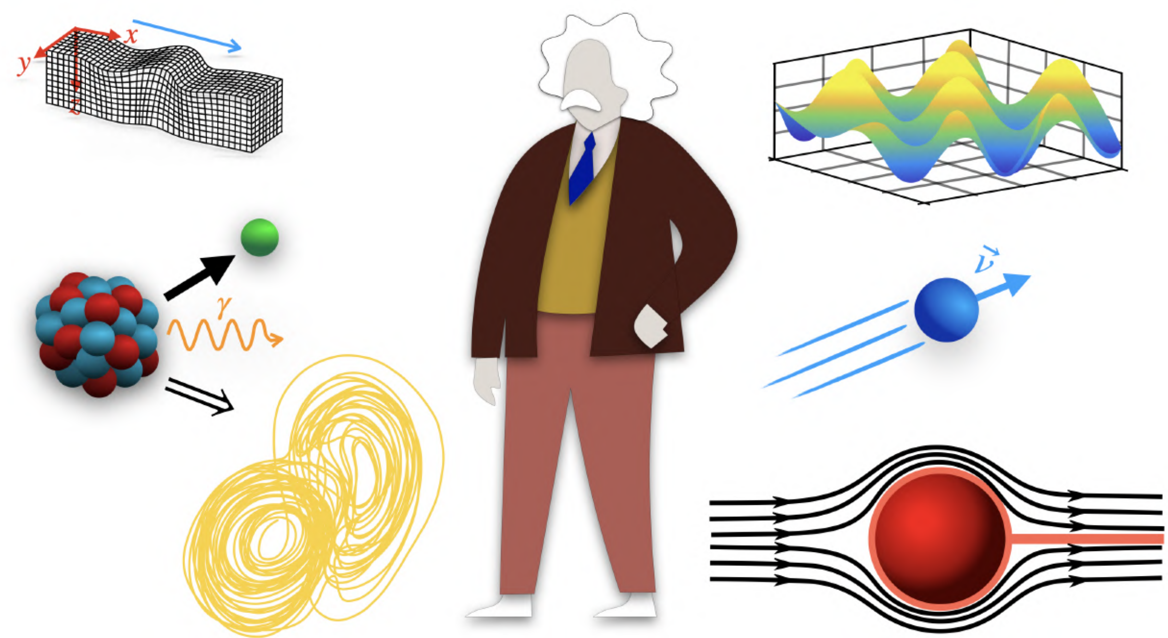
Does Math Overthink or Physics Oversimplify?

by DiBeos

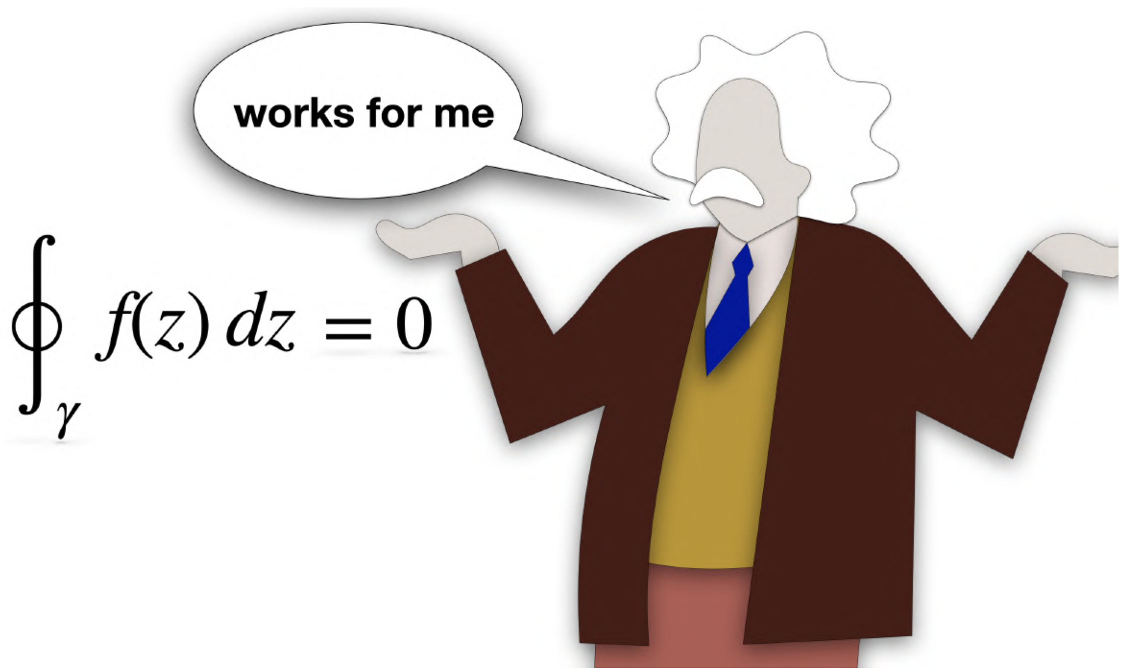
"If there is something very slightly wrong in our definition of the theories, then the full mathematical rigor may convert these errors into ridiculous conclusions." – **Richard P. Feynman**

