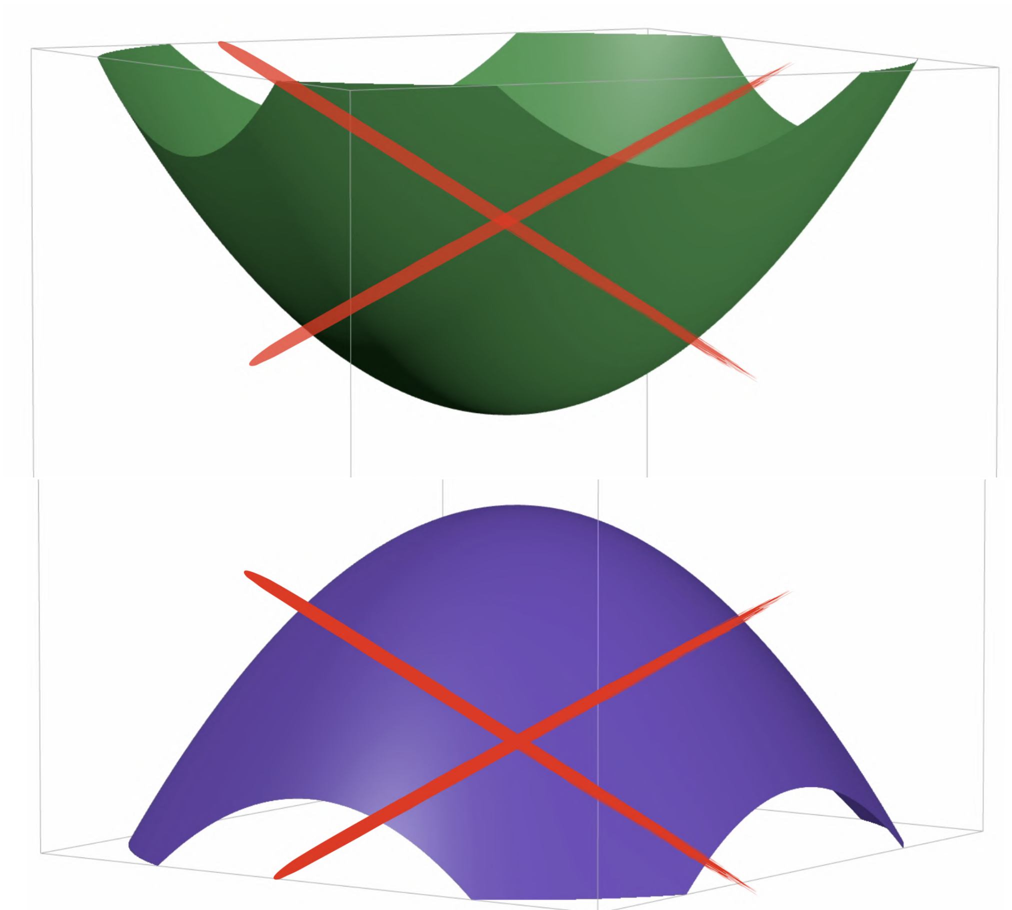
$$Q(x, y) = x^2 + 4xy - y^2$$

It has one **+** (represented by this happy face shape below) and one **-** (represented by this sad face shape below).



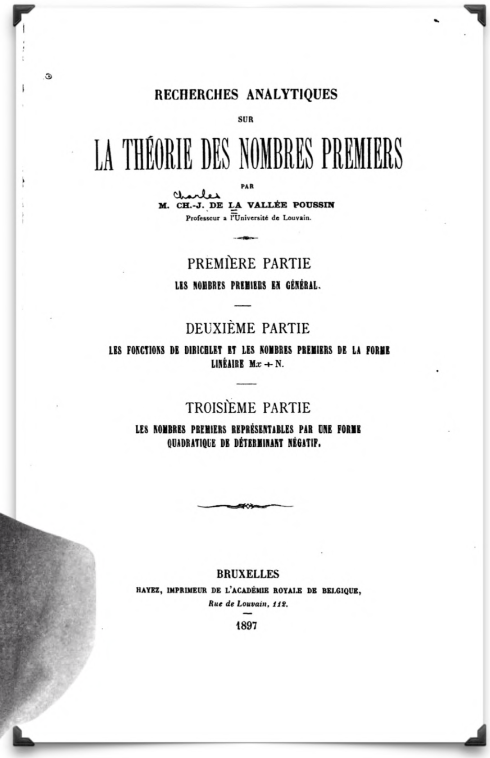
This is a *saddle*. No invertible change of variables can make it two pluses or two minuses. It will always stay one plus and one minus. That's the law.



Even though Sylvester co-laid invariant theory and even coined the term matrix, the grand results came from Cayley (1858 A Memoir on the Theory of Matrices, Cayley–Hamilton theorem, Cayley's theorem in group theory, Cayley tables, Cayley graphs). Because of that, historical credit in broad algebra often goes to Cayley, and Sylvester's role is sometimes noted but it's often not very remembered.

# 6 Charles-Jean de la Vallée Poussin

Charles-Jean de la Vallée Poussin was a Belgian mathematician whose main achievement was his 1896 proof of the **Prime Number Theorem**, which was first conjectured by Gauss.



