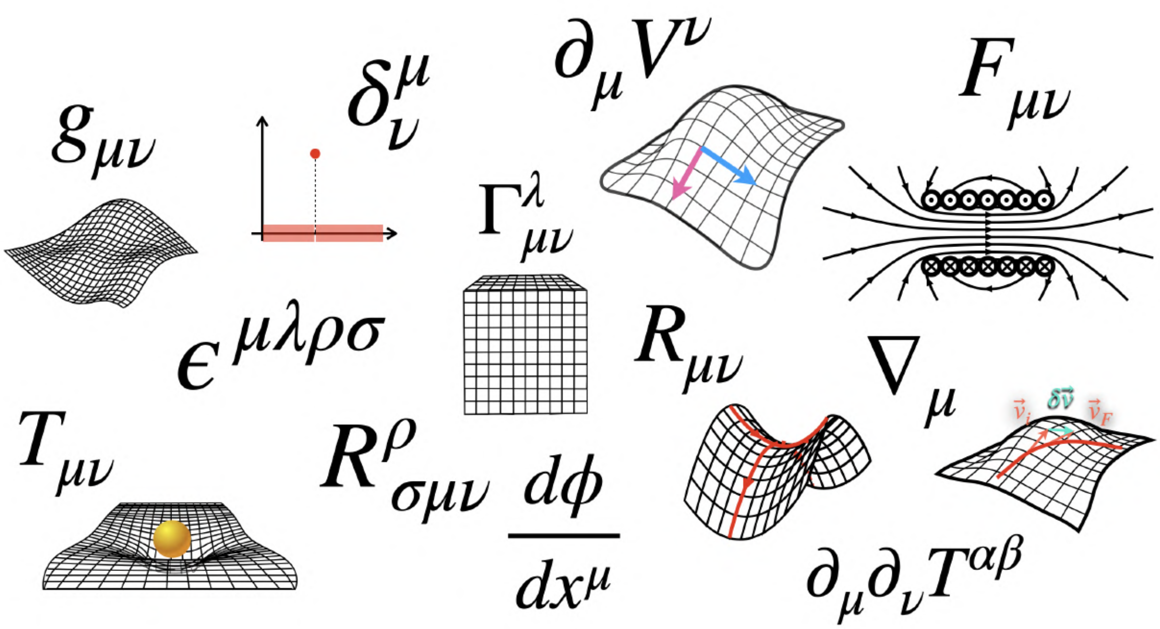
Tensor Impostors

by DiBeos



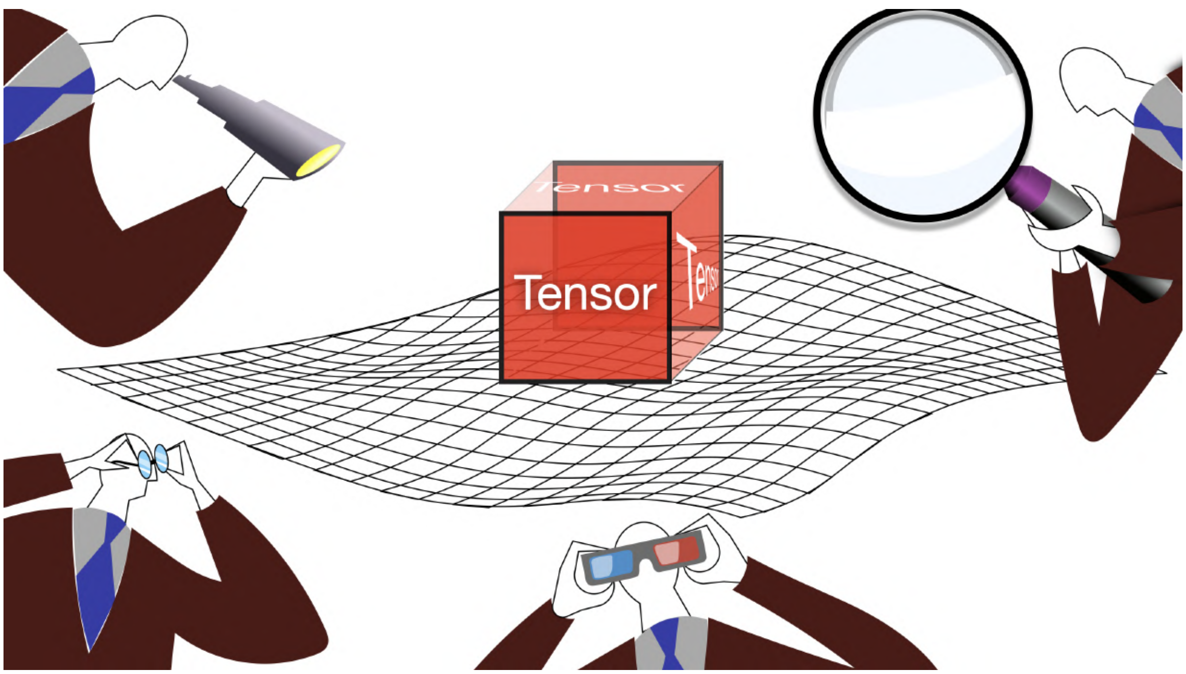
"Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change" – **Henri Poincaré**

Introduction

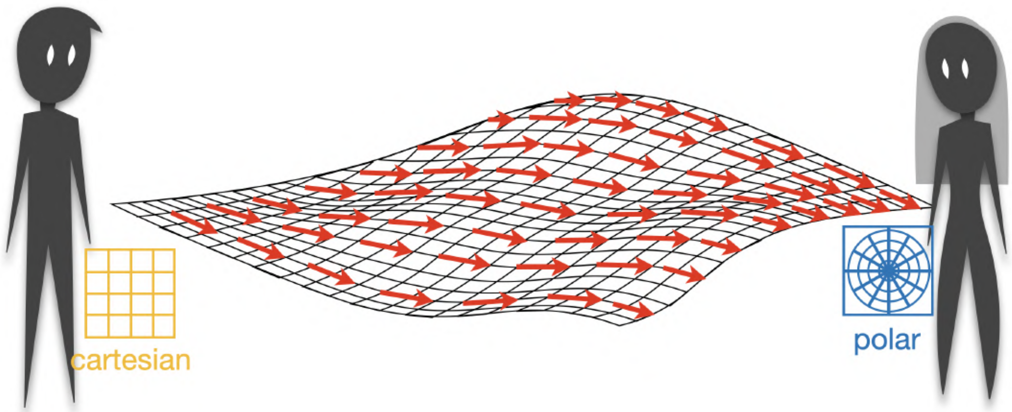
Look at the image above. Can you tell which of these mathematical objects are tensors and which are tensor impostors?

Before answering this question we need to define what a tensor is, and what we mean by "tensor impostors".

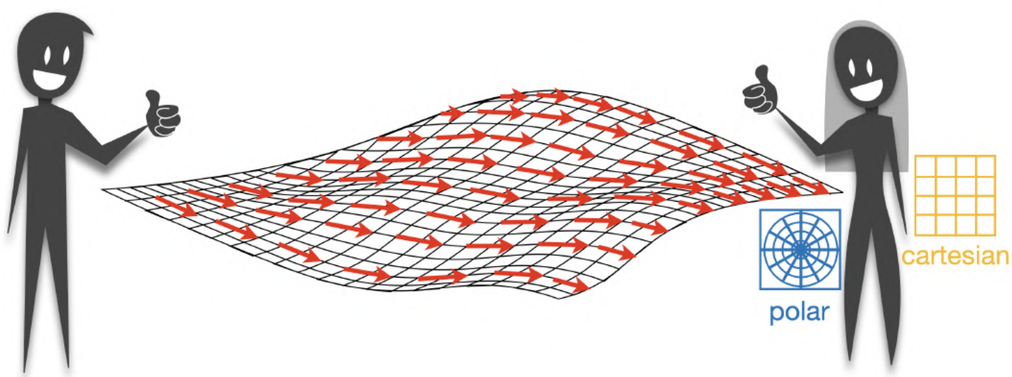
A tensor is just a mathematical object that lives on a space (like space-time or a curved surface, for example) and has a rule that tells it how to stay the same when you change the way you're looking at the space, i.e. when you change coordinates.



