How to Build Mathematical Intuition

by DiBeos

"It is by logic that we prove, but by intuition that we discover."
– Henri Poincaré

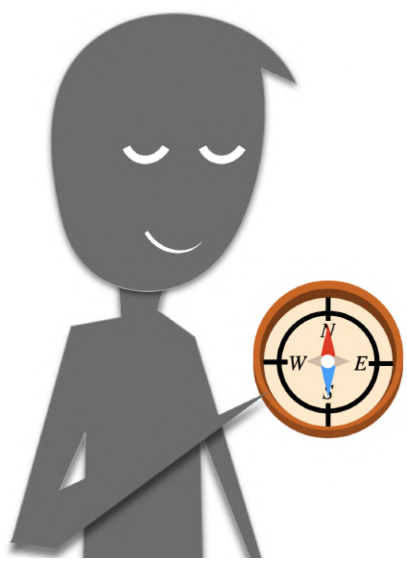
Introduction

Have you ever sat down to study a math book, looked at the theorems, lemmas, corollaries, and could not help but ask yourself: *"Why is this* *particular result so important? It looks so random... What was the original motivation for mathematicians to find these results?"*. You and I know that there's probably a good reason for its existence, but why isn't that reason obvious?!

The missing ingredient is simple, it's **intuition**.

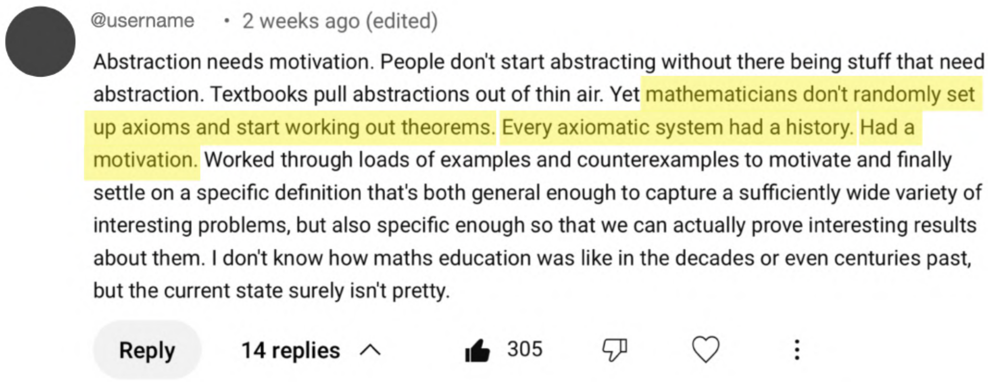
Intuition in mathematics is something like having a compass. It's not a replacement for technique and rigor, which are more equivalent to a complicated advanced GPS system that will get you out from anywhere, *but*, you won't have the slightest idea of what the symbols on the GPS mean. So, you have to start with a compass, with intuition, which gives you the right sense of direction that you need to be taking.

Or in other words, it gives you the reason you are seeing the math you're presented. And that makes everything easier. Learning the subject becomes easier, memorizing the results becomes easier, even developing creativity on how to prove them becomes easier.



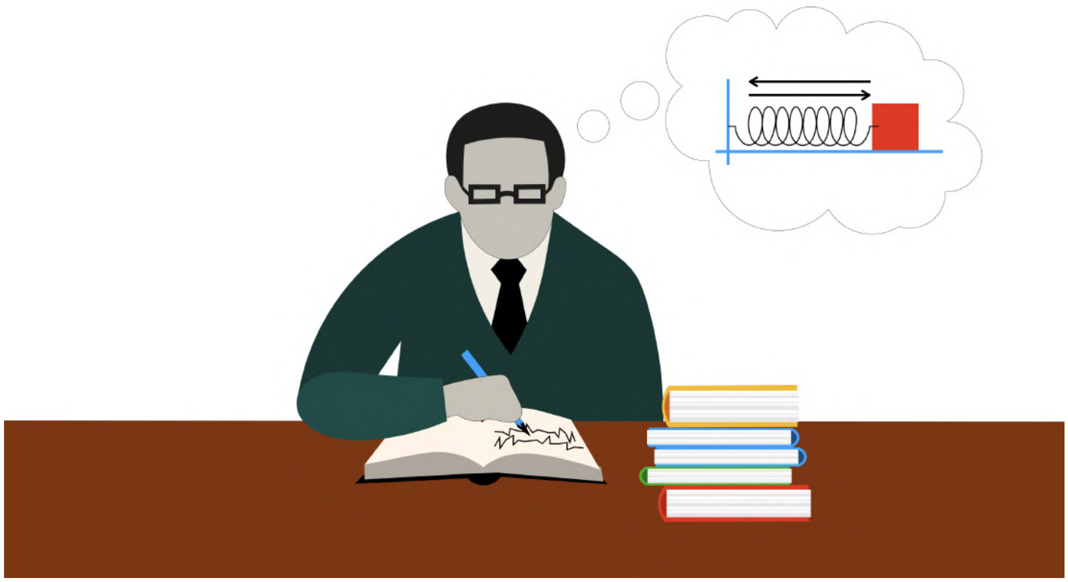
The Motive

A person once commented in one of our YouTube videos something that really stuck with me: that **abstraction needs motivation**.



Mathematicians don't randomly set up axioms and start working out theorems.