Introduction

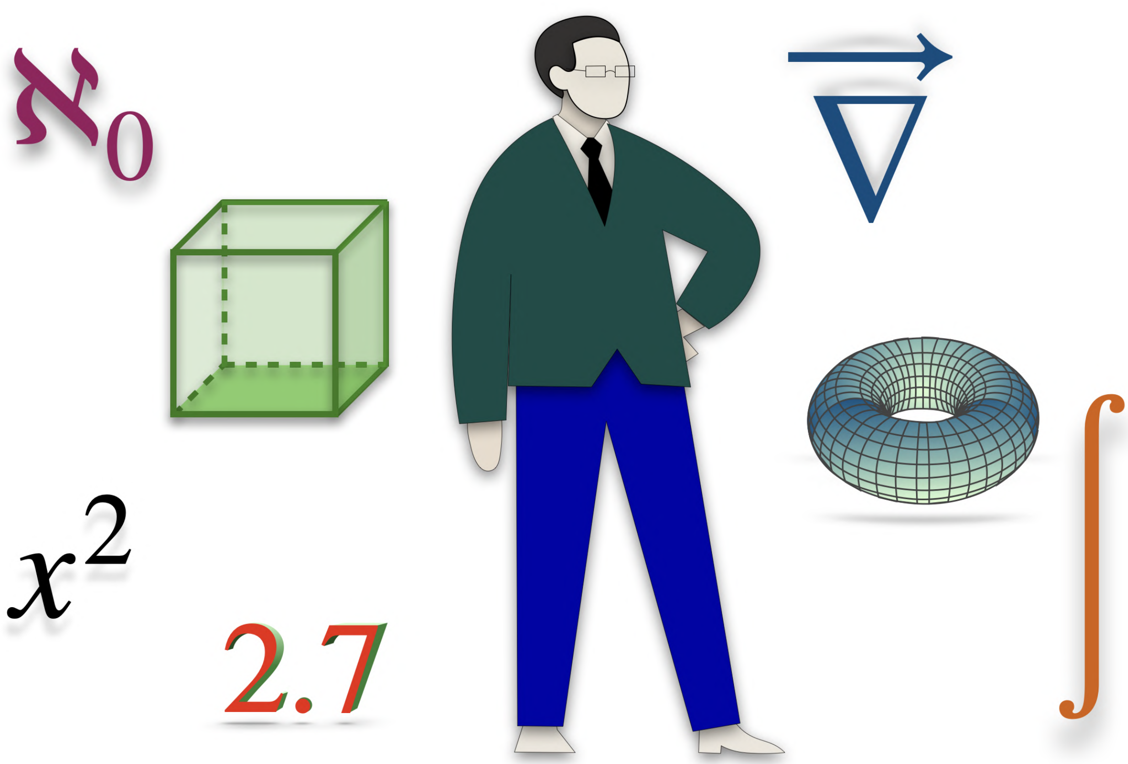
When we ask questions like *"Should mathematicians be more practical, or should physicists be more rigorous?"* or *"Does Math Overthink or Physics Oversimplify?"*, what we are really touching on is the deep tension between two communities that rely on the same language: **mathematics**.