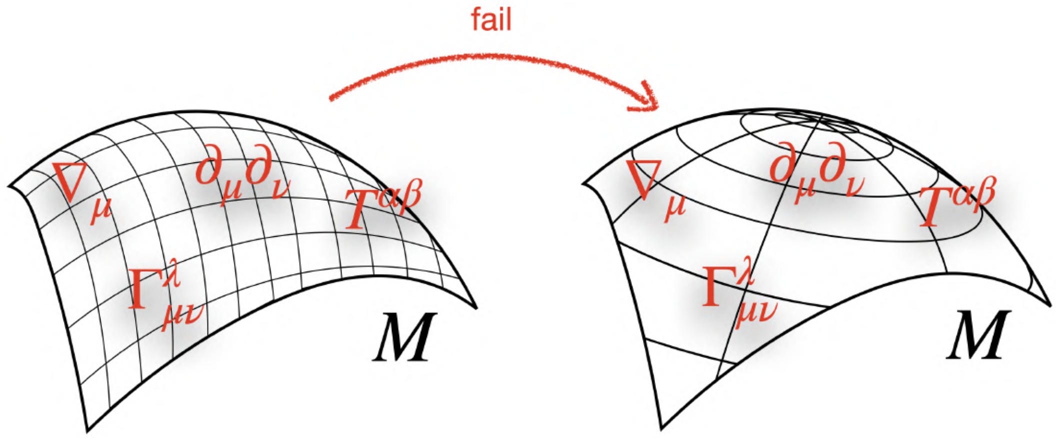
Think of a tensor as something whose values may change, but its meaning doesn't, no matter which coordinate system you're using. Even more concretely, say you're measuring something physical (like a force, a flow, or curvature) and hand your results to someone who uses a different map of the world (different axes, different rules).



If they recalculate your values using the correct transformation rule, they will get the same physical results from their own perspective.



A *tensor impostor* is a mathematical object that really looks like a tensor, especially because of the way it's written, with subscripts and superscripts, but when it comes to transforming it from one coordinate system to another, it fails to be consistent.

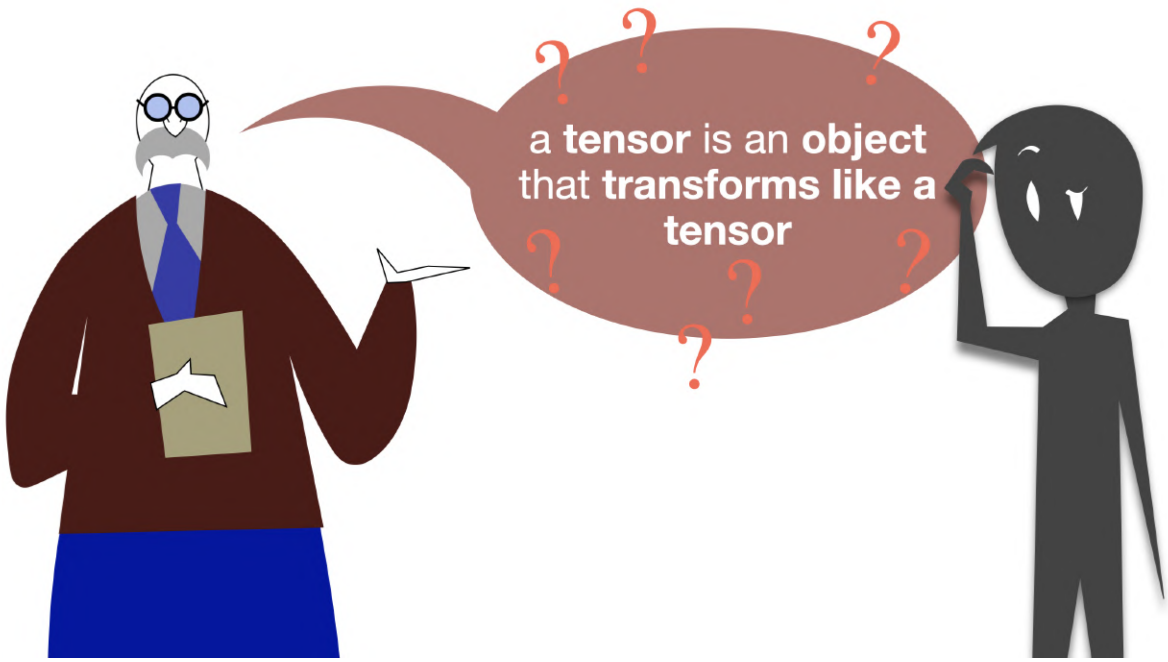


That's why they're impostors: their values are coordinate-dependent in a way that breaks the idea of being "the same in all coordinate systems".

Tensor Transformation Law

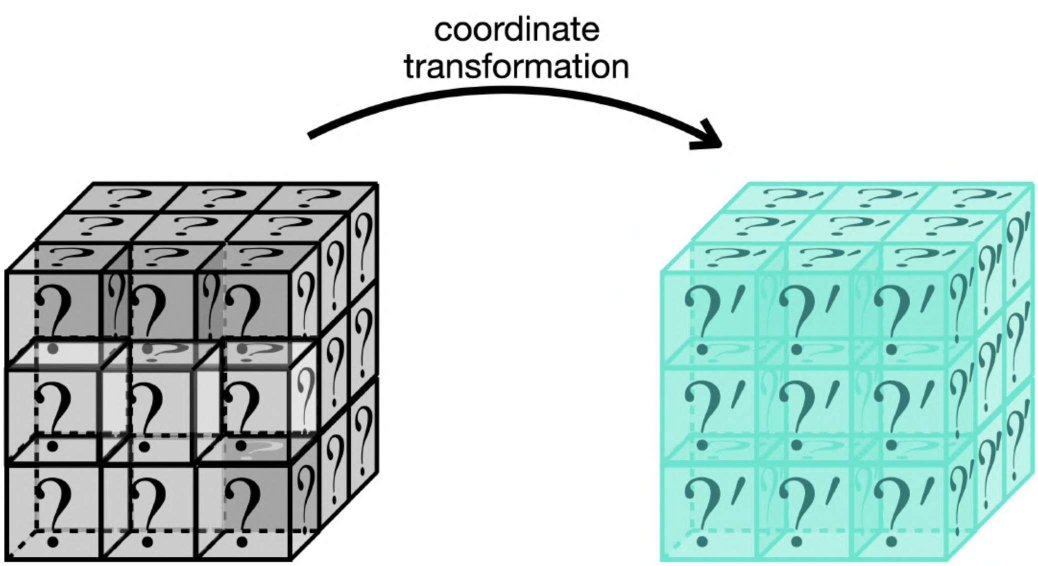
You'll often hear people saying: *"a tensor is an object that transforms like a tensor*". I mean, this phrase is not very useful...

I remember the very first time I heard a professor at my university saying that, and it got me more confused than before. The problem is that they are defining a tensor using the notion of *tensor transformation*, which is weird because if the student doesn't know what a tensor is (I mean, that's why you are trying to define it, right!?), then they probably don't know what a "tensor transformation" is as well.

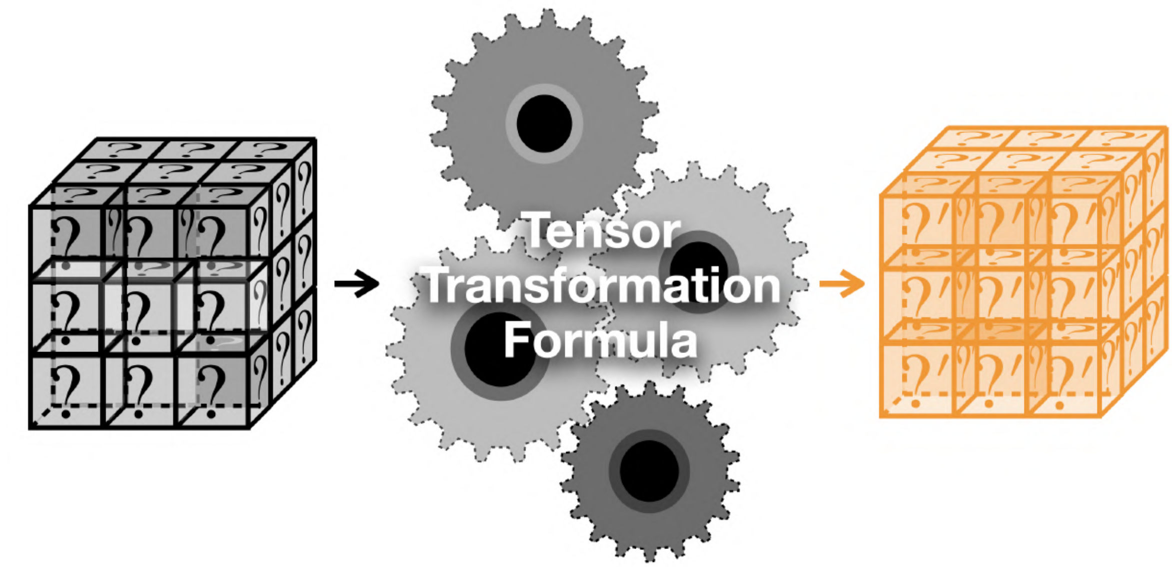


So, what do people mean by this unfortunate sentence, after all? Well, let's see a concrete example of a tensor transformation.