Every axiomatic system has a history. Had a motivation. First mathematicians come up with a picture on how parts of some problem fit together, with a clear intuitive idea in mind.

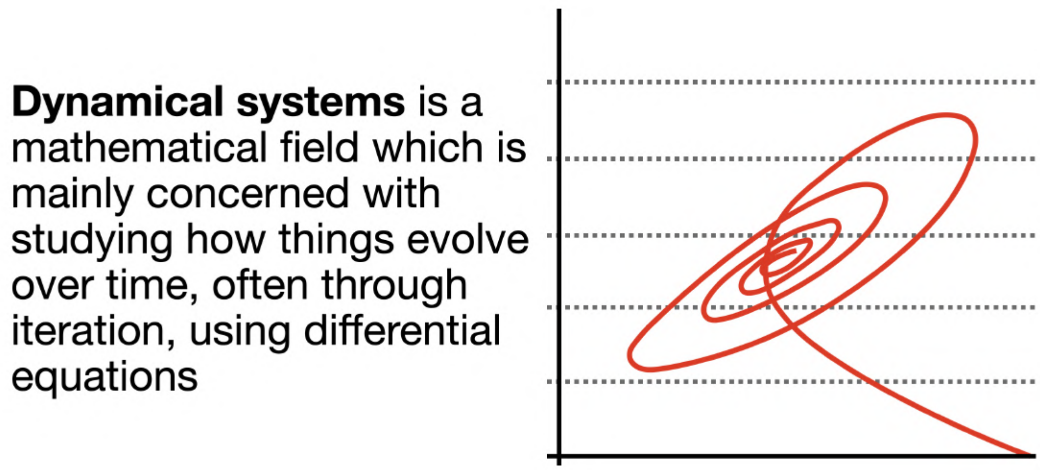


They don't just spew well worded theorems out of thin air, they come to a more rigorous idea through intuitive thinking.

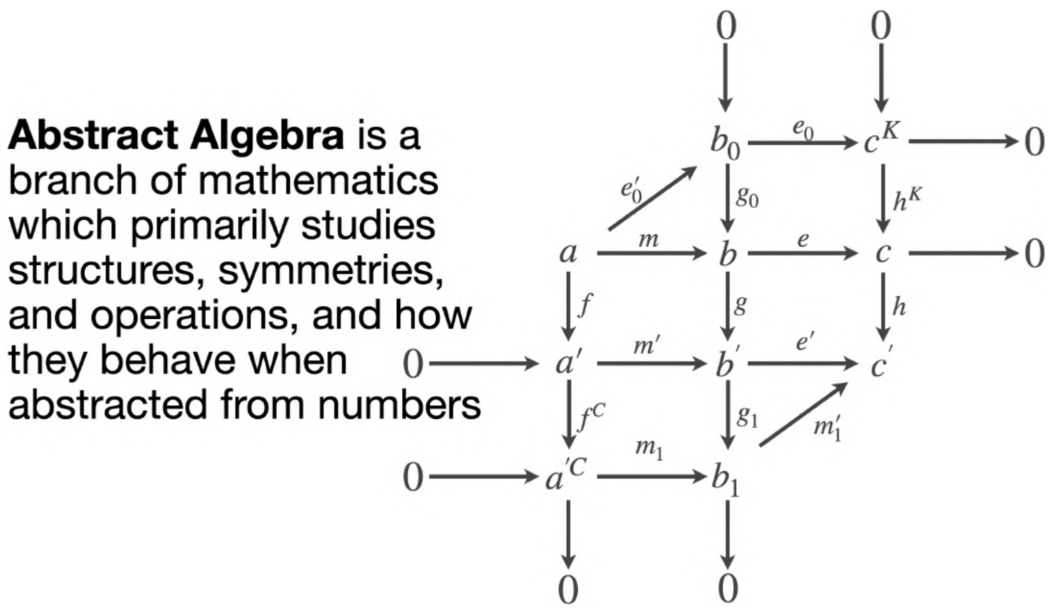
So let me ask you this: can you describe an entire field of math in just one sentence? This is probably impossible, or at least, that sentence won't be a complete description. But if you could go ahead and try to do it, you would naturally think about the *essence* (or the *core*) of the subject.

Of course, in trying to come up with one single sentence definition we would have to forget about a lot (and I mean a lot) of details. But this exercise does help you to develop an intuition about a specific subject.

For example, let me try and define dynamical systems:

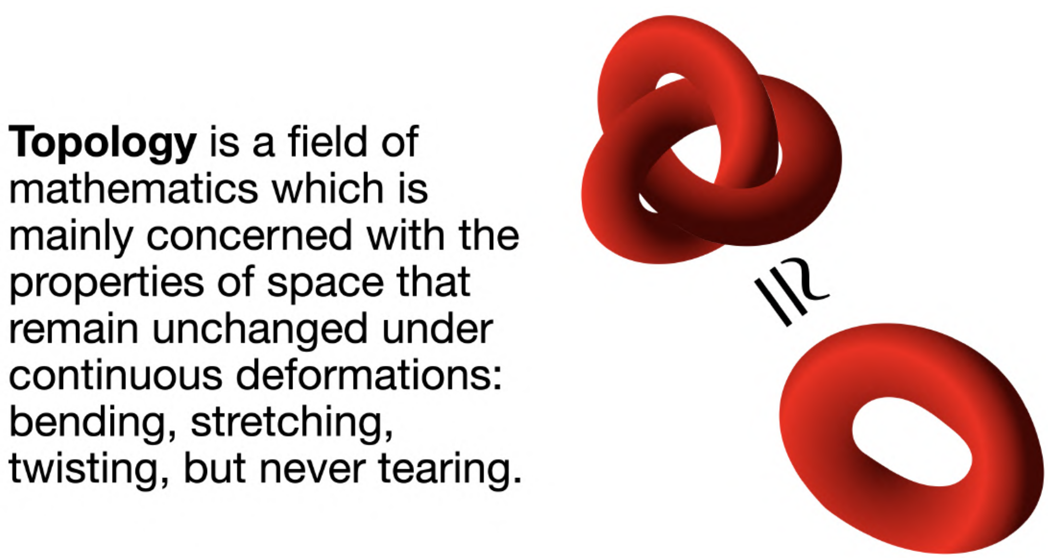


Obviously, I failed to mention many other things. That dynamical systems also happens to be the study of patterns in motion, like stability, chaos, periodicity, attractors, bifurcations, among other things, but the sentence does capture the core idea of dynamical systems. Now what about abstract algebra?



Again I’ve missed things like isomorphisms and homomorphisms, rings and fields, etc, but you got the main idea to just get a grasp of the field, right?

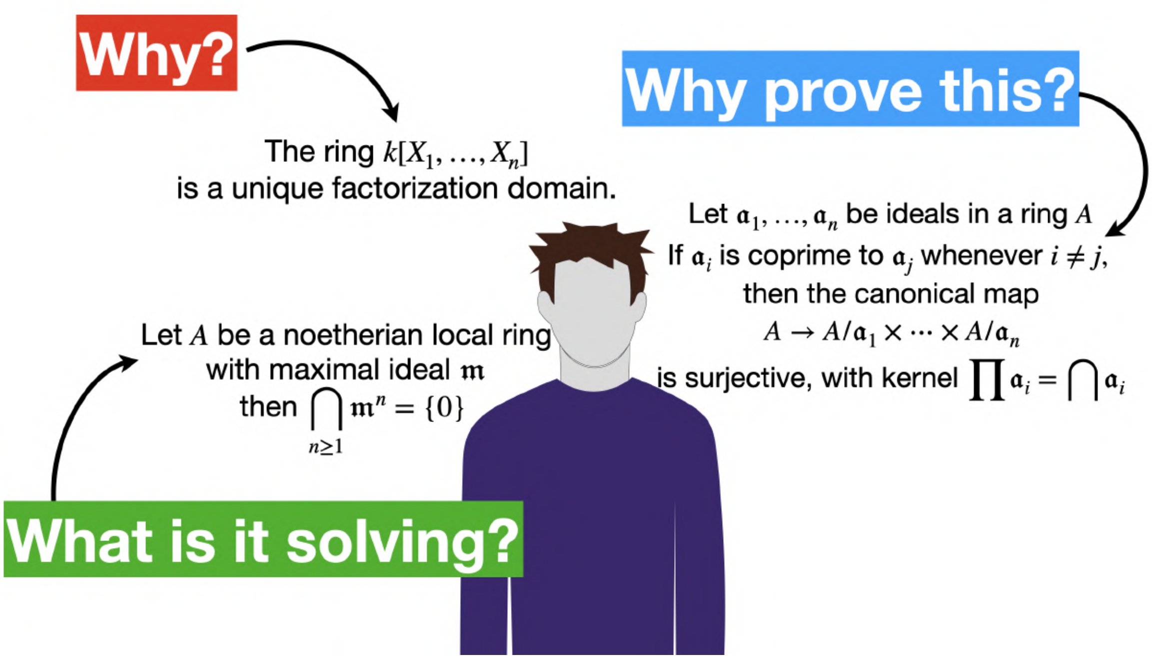
Now Topology.



These definitions are not perfect at all (I know that), and I'm pretty sure that if you asked 10 different mathematicians to define each of them in one sentence, you would get 10 similar but different sentences. It just shows how our brains grasp the totality of a subject giving different weights to different features of each.

So, for example, are there connections between topology and algebra? Absolutely. But does that mean algebraists think just like topologists? Not really. They tend to think very differently. Their *motives* are different. Their questions, and interests, are different.

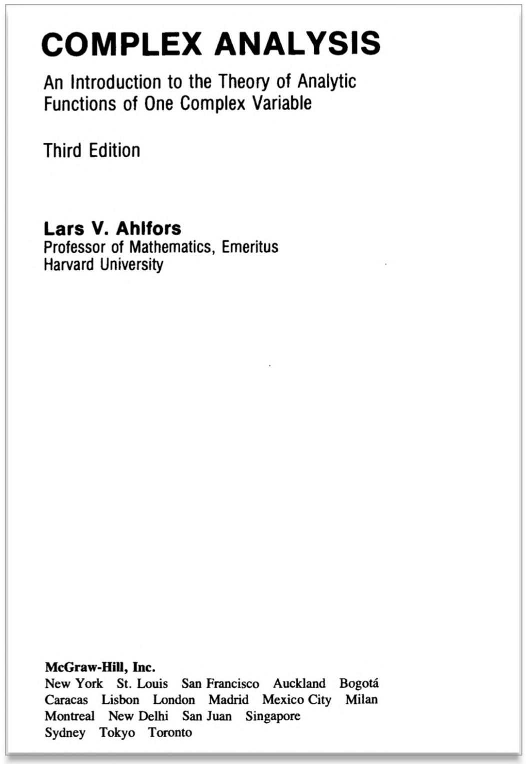
And that's the key. Every subject, every field of mathematics, is guided by some deep motive. If you're just memorizing theorems, you're completely missing the point. You need to be able to look at a theorem and ask why. Why is that there? Why was it worth proving? What's the problem that it's trying to solve?