RECHERCHES ANALYTIQUES

SUR

LA THÉORIE DES NOMBRES PREMIERS

PAR

Charles

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PREMIÈRE PARTIE

LES NOMBRES PREMIERS EN GÉNÉRAL.

DEUXIÈME PARTIE

LES FONCTIONS DE DIRICHLET ET LES NOMBRES PREMIERS DE LA FORME LINÉAIRE $Mx + N$.

TROISIÈME PARTIE

LES NOMBRES PREMIERS REPRÉSENTABLES PAR UNE FORME QUADRATIQUE DE DÉTERMINANT NÉGATIF.

BRUXELLES

HAYEZ, IMPRIMEUR DE L'ACADÉMIE ROYALE DE BELGIQUE,

*Rue de Louvain, 112.*

1897

If you ask yourself, how many prime numbers are there between 1 and $n$? A natural first reaction would be to define $\pi(n)$ to be a function that returns the number of prime numbers between 1 and the number $n$ you choose. Then, one would have to come up with a formula for $\pi(n)$. The thing is that we don't have an obvious pattern to primes, and we don't know if there's a way to find a formula that can spit out primes. So instead, Poussin looked for a good estimate.

The Prime Number Theorem tells us just that: as $n$ grows large, the number of primes less than $n$, written as $\pi(n)$, is approximately this:

$$\pi(n) \sim \frac{n}{\log n}$$

That is, primes become less frequent as numbers get larger, but they do so in a way that's closely ruled by logarithmic decay.

After proving this, Poussin realized that he could strengthen it. He showed that the better approximation is the logarithmic integral:

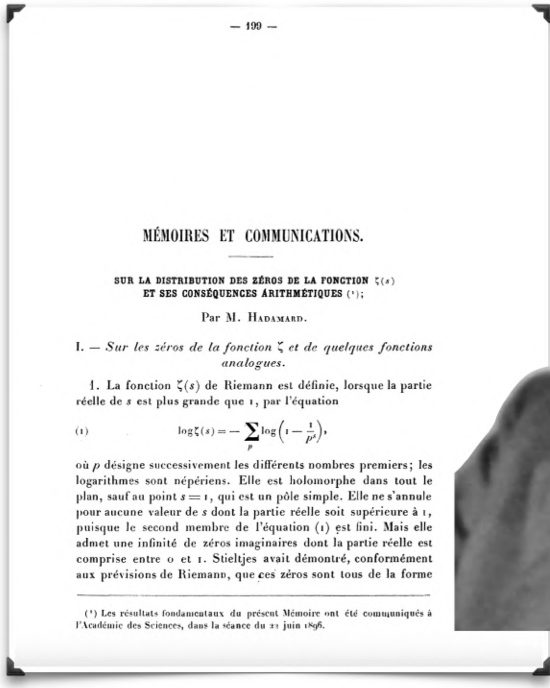
$$\mathrm{li}(n) = \int_2^n \frac{dt}{\log t}$$

And he proved an *error* that actually shrinks fast:

$$\pi(n) = \mathrm{li}(n) \; + \; O\big(n\, e^{-a\sqrt{\log n}}\big) \quad \text{for some } a > 0$$

So not only do we know the right scale, we also know how close $\pi(n)$ is to the real number of primes in this range.

Hadamard proved the Prime Number Theorem independently in the same year by similar complex methods, so many accounts lead with "Hadamard (and de la Vallée Poussin)". Yet the improvements shown above, the explicit zero-free region, the corresponding error bounds, and the progression version, are de la Vallée Poussin's and directly explain how the approximation $\pi(x) \approx \mathrm{li}(x)$ becomes precise.



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MÉMOIRES ET COMMUNICATIONS.

SUR LA DISTRIBUTION DES ZÉROS DE LA FONCTION $\zeta(s)$
ET SES CONSÉQUENCES ARITHMÉTIQUES [1];

Par M. HADAMARD.

I. — *Sur les zéros de la fonction $\zeta$ et de quelques fonctions analogues.*

1. La fonction $\zeta(s)$ de Riemann est définie, lorsque la partie réelle de $s$ est plus grande que 1, par l'équation

$$\log\zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right), \quad (1)$$

où $p$ désigne successivement les différents nombres premiers; les logarithmes sont népériens. Elle est holomorphe dans tout le plan, sauf au point $s = 1$, qui est un pôle simple. Elle ne s'annule pour aucune valeur de $s$ dont la partie réelle soit supérieure à 1, puisque le second membre de l'équation (1) est fini. Mais elle admet une infinité de zéros imaginaires dont la partie réelle est comprise entre 0 et 1. Stieltjes avait démontré, conformément aux prévisions de Riemann, que ces zéros sont tous de la forme

(1) Les résultats fondamentaux du présent Mémoire ont été communiqués à l'Académie des Sciences, dans la séance du 22 juin 1896.

As a consequence, usually, both mathematicians are credited with the proof, although Hadamard tends to be mentioned first because his name is more recognized. Poussin's proof is arguably stronger though, because he gave a sharper error term and extended it to primes in arithmetic progressions.

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This list is just a preview of some of the mathematicians that served as important figures in developing math but who don't get enough credit for their contribution. Let us know of any other mathematicians that you think don't get enough credit.

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