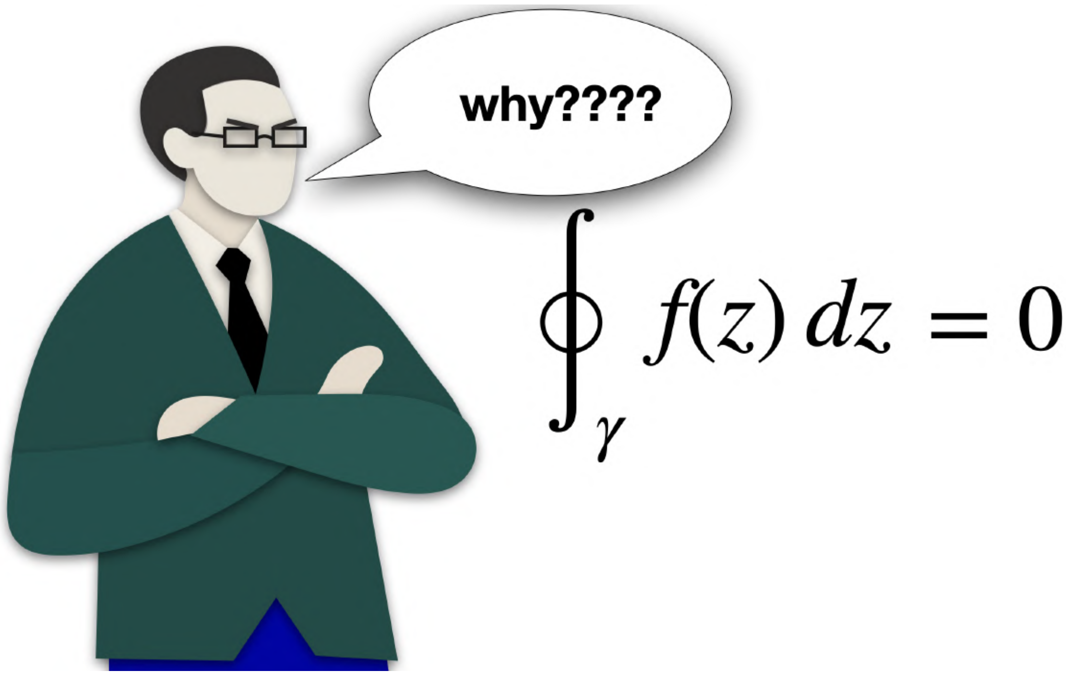
On one side, we have physicists, who often care more about describing the natural world quickly and effectively. Physicists can bend or even skip formal justifications when they want to arrive at an answer that works practically, because that's all that matters to them.



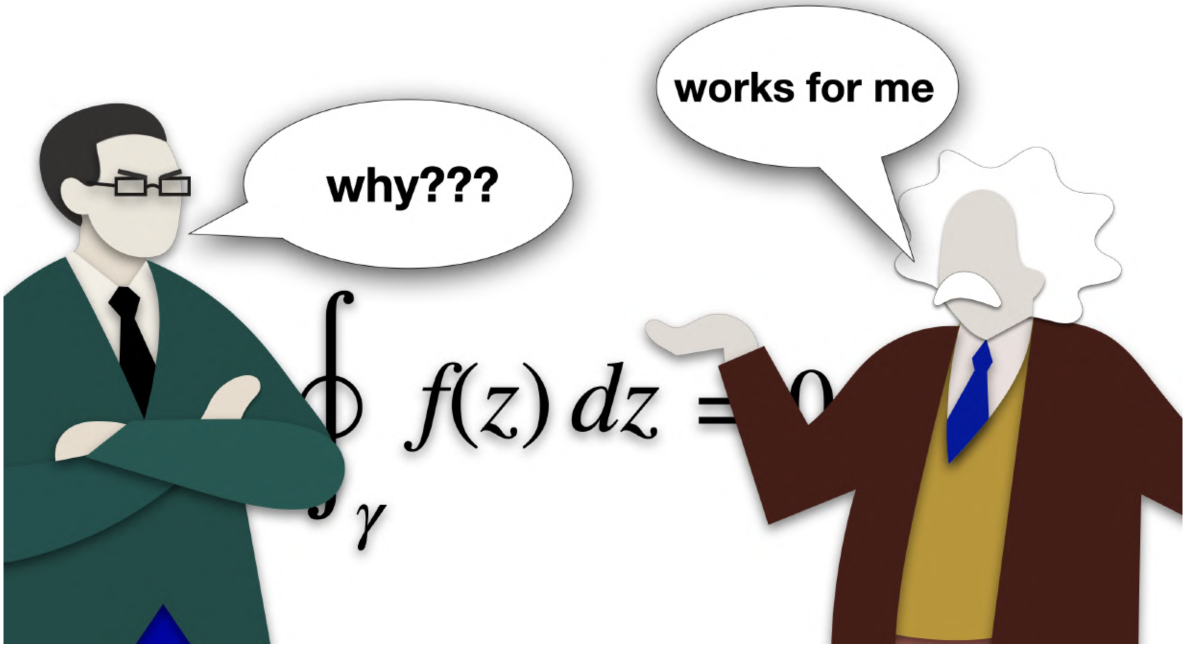
And then on the other side, we have mathematicians.