Let's see an illustration of this concept.

Cauchy’s Theorem in Complex Analysis

We will open up the book ”Complex Analysis” by Lars Ahlfors, and go to page 113.



COMPLEX ANALYSIS

An Introduction to the Theory of Analytic Functions of One Complex Variable

Third Edition

Lars V. Ahlfors
Professor of Mathematics, Emeritus
Harvard University

McGraw-Hill, Inc.
New York St. Louis San Francisco Auckland Bogotá
Caracas Lisbon London Madrid Mexico City Milan
Montreal New Delhi San Juan Singapore
Sydney Tokyo Toronto

There we find **Theorem 4** (i.e. **Cauchy’s theorem**), which says:

when C is a circle about a. In order to make sure that the integral vanishes, it is necessary to make a special assumption concerning the region Ω in which $f(z)$ is known to be analytic and to which the curve γ is restricted. We are not yet in a position to formulate this condition, and for this reason we must restrict attention to a very special case. In what follows we assume that Ω is an open disk $|z - z_0| < \rho$ to be de-

Theorem 4. *If $f(z)$ is analytic in an open disk Δ, then*

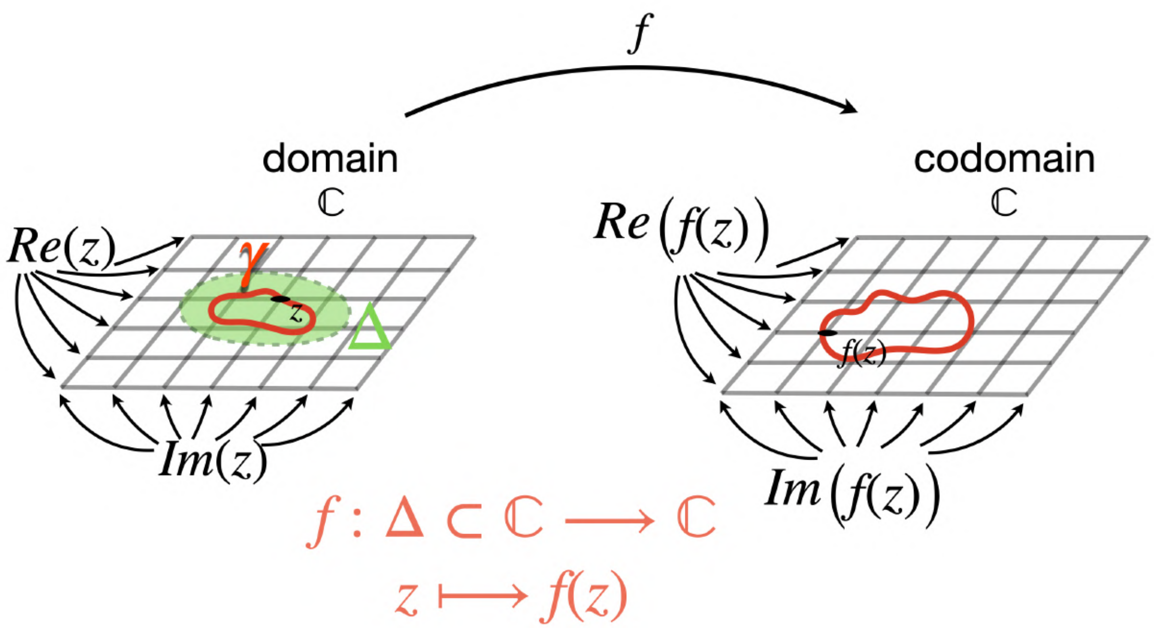
$$\int_\gamma f(z)\,dz = 0 \tag{18}$$

for every closed curve γ in Δ.