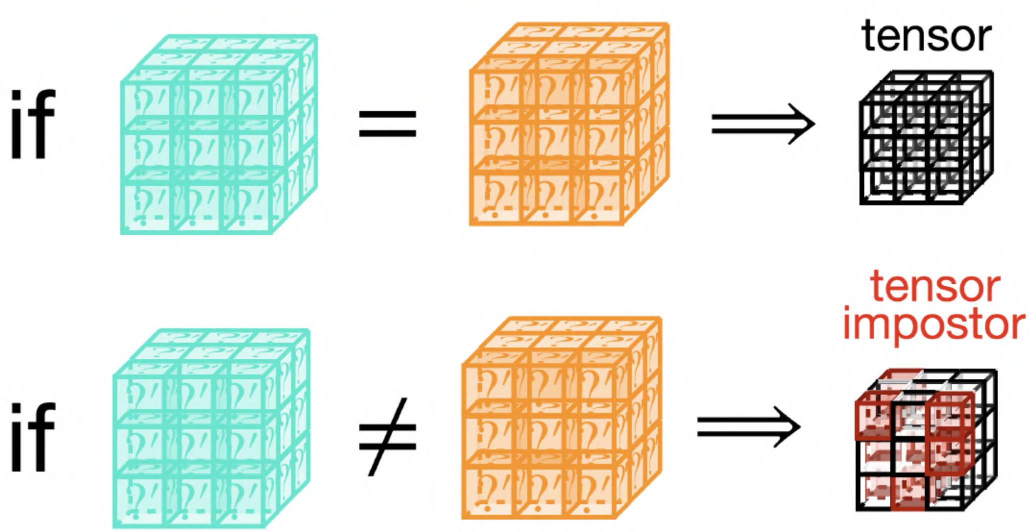
We want to pick a candidate for tensor, write it down in local coordinates, and transform it to a new coordinate system.



Then we take the original candidate once again, transform it using the tensor transformation formula that we will see in a moment, and finally compare the two results that we got from two different methods.

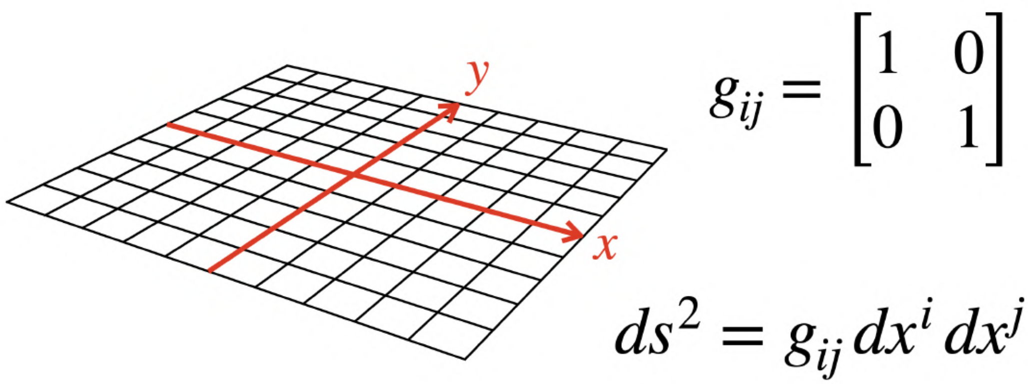


If they match, then we have a legitimate tensor, otherwise we just detected a tensor impostor! In other words, this is a coordinate-dependent object (not intrinsic, not fundamental).



The Metric Tensor

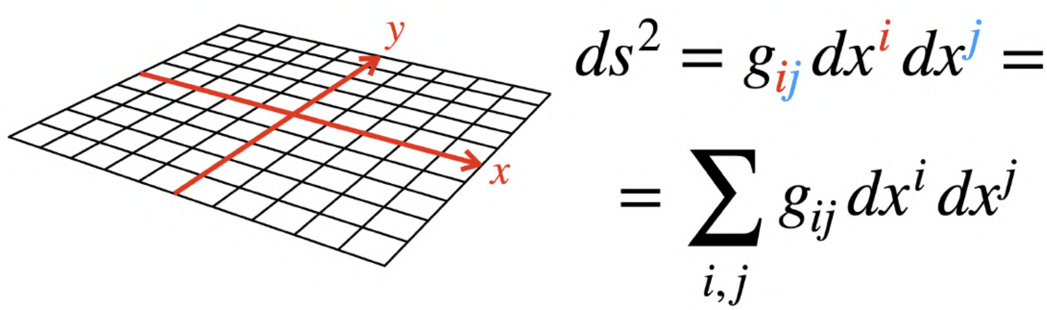
Let's pick a simple flat 2D space with Euclidean metric:



We can also rewrite it in the form

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

using the summation convention for matching upper and lower indices.



It's important to notice, as well, that in the matrix below we name the components in the following way:

$$g_{ij} = \begin{bmatrix} \underset{}{\overset{g_{11}}{\overset{\shortparallel}{\textcircled{1}}}} & \overset{g_{12}}{\overset{\shortparallel}{\textcircled{0}}} \\ \underset{g_{21}}{\underset{\shortparallel}{\textcircled{0}}} & \underset{g_{22}}{\underset{\shortparallel}{\textcircled{1}}} \end{bmatrix}$$

Thus applying the summation convention for these components, we get:

$$ds^2 = \overbrace{g_{11}}^{1}\, d\overbrace{x^1}^{x}\, d\overbrace{x^1}^{x} + \cancel{\overbrace{g_{12}}^{0}}\, d\overbrace{x^1}^{x}\, d\overbrace{x^2}^{y} +$$

$$+\cancel{\overbrace{g_{21}}^{0}}\, d\overbrace{x^2}^{y}\, d\overbrace{x^1}^{x} + \overbrace{g_{22}}^{1}\, d\overbrace{x^2}^{y}\, d\overbrace{x^2}^{y} =$$

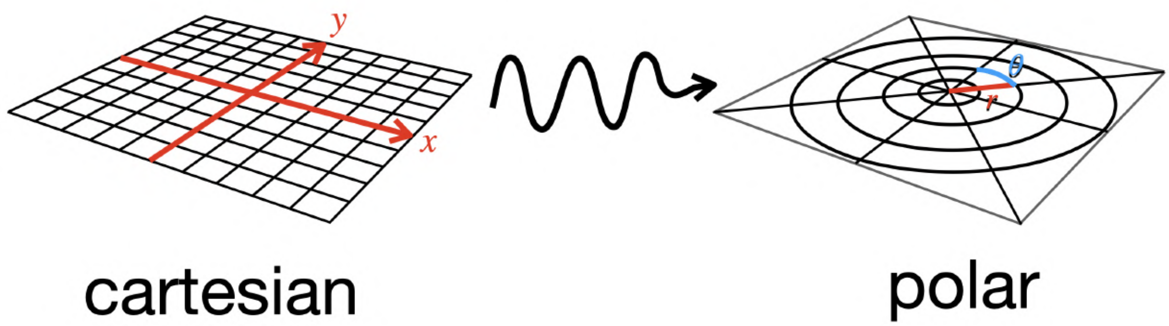
$g_{11} = 1$	$g_{21} = 0$
$g_{12} = 0$	$g_{22} = 1$

$x^1 = x$	$x^2 = y$

$$= dx\, dx + dy\, dy \implies \boxed{ds^2 = dx^2 + dy^2}$$

(Pythagorean Theorem)