Mathematicians demand rigor at every step, because they want to make sure that no argument will collapse later on in the logical chain of implications.



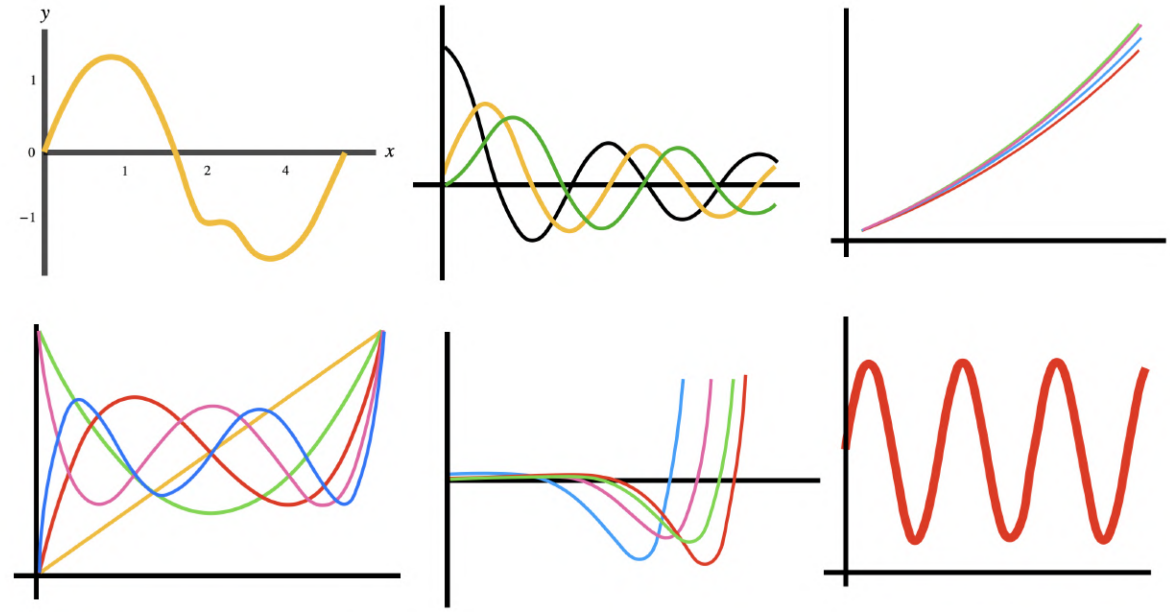
The approach of both sides can be justified, but when they are tackling similar problems with this sort of “conflict of interests”, mathematicians and physicists tend to disagree with each other very strongly, to say the least.