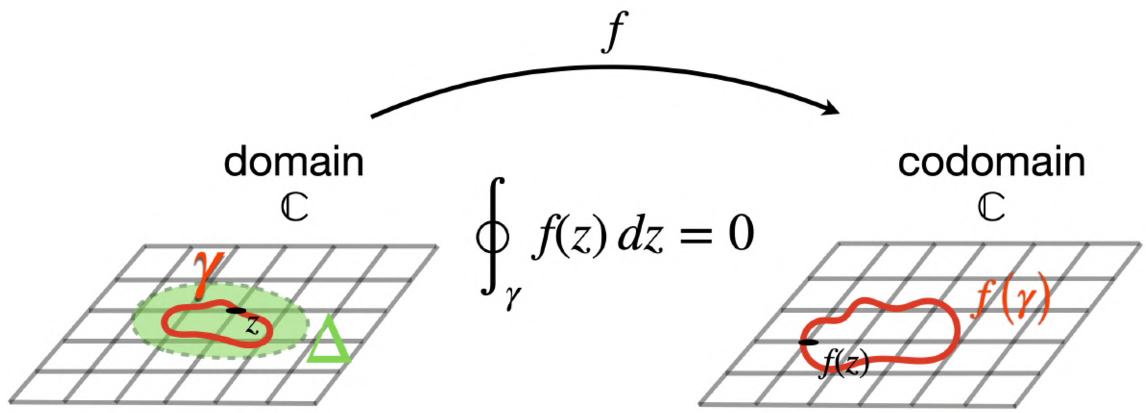
$$F(z) = \int_\sigma f\,dz \tag{19}$$

where σ consists of the horizontal line segment from the center (x_0, y_0) to (x, y_0) and the vertical segment from (x, y_0) to (x, y); it is immediately seen that $\partial F/\partial y = if(z)$. On the other hand, by Theorem 2 σ can be

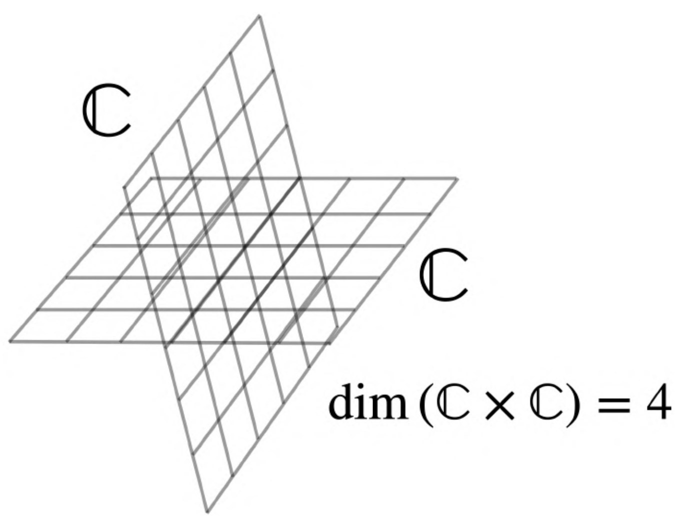
How do we interpret that?



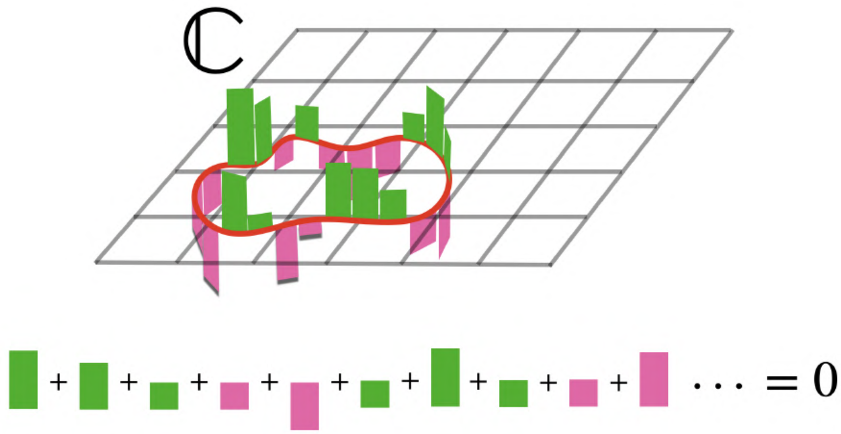
The domain is a copy of the complex plane, the codomain is another copy of the complex plane, and f is the function between them, which maps points z in the domain to points $f(z)$ in the codomain. Pick a disk and call it Δ, then pick any closed curve γ inside of it. If we map this curve using the function f, we get another curve, $f(\gamma)$. Now, we integrate it over γ and the result must be zero.



Drawing a graph, like is usually done in real analysis, would require a 4-dimensional space ($\mathbb{C} \times \mathbb{C}$). It is impossible to draw it…



The theorem says that if a function is analytic in an open neighborhood Δ, then integrating it along any closed curve $\gamma \in \Delta$ gives zero. So, the infinitesimal contributions perfectly cancel.



Ok, there is a lot to unpack here, like the *definition of an analytic function*, how exactly these quantities *perfectly add up to zero*, maybe we could also see a *concrete example*, and so on… But the point I'm trying to make is another one: you need to look for the **motive**. Always look for the motive.

The motive here is to simplify something that is already very complex (literally complex, in both senses, haha). Just think about it, we can't even draw the function because it would require 4 dimensions. Imagine trying to visualize its integration process, for example! It's even harder. However, when we realize that in many cases, the integrals will automatically turn out to be zero, as long as the theorem's conditions are met, we begin to appreciate just how powerful this result is. It saves us a lot of unnecessary calculations, since the final result will be zero anyway.