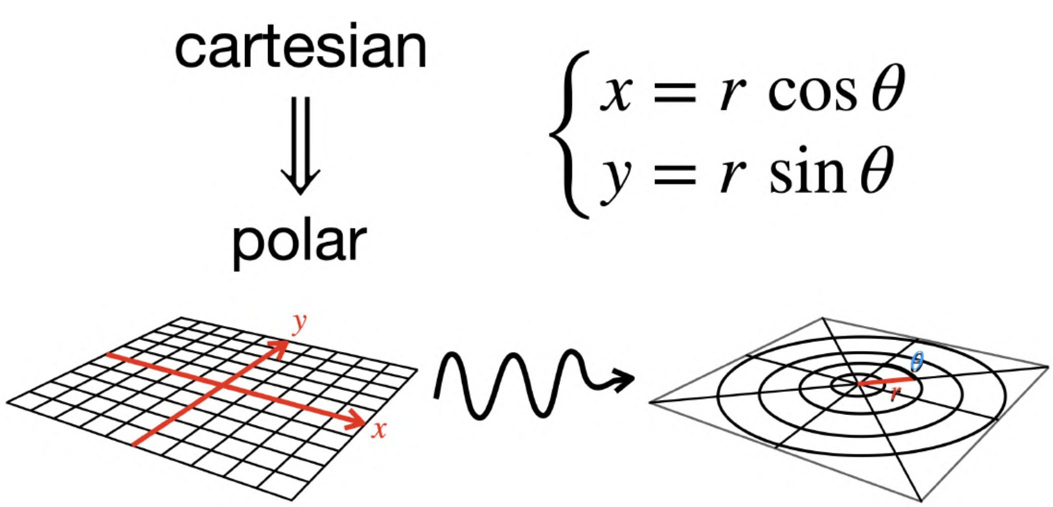
Nice! Now, we want to see the problem from a different perspective, i.e. we want to perform a coordinate transformation, from Cartesian to polar coordinates.



The formulas that translate the coordinates $(x\,,y)$ into $(r\,,\theta)$ are:

$$(*)\begin{cases} x = r\ \cos\theta \\ y = r\ \sin\theta \end{cases}$$

Notice that the space is still the same. The only thing that changed was the grid we picked to measure things in it. The most fundamental characteristics about this flat sheet cannot change in the transformation. We usually call these unchanged characteristics: **intrinsic properties**.



A good definition of intrinsic property is: any geometric object, or quantity, that can be derived from the metric (including the metric itself). In simple terms, the metric, or the "ruler" we use to measure distances, is the most fundamental geometric feature of a space.

Going back to our metric $\boxed{ds^2 = dx^2 + dy^2}$, we can use the substitutions $(*)$:

$$ds'^2 = \left[d\left(r\cos\theta\right)\right]^2 + \left[d\left(r\sin\theta\right)\right]^2$$

Ok... but what do we do with these guys ($d\left(r\cos\theta\right)$ and $d\left(r\sin\theta\right)$)!?

Well, these are **differential** 1**-forms**, and they can be expressed in the new local (polar) coordinates using the chain rule of *Multivariable Calculus*:

$$d\left(r\cos\theta\right) = \frac{\partial}{\partial r}\left(r\cos\theta\right)\,dr + \frac{\partial}{\partial\theta}\left(r\cos\theta\right)\,d\theta$$

$$d\left(r\sin\theta\right) = \frac{\partial}{\partial r}\left(r\sin\theta\right)\,dr + \frac{\partial}{\partial\theta}\left(r\sin\theta\right)\,d\theta$$

After working on all the derivatives and simplifications here, we find the following:

$$ds'^2 = \left[\frac{\partial}{\partial r}\left(r\cos\theta\right)\,dr + \frac{\partial}{\partial\theta}\left(r\cos\theta\right)\,d\theta\right]^2 +$$

$$+\left[\frac{\partial}{\partial r}\left(r\sin\theta\right)\,dr + \frac{\partial}{\partial\theta}\left(r\sin\theta\right)\,d\theta\right]^2 =$$

$$= \left[\cos\theta\,dr - r\sin\theta\,d\theta\right]^2 + \left[\sin\theta\,dr + r\cos\theta\,d\theta\right]^2 =$$

$$= \cos^2\theta\,dr^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\sin^2\theta\,d\theta^2 +$$

$$+\sin^2\theta\, dr^2 + 2r\ \sin\theta\cos\theta\, dr\, d\theta + r^2\ \cos^2\theta\, d\theta^2 =$$

$$= \left(\cos^2\theta + \sin^2\theta\right)\ dr^2 + r^2\left(\sin^2\theta + \cos^2\theta\right) d\theta^2 =$$

$$= dr^2 + r^2 d\theta \quad \Longrightarrow \quad \boxed{ds'^{\,2} = dr^2 + r^2 d\theta}$$

Or, written in matrix form:

$$g'_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

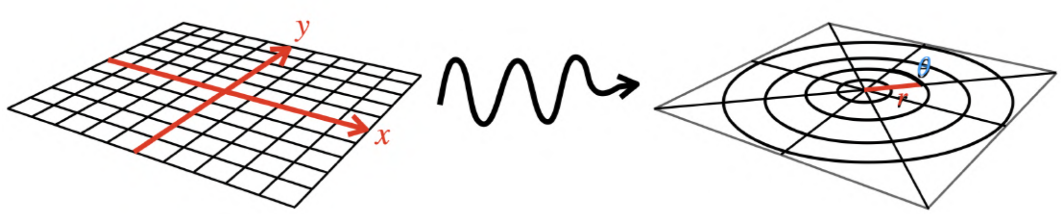
Suggestion: *If you guys find the concept of differential forms confusing, check out the video and PDF below, where we break down the topic in very simple terms:*

The Core of Differential Forms

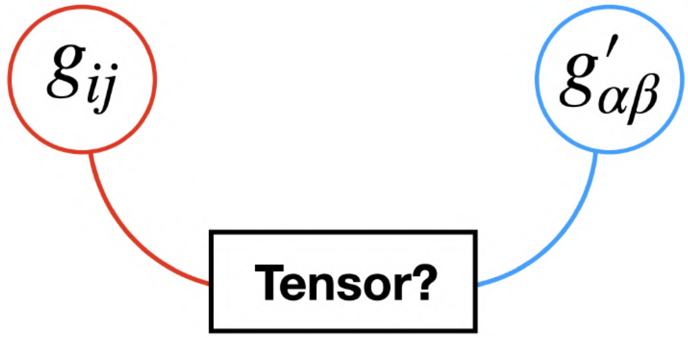
PDF link: Differential Forms

Back to our example, we basically performed a coordinate transformation to express the same (identical) metric in two different local coordinate systems:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad g'_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

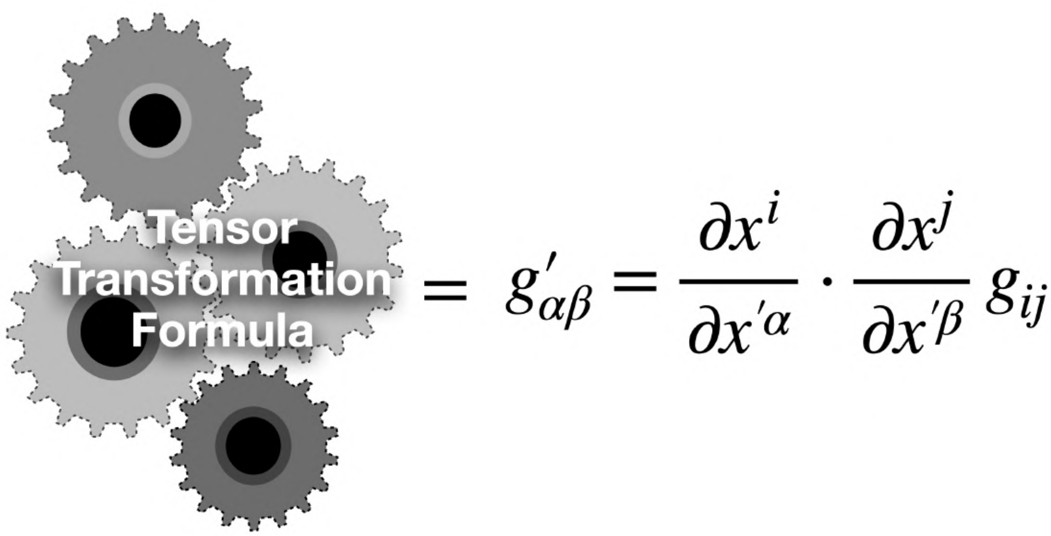


But the question remains: is it a legitimate tensor?