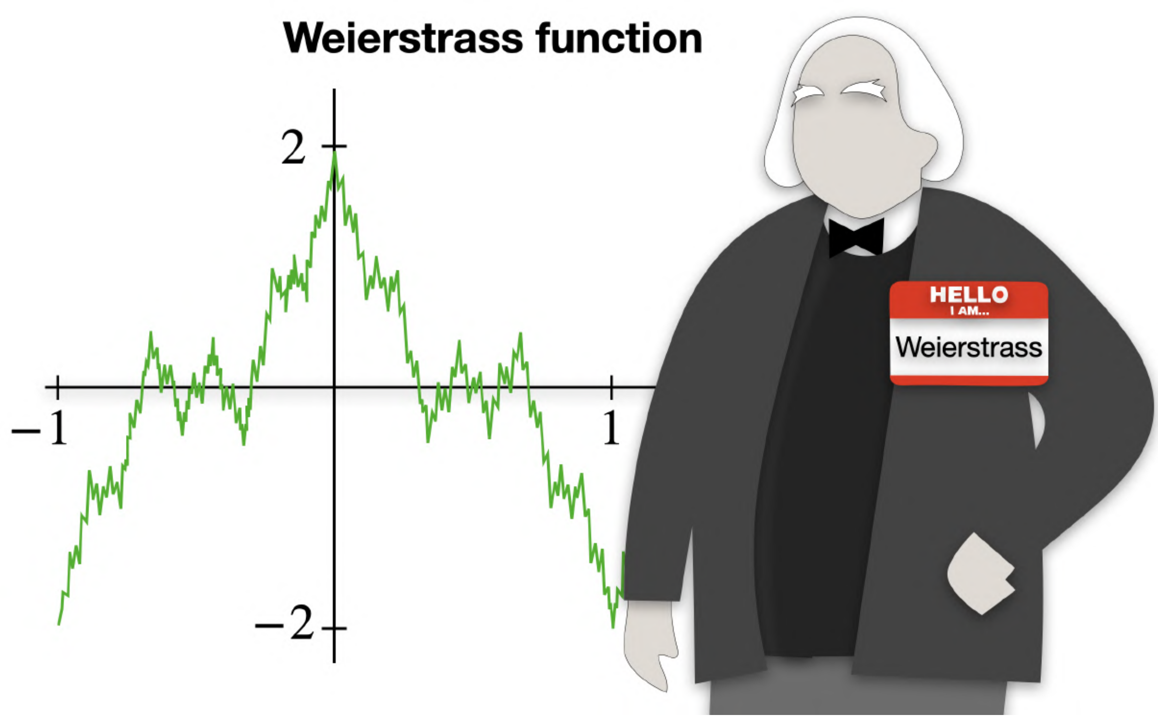
Rigor

Let's start with rigor. Rigor is not something that mathematicians impose just because they are annoying, it is the absolute foundation of mathematics. Rigor is what stops entire subjects from falling apart. Without it, we could very well make assumptions that can feel intuitive but are actually completely false, and from sloppy assumptions, we could derive results that mislead many people.

A classic example of this is from the 19th century, when many mathematicians believed that continuous functions were differentiable almost everywhere, because, intuitively, that's how most functions they dealt with behaved.



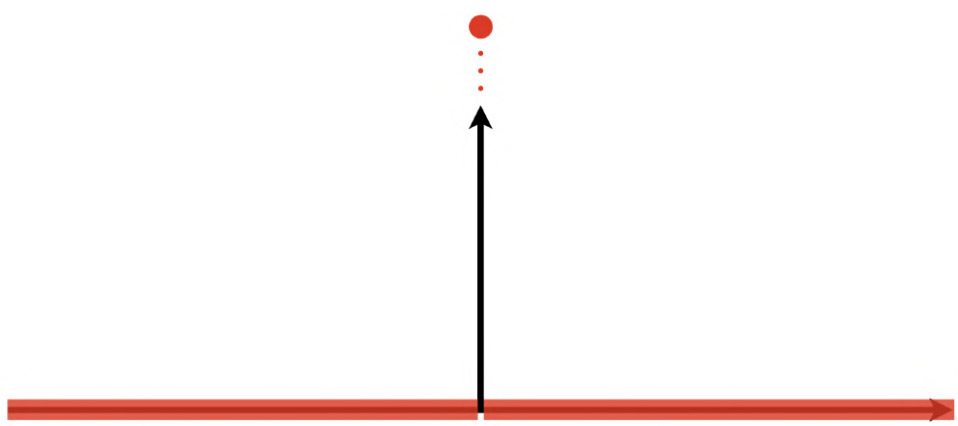
This idea felt true, especially for the physicists, because continuity and differentiability were almost synonyms. But then, later on, we found counterexamples like the famous *Weierstrass's function*, which is a function that's continuous everywhere but differentiable nowhere. And it completely destroyed the initial assumption.



So the entire framework of analysis had to be re-examined, and that's what happens when we trust intuition too much without having the demand for rigor: we end up *"building a castle on sand"*.