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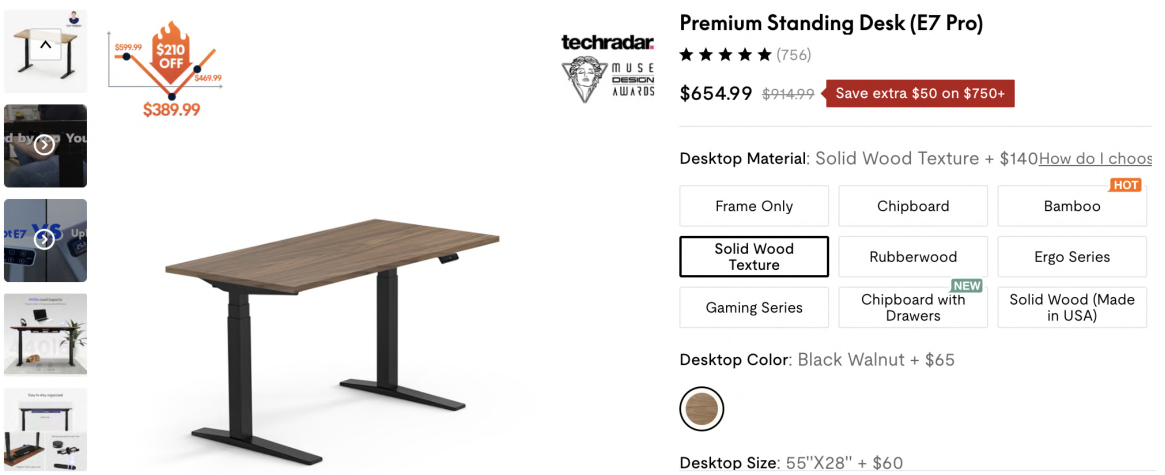
Right here is the E7 PRO Model.



You can customize the desk to your liking: size, color, material, and additional accessories. It is great value for the money because once you get the E7 PRO Model, you won't be able to live without it. Not to mention that they have a huge selection of other products or desks that you can check out.

Check out our website and if you want a premium standing desk. And you can check their E7 PRO model which is more stable for heavy equipment on the desk. Plus, you can use our code to get a discount **YTE730**.

Now let's get back to build mathematical intuition.

Concrete Example

For example, for $f(z) = z^2$, where $z \in \mathbb{C}$, what is the integral of this function over this path $\gamma(t)$ show below? $(t \in [0\,,\,2\pi])$
imagine

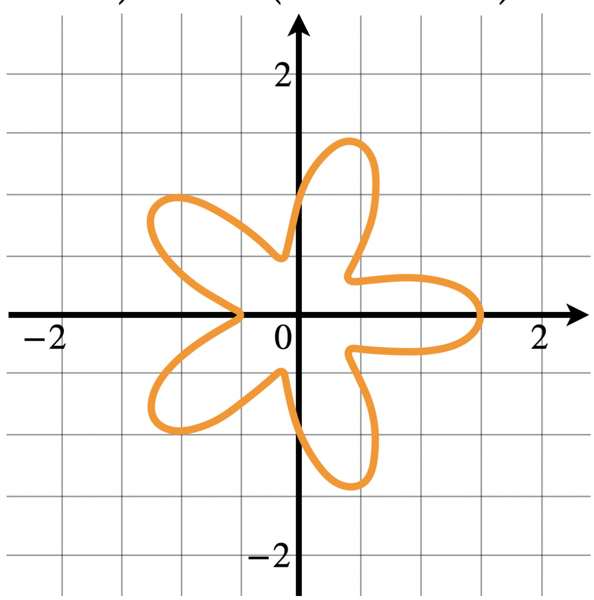
$$\boxed{\gamma(t) = \left(1 + \frac{1}{2}\cos(5t)\right) e^{it}}$$

$$\Updownarrow$$

$$\boxed{\gamma(t) = \left(\left(1 + \frac{1}{2}\cos(5t)\right)\cos t\,,\; \left(1 + \frac{1}{2}\cos(5t)\right)\sin t\right)}$$

$$f(z) = z^2 \qquad\qquad z \in \mathbb{C}$$

$$\gamma(t) = \left(\left(1 + \frac{1}{2}\cos(5t)\right)\cos t\,,\; \left(1 + \frac{1}{2}\cos(5t)\right)\sin t\right) \qquad t \in \left[0\,,2\pi\right]$$



All we need is to calculate this *"simple"* integral:

$$\int_\gamma f(z)\,dz = \int_\gamma z^2\,dz$$

In parametric form we can write:

$$\gamma(t) = \left(\left(1+\frac{1}{2}\cos(5t)\right)\cos t\,,\ \left(1+\frac{1}{2}\cos(5t)\right)\sin t \right) = (\,(x(t), y(t)\,)$$

So:

$$x(t) = \left(1+\frac{1}{2}\cos(5t)\right)\cos t$$
$$y(t) = \left(1+\frac{1}{2}\cos(5t)\right)\sin t$$

Now we can write:

$$z(t) = x(t) + iy(t)$$
$$z(t)^2 = (x(t)+iy(t))^2$$
$$dz = \frac{d}{dt}(x(t)+iy(t))\,dt = \left(\frac{dx}{dt} + i\frac{dy}{dt}\right)dt$$

$\therefore$ The integral in full (explicit format) is:

$$\int_\gamma z^2\,dz = \int_0^{2\pi} (x(t)+iy(t))^2 \cdot \left(\frac{dx}{dt} + i\frac{dy}{dt}\right)\,dt =$$

$$= \int_0^{2\pi} \left[\left(1+\frac{1}{2}\cos(5t)\right)\cos t + i\left(1+\frac{1}{2}\cos(5t)\right)\sin t \right]^2 \cdot$$

$$\cdot \left[\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\cos t \right) + i\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\sin t \right) \right] dt$$