And the answer is given by the formula for a legitimate tensor transformation:

$$g'_{\alpha\beta} = \frac{\partial x^i}{\partial x'^\alpha} \cdot \frac{\partial x^j}{\partial x'^\beta} g_{ij}$$



Failing to satisfy this equation would automatically disqualify an object as a candidate for tensor.

Let's analyse each term:

$$g'_{\alpha\beta} = \frac{\partial x^i}{\partial x'^\alpha} \cdot \frac{\partial x^j}{\partial x'^\beta} g_{ij}$$

g_{ij} is our original metric, before the coordinate transformation. In our case

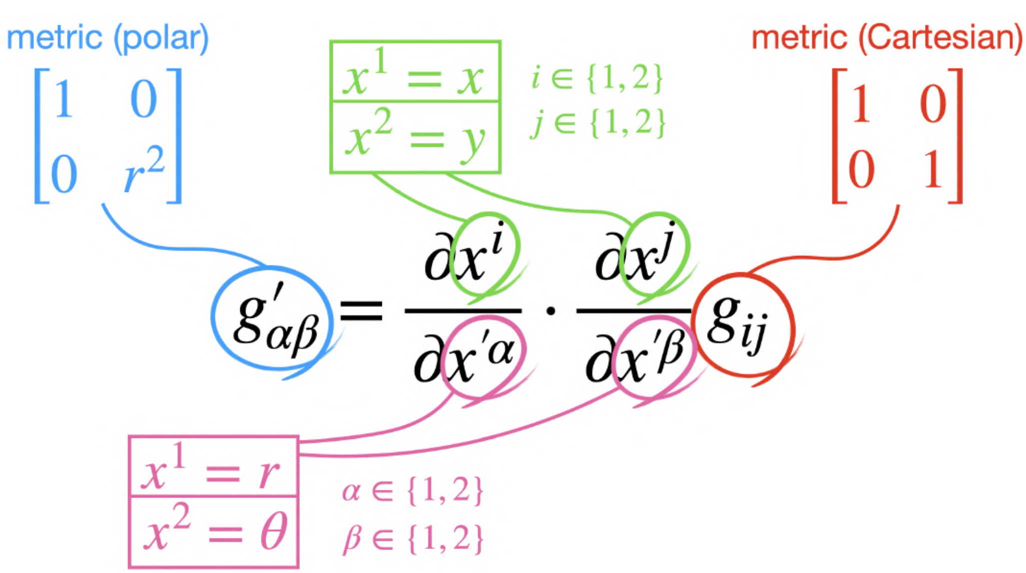
$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$g'_{\alpha\beta}$ is the same metric, but after the transformation:

$$g'_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

x^i are the original coordinates $x^1 = x$ and $x^2 = y$ (for $i \in \{1,2\}$). Same thing for x^j (with $j \in \{1,2\}$).

x'^α and x'^β are the new coordinates $x'^1 = r$ and $x'^2 = \theta$ (for $\alpha, \beta \in \{1,2\}$).



The expectation is that, after using this tensor transformation formula for the original metric g_{ij}, we get as an output the matrix $g'_{\alpha\beta}$ that we found before. If this fails to happen, then we just detected a tensor impostor!

Let's try to prove that:

$$g'_{\alpha\beta} = \frac{\partial x^i}{\partial x'^\alpha} \cdot \frac{\partial x^j}{\partial x'^\beta} g_{ij} = \frac{\partial x^1}{\partial x'^\alpha} \cdot \frac{\partial x^j}{\partial x'^\beta} g_{1j} + \frac{\partial x^2}{\partial x'^\alpha} \cdot \frac{\partial x^j}{\partial x'^\beta} g_{2j} =$$

$$= \frac{\partial x^1}{\partial x'^\alpha} \cdot \frac{\partial x^1}{\partial x'^\beta} g_{11} + \frac{\partial x^1}{\partial x'^\alpha} \cdot \frac{\partial x^2}{\partial x'^\beta} g_{12} + \frac{\partial x^2}{\partial x'^\alpha} \cdot \frac{\partial x^1}{\partial x'^\beta} g_{21} + \frac{\partial x^2}{\partial x'^\alpha} \cdot \frac{\partial x^2}{\partial x'^\beta} g_{22} =$$

$$= \frac{\partial x}{\partial x'^\alpha} \cdot \frac{\partial x}{\partial x'^\beta} \cdot 1 + \cancel{\frac{\partial x}{\partial x'^\alpha} \cdot \frac{\partial y}{\partial x'^\beta}} \cdot 0 + \cancel{\frac{\partial y}{\partial x'^\alpha} \cdot \frac{\partial x}{\partial x'^\beta}} \cdot 0 + \frac{\partial y}{\partial x'^\alpha} \cdot \frac{\partial y}{\partial x'^\beta} \cdot 1 =$$

$$= \frac{\partial x}{\partial x'^\alpha} \cdot \frac{\partial x}{\partial x'^\beta} + \frac{\partial y}{\partial x'^\alpha} \cdot \frac{\partial y}{\partial x'^\beta}$$

Here, $x'^\alpha,\ x'^\beta \in \{r, \theta\}$.

There are 4 entries in this matrix:

$$g'_{\alpha\beta} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

<u>For $x'^\alpha = x'^\beta = r$</u>: $\quad (\alpha = \beta = 1)$

$$g'_{11} = \frac{\partial x}{\partial r} \cdot \frac{\partial x}{\partial r} + \frac{\partial y}{\partial r} \cdot \frac{\partial y}{\partial r} = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 =$$

$$= \left[\frac{\partial}{\partial r}(r\cos\theta)\right]^2 + \left[\frac{\partial}{\partial r}(r\sin\theta)\right]^2 = [\cos\theta]^2 + [\sin\theta]^2 =$$

$$= \cos^2\theta + \sin^2\theta = 1$$

<u>For $x'^{\alpha} = r$ and $x'^{\beta} = \theta$</u>: $(\alpha = 1 \text{ and } \beta = 2)$

$$g'_{12} = \frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial r}(r\cos\theta)\cdot\frac{\partial}{\partial \theta}(r\cos\theta) + \frac{\partial}{\partial r}(r\sin\theta)\cdot\frac{\partial}{\partial \theta}(r\sin\theta) =$$

$$= \cos\theta\cdot(-r\sin\theta) + \sin\theta\cdot r\cos\theta = 0$$

<u>For $x'^{\alpha} = \theta$ and $x'^{\beta} = r$</u>: $(\alpha = 2 \text{ and } \beta = 1)$

$$g'_{21} = \frac{\partial x}{\partial \theta}\cdot\frac{\partial x}{\partial r} + \frac{\partial y}{\partial \theta}\cdot\frac{\partial y}{\partial r} = -r\sin\theta\cos\theta + r\sin\theta\cos\theta = 0$$

<u>For $x'^{\alpha} = \theta$ and $x'^{\beta} = \theta$</u>: $(\alpha = 2 \text{ and } \beta = 2)$

$$g'_{22} = \frac{\partial x}{\partial \theta}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\cdot\frac{\partial y}{\partial \theta} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 =$$

$$= \left[\frac{\partial}{\partial \theta}(r\cos\theta)\right]^2 + \left[\frac{\partial}{\partial \theta}(r\sin\theta)\right]^2 = r^2\sin^2\theta + r^2\cos^2\theta =$$