Physicists tend to fall into this trap. There are so many historical examples of physicists using mathematics in a sloppy way. They treat divergent series as though they converge, or swap limits without showing why they did so… Sometimes, this sloppiness works just fine, and leads to predictions that actually match experiments. But in other times, this sloppiness creates confusion or mistakes that can take decades to fix. So yeah, it is tempting to admire how elegant intuition is in physics, but we absolutely need rigor to stop us from falling into a trap of errors. And in this sense, physicists should absolutely be more rigorous, at least to the point of understanding where their arguments stand on firm ground and where they are clearly stretching things.

An example would be the **Dirac delta function**. This function (if you even want to call it that) is not actually a function in the mathematical sense. It's this weird object that's zero everywhere, except at a single point where it's infinite, but the total integral is somehow 1.

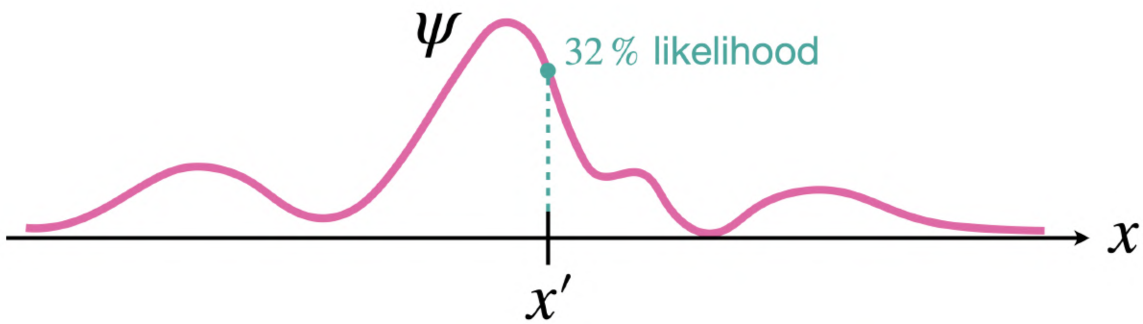


$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

*just a representation

I mean, very weird. Physicists used it like a normal function because it gave them the answers they needed. But mathematicians were like, "Wait, what even *is* this thing?" It wasn't until way later that the rigorous theory of distributions was developed to give the Dirac delta a proper mathematical definition. So the intuition came first, and it worked, but it was messy. And somebody had to clean the mess...

In quantum mechanics, the Dirac delta is used all over the place, like in representing point particles, or the probability density of a particle being exactly at some position x'. It shows up in things like the *orthonormality* of wavefunctions, where you'll see something like this:



$$\langle x \,|\, x' \rangle = \delta(x - x')$$

This tells you that the position eigenstates are orthogonal, which means that the result is zero if $x \neq x$. But when $x = x$, the expression "blows up" to infinity, even though in a way that still makes the math work, because the delta function is normalized, which means that its integral over all space is 1. So when we measure the particle at position x', we have 1 (or 100%) certainty of finding it there.

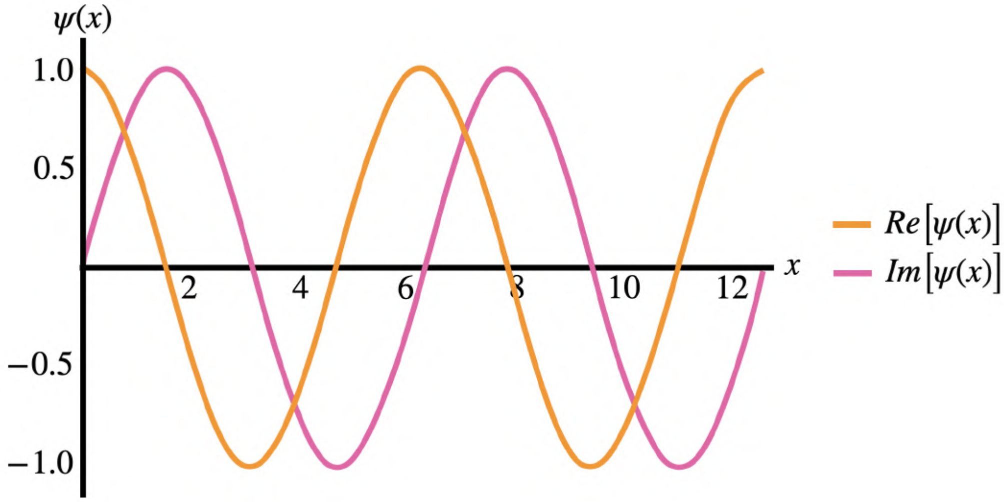
$$\boxed{\langle x | x' \rangle = \delta(x - x')} \quad \xRightarrow{x \neq x'} 0, \qquad \xRightarrow{x = x'} \infty$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \implies \text{At } x' : \; 100\,\% \text{ likelihood}$$