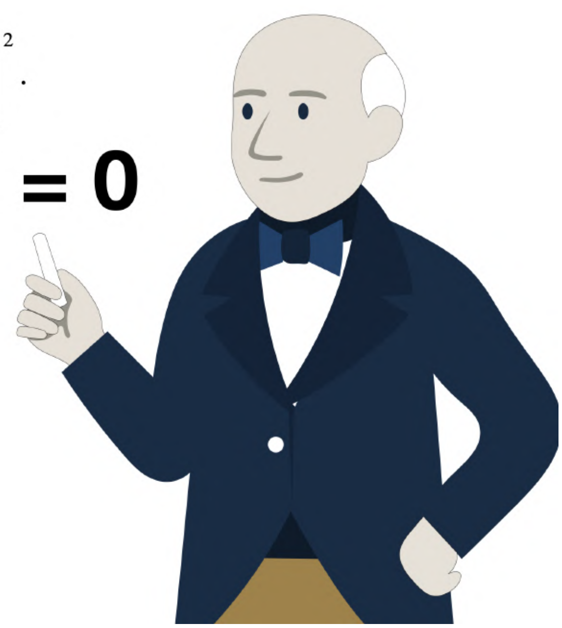
Are you guys ready to do it?! Cuz I am not!!! (haha) Instead, I'm just gonna do as most books do:

"The details are left as an exercise for the reader."

$$\int_0^{2\pi}\left[\left(1+\frac{1}{2}\cos(5t)\right)\cos t + i\left(1+\frac{1}{2}\cos(5t)\right)\sin t\right]^2 \cdot$$
$$\cdot\left[\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\cos t\right)+ i\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\sin t\right)\right]dt$$

Thankfully, there is Cauchy to save us, and we can simply say that the result is *zero*.

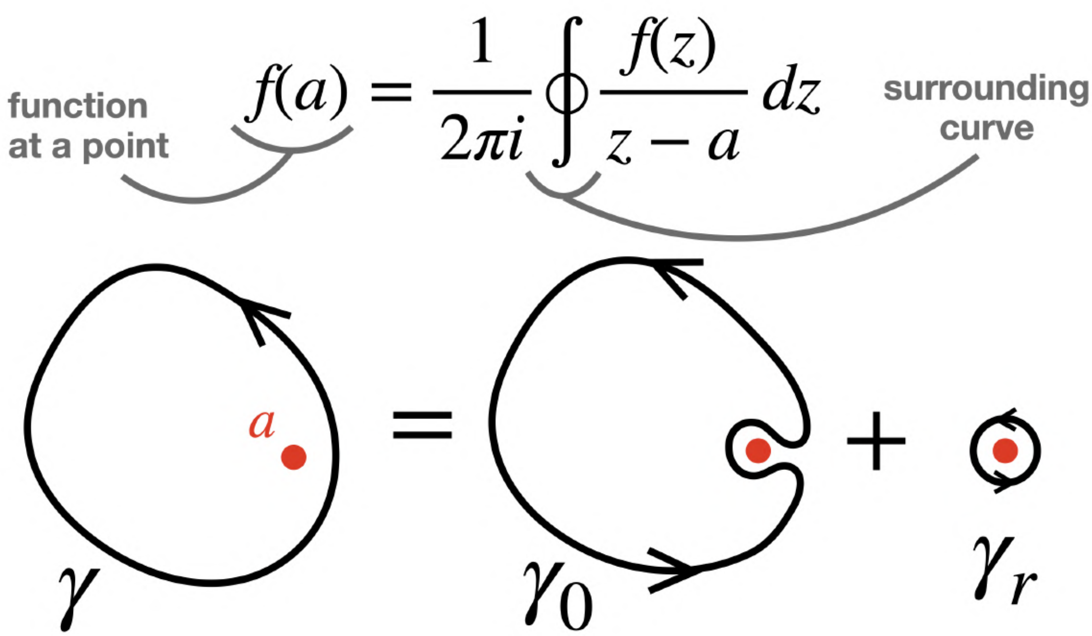
$$\int_0^{2\pi}\left[\left(1+\frac{1}{2}\cos(5t)\right)\cos t + i\left(1+\frac{1}{2}\cos(5t)\right)\sin t\right]^2 \cdot$$
$$\cdot\left[\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\cos t\right)+ i\frac{d}{dt}\left(\left(1+\frac{1}{2}\cos(5t)\right)\sin t\right)\right]dt = 0$$