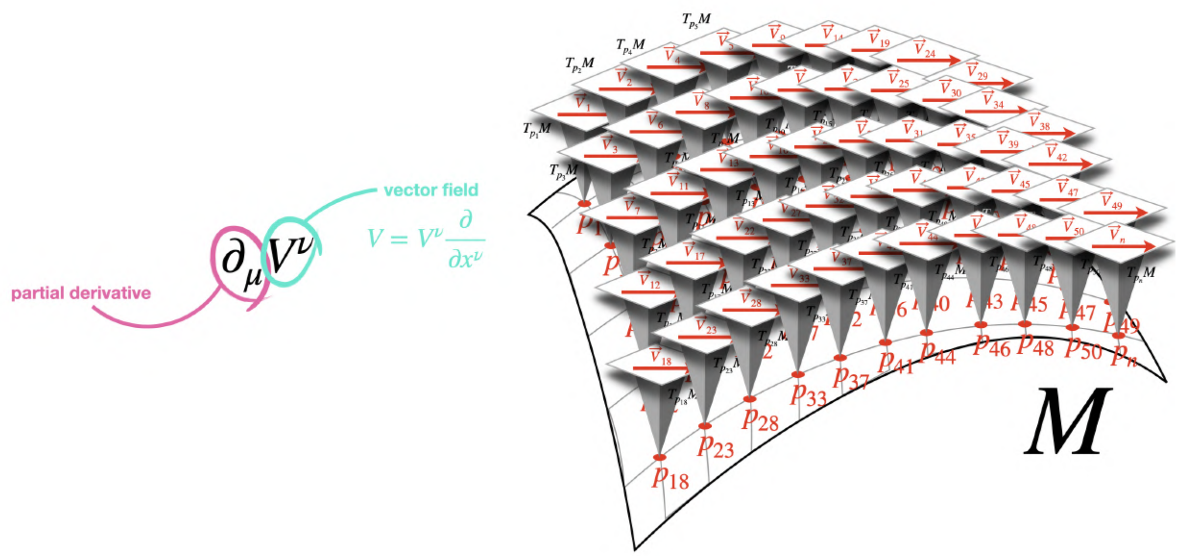
$$= r^2\left(\sin^2\theta + \cos^2\theta\right) = r^2$$

$$\therefore\ g'_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

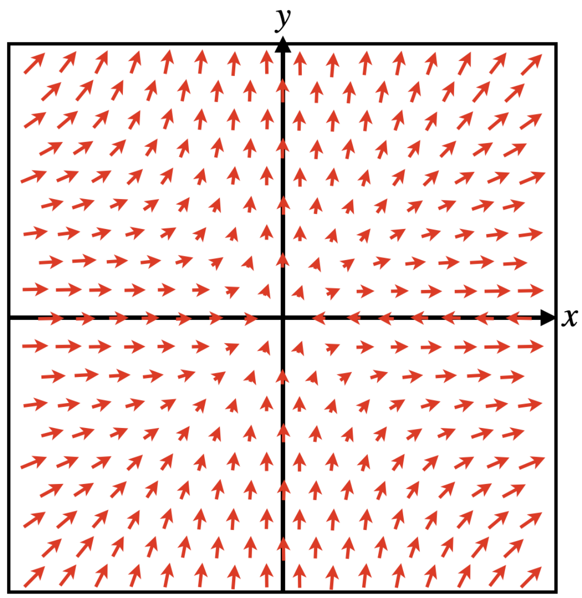
Therefore, the metric tensor is indeed a **legitimate tensor**!

The Partial Derivative of a Vector Field

Let's try the same (identical) thing with the object $\partial_\mu V^\nu$, which is just the partial derivative of a vector field $\vec{V} = V^\nu \frac{\partial}{\partial x^\nu}$, i.e. the rate of change of the components of a vector field defined throughout the manifold, at each point, and living in the tangent space $T_p M$, for each point $p \in M$.



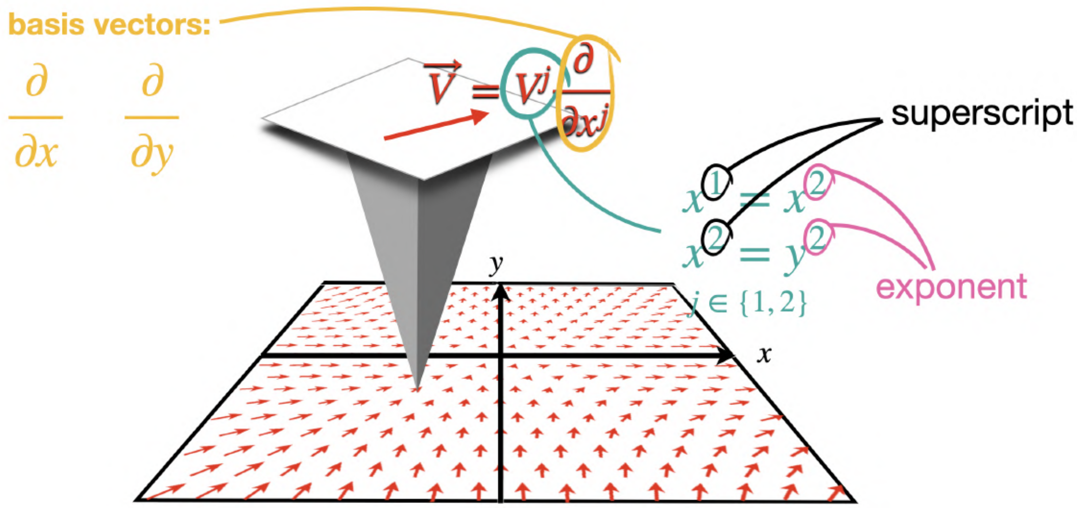
$$\overrightarrow{V} = \begin{bmatrix} V^1(x,y) \\ V^2(x,y) \end{bmatrix} = \underbrace{V^1(x,y)}_{\overset{\|}{x^2}} \frac{\partial}{\partial x} + \underbrace{V^2(x,y)}_{\overset{\|}{y^2}} \frac{\partial}{\partial y}$$



This is a vector field defined on a $2D$ flat plane in Cartesian coordinates (x, y). To make things less abstract, we picked the specific case in which the vector field is:

$$\vec{V} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$$

Since each of these vectors actually live in the tangent spaces at each point, they can be written as $\vec{V} = V^j \frac{\partial}{\partial x^j}$, where V^j are the components (in this case either $x^1 = x^2$ (x squared in the right hand side) or $x^2 = y^2$ (y squared in the right hand side), for $j \in \{1, 2\}$), and $\frac{\partial}{\partial x^j}$ are the basis vectors of each tangent space ($x^j \in \{x, y\}$) written in local (Cartesian) coordinates (x, y).



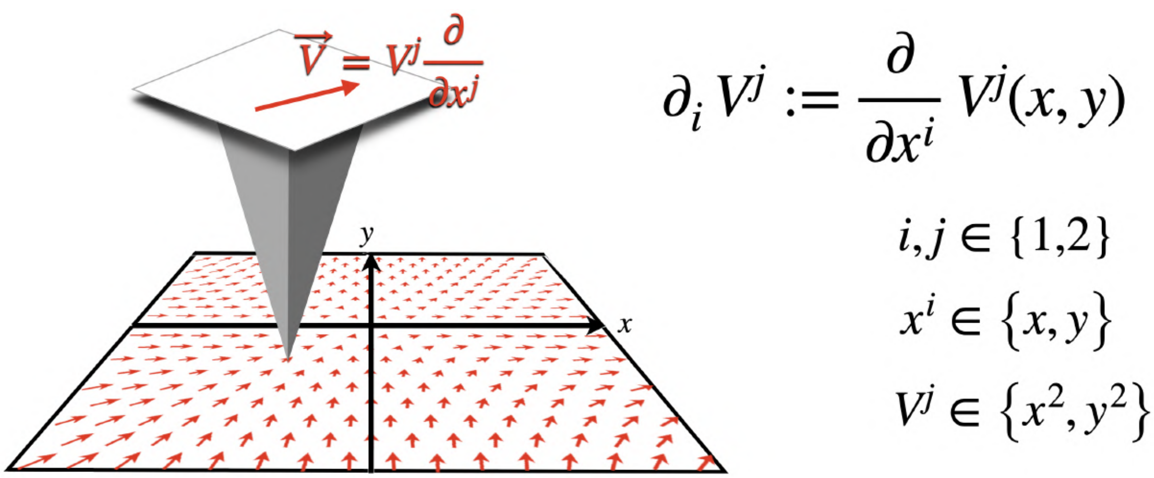
Now, we can take the partial derivative of the vector field, component by component:

$$\boxed{\partial_i V^j := \frac{\partial}{\partial x^i} V^j(x, y)}$$

$$i, j \in \{1, 2\}$$

$$x^i \in \{x, y\}$$

$$V^j \in \{x^2, y^2\} \text{ (} x \text{ squared and } y \text{ squared)}$$



For $i = j = 1$:

$$\partial_1 V^1 = \frac{\partial}{\partial x^1} V^1(x, y) = \frac{d}{dx}(x^2) = 2x$$