Even though the delta isn't a function in the traditional sense, it gives us some very powerful predictions, because it lets physicists calculate things like the outcomes of *quantum measurements* in a way that matches experimental data. And yes, it worked beautifully.

But here's the problem.

Let's say that you have a wavefunction like this:



$$\psi_k(x) = Ae^{ikx} = \underbrace{A\cos(kx)}_{Re(\psi)} + \underbrace{iA\sin(kx)}_{Im(\psi)}$$

This wavefunction doesn't represent a real (physical) particle by itself. It spreads out infinitely across space and cannot be normalized.

If we want to normalize a function, we would need to impose this condition:

$$\int_{-\infty}^{\infty} \left|\psi_k(x)\right|^2 dx = 1$$

In other words, the integral over all space of the squared magnitude of the wavefunction has to be equal to 1, meaning that the total probability of finding the particle somewhere is 100%.

Let's try to normalize it, real quick:

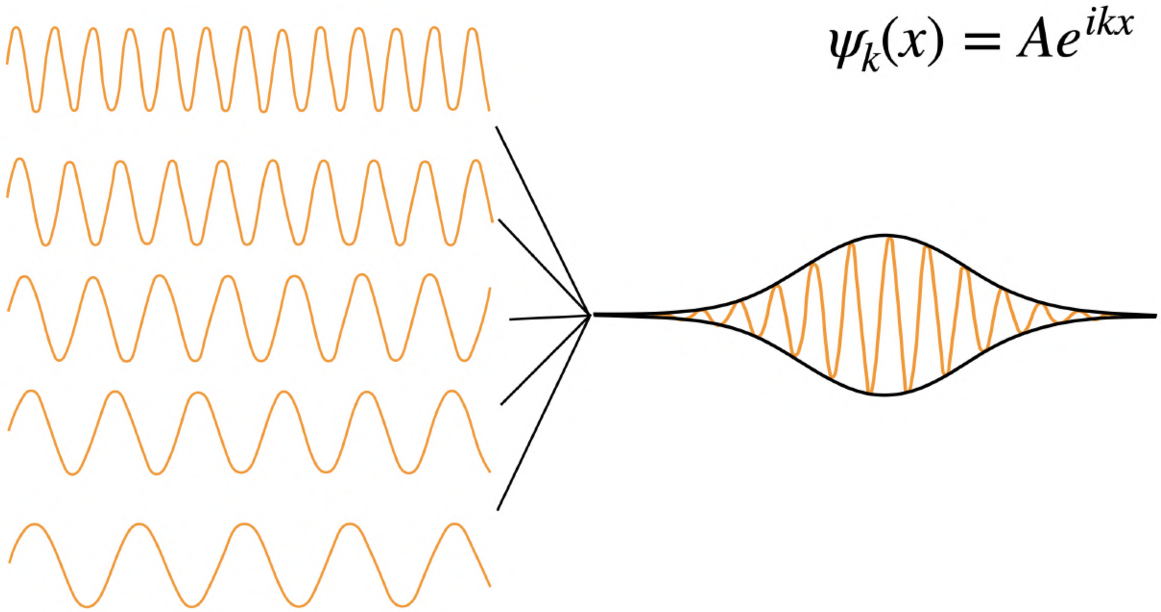
$$\int_{-\infty}^{\infty} \left| Ae^{ikx} \right|^2 dx = |A|^2 \int_{-\infty}^{\infty} \left| e^{ikx} \right|^2 dx = 1 \quad \Longrightarrow$$

$$\Longrightarrow \quad |A|^2 \int_{-\infty}^{\infty} 1 \, dx = \boxed{|A|^2 \cdot \infty = 1} \quad \lightning$$