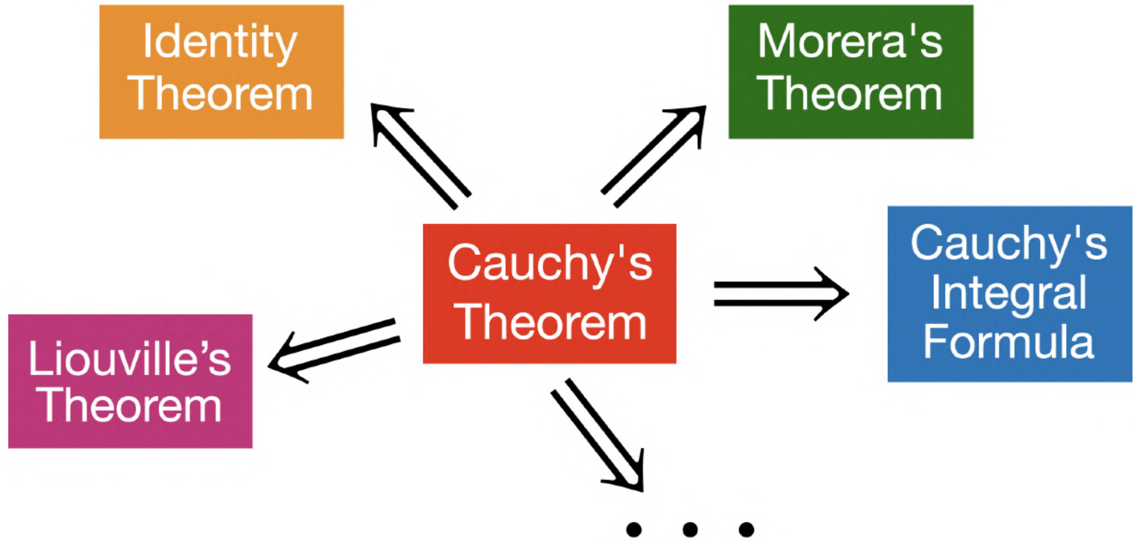
Cauchy's theorem is actually the foundation of many other deep results in complex analysis. Like **Cauchy's Integral Formula**:

$$\boxed{f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz}$$

Which says that you can recover the value of a function at a point using its values on a surrounding curve.



Cauchy's theorem is also the foundation of **Morera's theorem**, the **Identity theorem**, **Liouville's theorem**, and many other results in complex analysis.



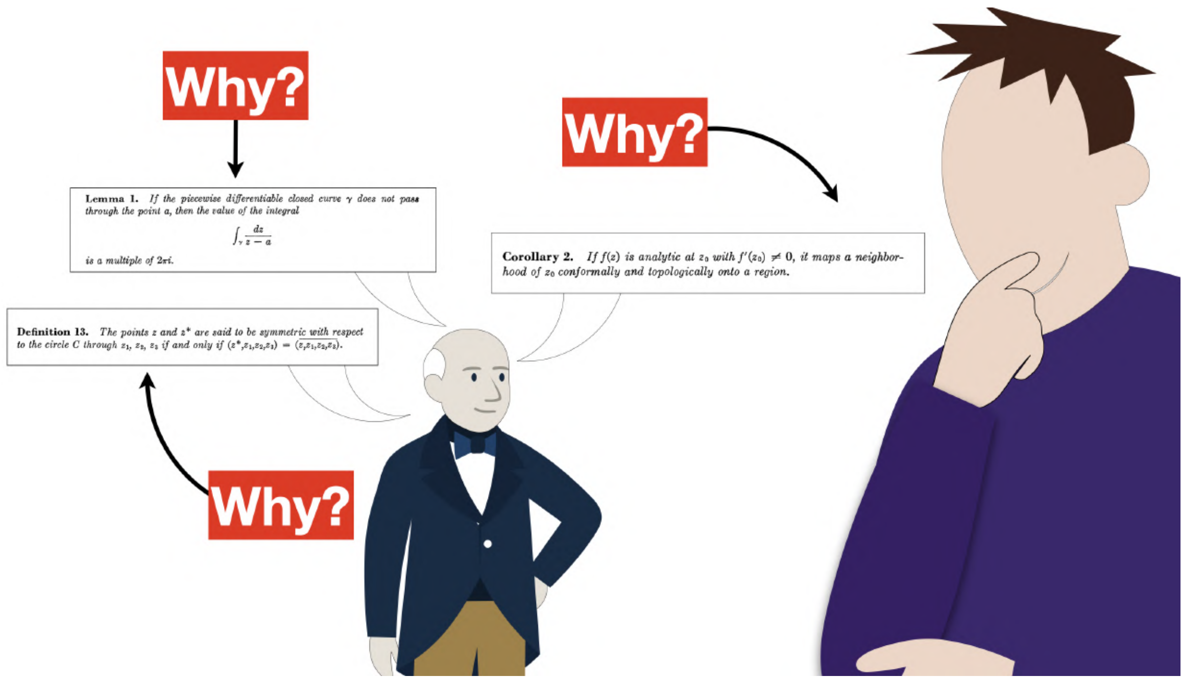
And, by the way, that's just in pure mathematics, there are also many many applications in physics.

Conclusion

The point is: every lemma, every corollary, and every definition we encounter is part of an idea some mathematicians had decades, or even centuries ago, and you're now part of understanding why these particular mathematicians think of that. Why did they come to this conclusion? And that means always looking for the motive.

You can memorize every single proof in a textbook and still fail to truly understand the subject. But if you capture the intent behind the math, and the questions it's built to answer, then even the most abstract theory will begin to feel intuitive.

And intuition isn't something you're born with. It's something you develop when you stretch your brain and force it to dig deep to find the motive behind every result in mathematics.



Try to look for the motive next time you bump into a theorem, and you will for sure see how this little habit will boost your learning process.

So, in conclusion: how do you build mathematical intuition?

Look for the motive every time you read a math result. You can discuss it with friends, look for *reddit* posts, use *forums*, *Mathematics Stack Exchange*, *MathOverflow*, you can use *AI*, you can look up *YouTube videos*... There are literally so many ways to find out the motive behind a theorem or definition nowadays...

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