For $i = 1$ and $j = 2$:

$$\partial_1 V^2 = \frac{\partial}{\partial x^1} V^2(x, y) = \frac{\partial}{\partial x}(y^2) = 0$$

For $i = 2$ and $j = 1$:

$$\partial_2 V^1 = \frac{\partial}{\partial x^2} V^1(x, y) = \frac{\partial}{\partial y}(x^2) = 0$$

For $i = j = 2$:

$$\partial_2 V^2 = \frac{\partial}{\partial x^2} V^2(x, y) = \frac{d}{dy}(y^2) = 2y$$

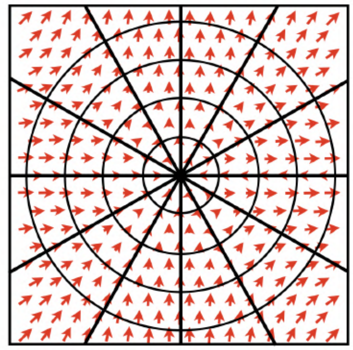
$$\therefore\ \partial_i V^j = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

(Cartesian coordinates)

Good! Now, let's transform it to polar coordinates.

First, we need to rewrite the vector field in polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



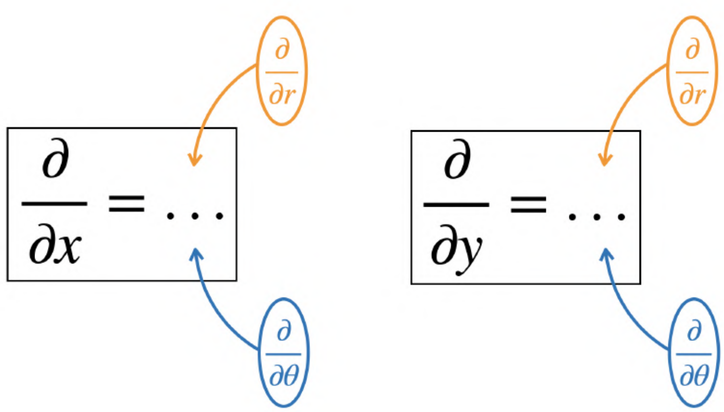
$$\vec{V} = \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = V^j \frac{\partial}{\partial x^j} \rightsquigarrow \vec{V}' = \begin{bmatrix} V'^1 \\ V'^2 \end{bmatrix} = V'^\beta \frac{\partial}{\partial x'^\beta}$$

cartesian polar

This is not an easy task, because not only the components V^j and V'^β are written in different coordinate systems, but the basis of the vectors $\vec{V}$ and $\vec{V}'$ are also written in different coordinates:

$$\frac{\partial}{\partial x^j} \neq \frac{\partial}{\partial x'^\beta}$$

And as we know from *Linear Algebra*, when we change coordinates, both the components and the basis vectors transform.
So, let's find a way to express the Cartesian basis $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in terms of polar basis $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$:



$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \cdot \frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \quad \Longrightarrow$$

$$\Longrightarrow \boxed{\frac{\partial}{\partial x} = \frac{1}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{\cos\theta} \cdot \frac{\partial}{\partial y}} \qquad (\mathrm{I})$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \cdot \frac{\partial}{\partial y} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y} \ (\mathrm{I})$$

$$(\mathrm{I})\ \frac{\partial}{\partial \theta} = -r\sin\theta \cdot \left[\frac{1}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{\cos\theta} \cdot \frac{\partial}{\partial y}\right] + r\cos\theta \frac{\partial}{\partial y} =$$

$$= -\frac{r\sin\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} + \frac{r\sin^2\theta}{\cos\theta} \cdot \frac{\partial}{\partial y} + r\cos\theta \frac{\partial}{\partial y} =$$

$$= -\frac{r\sin\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} + \left(\frac{r\sin^2\theta + r\cos^2\theta}{\cos\theta}\right) \frac{\partial}{\partial y} =$$

$$= -\frac{r\sin\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} + \frac{r}{\cos\theta} \cdot \frac{\partial}{\partial y} \quad \Longrightarrow$$

$$\Longrightarrow \boxed{\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}} \qquad (\mathrm{II})$$

<u>(II) in (I)</u>:

$$\frac{\partial}{\partial x} = \frac{1}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{\cos\theta} \cdot \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial \theta}\right) =$$

$$= \frac{1}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin^2\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta} =$$

$$= \frac{1 - \sin^2\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta} = \frac{\cos^{2}\theta}{\cos\theta} \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta} \quad \Longrightarrow$$

$$\Longrightarrow \boxed{\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}}$$

$$\therefore \quad \boxed{\frac{\partial}{\partial x} = \cos\theta \, \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial}{\partial y} = \sin\theta \, \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}}$$

At this point, we can finally write $\vec{V}$ (Cartesian) as $\vec{V}'$ (polar):

$$\vec{V}' = \vec{V} = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} = (r\cos\theta)^2 \frac{\partial}{\partial x} + (r\sin\theta)^2 \frac{\partial}{\partial y} =$$

$$= r^2 \cos^2\theta \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta} \right) + r^2 \sin^2\theta \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial \theta} \right) =$$

$$= r^2 \cos^3\theta \frac{\partial}{\partial r} - r\cos^2\theta \, \sin\theta \frac{\partial}{\partial \theta} + r^2 \sin^3\theta \frac{\partial}{\partial r} + r\sin^2\theta \, \cos\theta \frac{\partial}{\partial \theta} =$$

$$= r^2 \left(\sin^3\theta + \cos^3\theta \right) \frac{\partial}{\partial r} + r\sin\theta \, \cos\theta \, (\sin\theta - \cos\theta) \frac{\partial}{\partial \theta} \implies$$

$$\implies \quad \vec{V}' = \begin{bmatrix} r^2 \left(\sin^3\theta + \cos^3\theta \right) \\ r \, \sin\theta \, \cos\theta \, \left(\sin\theta - \cos\theta \right) \end{bmatrix}$$

Now, take the partial derivative of the components of the vector in order to get our candidate for tensor:

$$\boxed{\left(\partial_\alpha V^\beta \right)' = \frac{\partial}{\partial x'^\alpha} \, V'^\beta (r, \theta)} \qquad \begin{array}{r} \alpha, \beta \in \{1, 2\} \\ x^{\alpha'} \in \{1, 2\} \end{array}$$

polar coordinates

For $\alpha = \beta = 1$:

$$\left(\partial_1 V^1\right)' = \frac{\partial}{\partial x'^1} V'^1(r,\theta) = \frac{\partial}{\partial r} \left(r^2 \sin^3\theta + r^2 \cos^3\theta\right) =$$

$$= 2r \left(\sin^3\theta + \cos^3\theta\right)$$

For $\alpha = 1$ and $\beta = 2$:

$$\left(\partial_1 V^2\right)' = \frac{\partial}{\partial x'^1} V'^2(r,\theta) = \frac{\partial}{\partial r} \left[r \sin\theta \, \cos\theta \, (\sin\theta - \cos\theta)\right] =$$

$$= \sin\theta \, \cos\theta \, (\sin\theta - \cos\theta)$$

For $\alpha = 2$ and $\beta = 1$:

$$\left(\partial_2 V^1\right)' = \frac{\partial}{\partial x'^2} V'^1(r,\theta) = \frac{\partial}{\partial \theta} \left(r^2 \sin^3\theta + r^2 \cos^3\theta\right) =$$

$$= 3r^2 \sin^2\theta \, \cos\theta - 3r^2 \cos^2\theta \, \sin\theta =$$

$$= 3r^2 \sin\theta \, \cos\theta \, (\sin\theta - \cos\theta)$$

For $\alpha = \beta = 2$:

$$\left(\partial_2 V^2\right)' = \frac{\partial}{\partial x'^2} V'^2(r,\theta) = \frac{\partial}{\partial \theta} \left[r \sin\theta \, \cos\theta \, (\sin\theta - \cos\theta)\right] =$$

$$= \frac{\partial}{\partial \theta} \left(r \sin^2\theta \, \cos\theta - r \sin\theta \, \cos^2\theta\right) =$$

$$= r \frac{\partial}{\partial \theta} \left(\sin^2\theta \cdot \cos\theta\right) - r \frac{\partial}{\partial \theta} \left(\sin\theta \cdot \cos^2\theta\right) =$$

$$= r\ \left(2\sin\theta\ \cos^2\theta - \sin^3\theta\right) - r\ \left(\cos^3\theta - 2\sin^2\theta\ \cos\theta\right) =$$

$$= 2r\ \sin\theta\cos^2\theta - r\ \sin^3\theta - r\ \cos^3\theta + 2r\ \sin^2\theta\cos\theta =$$

$$= r\ (\sin\theta + \cos\theta)\cdot(3\sin\theta\ \cos\theta - 1)$$

Thus we find this matrix describing the object we are studying, but this time written in polar coordinates:

$$\left(\partial_\alpha\, V^\beta\right)' = \begin{bmatrix} 2r\ \left(\sin^3\theta + \cos^3\theta\right) & \sin\theta\ \cos\theta\,(\sin\theta - \cos\theta) \\ 3r^2\sin\theta\ \cos\theta\ (\sin\theta - \cos\theta) & r\ (\sin\theta + \cos\theta)\cdot(3\sin\theta\ \cos\theta - 1) \end{bmatrix}$$

(Polar coordinates)

Nice! Now, the question is: **is it consistent with the tensor transformation rule?** We need to check it out:

$$\left(\partial_\alpha\, V^\beta\right)' = \frac{\partial x^i}{\partial x'^\alpha}\cdot\frac{\partial x'^\beta}{\partial x^j}\cdot\partial_i\, V^j$$

The vector field and its partial derivatives are:

$$V^j(x,y) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \quad \text{and} \quad \partial_i V^j = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

We'll label the matrix components:

$$(\partial_i V^j) = \begin{bmatrix} \partial_1 V^1 & \partial_1 V^2 \\ \partial_2 V^1 & \partial_2 V^2 \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

From the transformation:

$$x = r\cos\theta, \quad y = r\sin\theta$$

We compute:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta & \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial r} &= \sin\theta & \frac{\partial y}{\partial \theta} &= r\cos\theta \end{aligned}$$

So:

$$\left(\frac{\partial x^i}{\partial x'^\alpha}\right) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

We invert the transformation:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

Compute:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta \\ \frac{\partial \theta}{\partial x} &= \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r} \end{aligned}$$

So:

$$\left(\frac{\partial x'^\beta}{\partial x^j}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

We now compute all 4 entries using:

$$(\partial_\alpha V^\beta)' = \frac{\partial x^i}{\partial x'^\alpha} \cdot \frac{\partial x'^\beta}{\partial x^j} \cdot \partial_i V^j$$

We organize this as a **matrix multiplication**:

$$(\partial_\alpha V^\beta)' = A \cdot (\partial_i V^j) \cdot B$$

Where:

- $A = \left(\frac{\partial x^i}{\partial x'^\alpha}\right)$
- $B = \left(\frac{\partial x'^\beta}{\partial x^j}\right)$
- $\partial_i V^j = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$

So we compute:

$$(\partial_\alpha V^\beta)' = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \cdot \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

Step 1: Multiply $A \cdot \partial_i V^j$

$$A \cdot \partial_i V^j = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \cdot \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix} = \begin{bmatrix} 2x\cos\theta & -2ry\sin\theta \\ 2x\sin\theta & 2ry\cos\theta \end{bmatrix}$$

Step 2: Multiply result with B

$$\begin{bmatrix} 2x\cos\theta & -2ry\sin\theta \\ 2x\sin\theta & 2ry\cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{bmatrix}$$

Compute entries:

(1,1):

$$2x\cos\theta \cdot \cos\theta + (-2ry\sin\theta) \cdot \left(-\frac{\sin\theta}{r}\right) = 2x\cos^2\theta + 2y\sin^2\theta =$$

$$= 2r\cos\theta\cos^2\theta + 2r\sin\theta\sin^2\theta = 2r(\cos^3\theta + \sin^3\theta)$$

(1,2):

$$2x\cos\theta \cdot \sin\theta + (-2ry\sin\theta) \cdot \left(\frac{\cos\theta}{r}\right) = 2x\cos\theta\sin\theta - 2y\sin\theta\cos\theta =$$

$$= 2(x-y)\cos\theta\sin\theta = 2r(\cos\theta-\sin\theta)\cos\theta\sin\theta = 2r\cos\theta\sin\theta(\cos\theta-\sin\theta)$$

(2,1):

$$2x\sin\theta\cdot\cos\theta + 2ry\cos\theta\cdot\left(-\frac{\sin\theta}{r}\right) = 2x\sin\theta\cos\theta - 2y\sin\theta\cos\theta =$$

$$= 2(x-y)\sin\theta\cos\theta = 2r(\cos\theta-\sin\theta)\cos\theta\sin\theta = 2r\cos\theta\sin\theta(\cos\theta-\sin\theta)$$

(2,2):

$$2x\sin\theta\cdot\sin\theta + 2ry\cos\theta\cdot\left(\frac{\cos\theta}{r}\right) = 2x\sin^2\theta + 2y\cos^2\theta =$$

$$= 2r\cos\theta\sin^2\theta + 2r\sin\theta\cos^2\theta = 2r(\cos\theta\sin^2\theta + \sin\theta\cos^2\theta)$$

$$\boxed{(\partial_\alpha V^\beta)' = \begin{bmatrix} 2r(\cos^3\theta+\sin^3\theta) & 2r\cos\theta\sin\theta(\cos\theta-\sin\theta) \\ 2r\cos\theta\sin\theta(\cos\theta-\sin\theta) & 2r(\cos\theta\sin^2\theta+\sin\theta\cos^2\theta) \end{bmatrix}}$$

After a lot of calculations, we found a different matrix from before!

$$\begin{bmatrix} 2r(\cos^3\theta+\sin^3\theta) & 2r\cos\theta\sin\theta(\cos\theta-\sin\theta) \\ 2r\cos\theta\sin\theta(\cos\theta-\sin\theta) & 2r(\cos\theta\sin^2\theta+\sin\theta\cos^2\theta) \end{bmatrix}$$

$$\not\parallel$$

$$\begin{bmatrix} 2r\,\left(\sin^3\theta+\cos^3\theta\right) & \sin\theta\,\cos\theta\,(\sin\theta-\cos\theta) \\ 3r^2\sin\theta\,\cos\theta\,\left(\sin\theta-\cos\theta\right) & r\,\left(\sin\theta+\cos\theta\right)\cdot\left(3\sin\theta\,\cos\theta-1\right) \end{bmatrix}$$

The conclusion is:

$\partial_i V^j$ is coordinate-dependent $\implies$ $\partial_i V^j$ does not obey the tensor transformation rule $\implies$ $\partial_i V^j$ is a **tensor impostor** $\implies$ $\partial_i V^j$ is not intrinsic to the manifold.

Legitimate Tensors *vs* Tensor Impostors

These are tables of some tensor impostors and some legitimate tensors:

Tensor Impostors

Name	Notation
Christoffel symbols	$\Gamma^\lambda_{\mu\nu}$
Partial derivative of a vector	$\partial_\mu V^\nu$
Second derivatives of tensor fields	$\partial_\mu \partial_\nu T^{\alpha\beta}$
Affine connection (general)	∇_μ
Levi-Civita symbol	$\epsilon^{\mu\lambda\rho\sigma}$

Legitimate Tensors

Name	Notation
Metric tensor	$g_{\mu\nu}$
Inverse metric tensor	$g^{\mu\nu}$
Riemann curvature tensor	$R^\rho{}_{\sigma\mu\nu}$
Ricci tensor	$R_{\mu\nu}$
Einstein tensor	$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
Energy-momentum tensor	$T_{\mu\nu}$
Electromagnetic field strength tensor	$F_{\mu\nu}$
Kronecker delta	δ^μ_ν

We highly encourage you guys to use the same method that we've seen here and apply it to prove if some of these mathematical objects are legitimate tensors or tensor impostors. Doing so will **really** help you to master the subject.

If you found this document useful let us know. If you found typos or things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com