That's a contradiction. That's just wrong. The integral diverges for any nonzero values of A. Even if we tried to set $A = 0$, the wavefunction would be just zero everywhere, which means that there isn't even any particle to begin with. So this wavefunction is not normalizable.

$$|A|^2 \cdot \infty = 1 \qquad (A \to 0)$$

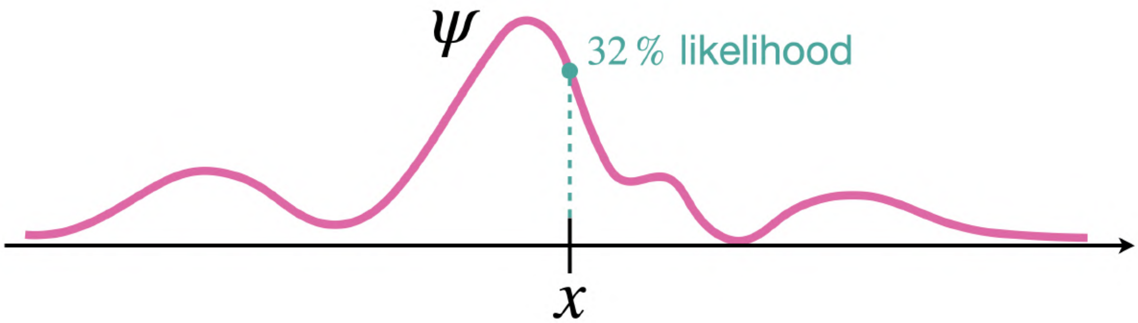
BUT, it is still useful in theory, because we can use it to build more realistic, localized, *wave packets*, and that's where the delta function comes in.



A wave packet is a localized quantum state that is formed by the superposition of many wavefunctions.

If we compute the expression $\psi_k^*(x)\,\psi_{k'}(x)$, which is the probability density (so the likelihood of finding the particle at position x), and then we integrate it over x, we get the Dirac delta:

$$\int_{-\infty}^{\infty} \psi_k^*(x)\, \psi_{k'}(x)\, dx = \delta(k - k')$$



$\psi^*(x)$ is the **complex conjugate** of the wavefunction.

complex conjugate

$$\int_{-\infty}^{\infty} \psi_k^*(x)\, \psi_{k'}(x)\, dx = \delta(k - k')$$

$$\psi_k^*(x) = A^* e^{-ikx}$$

If $A = \mathrm{Re}(A) + i\,\mathrm{Im}(A) \implies A^* = \mathrm{Re}(A) - i\,\mathrm{Im}(A)$

These two wavefunctions are *orthogonal*.

Now, try to catch my logical mistake:

"Well, let me write these wavefunctions explicitly:

$$\int_{-\infty}^{\infty} \psi_k^*(x)\,\psi_{k'}(x)\,dx = \int_{-\infty}^{\infty} |A|^2\, e^{i(k'-k)x}\,dx = \delta(k-k') \quad \Longrightarrow$$

$$\Longrightarrow \quad |A|^2 \underbrace{\int_{-\infty}^{\infty} e^{i(k'-k)x}\,dx}_{=\,2\pi\,\delta(k-k')\ \text{(distribution theory)}} = \delta(k-k')$$

So I guess I can write this:

$$|A|^2 \cdot 2\pi\,\delta(k-k') = \delta(k-k')$$

Now, I divide both sides by $\delta(k-k')$:

$$\frac{|A|^2 \cdot 2\pi\, \cancel{\delta(k-k')}}{\cancel{\delta(k-k')}} = \frac{\cancel{\delta(k-k')}}{\cancel{\delta(k-k')}} \Rightarrow \boxed{|A|^2 = \frac{1}{2\pi}}$$

″
But… wait a second...

When $k = k'$ these two expressions are the same:

$$\int_{-\infty}^{\infty} \left|\psi_k(x)\right|^2 dx = \int_{-\infty}^{\infty} \psi_k^*(x)\, \psi_{k'}(x)\, dx$$

$$\left(k = k'\right)$$

And therefore, we just found a *normalization constant*:

$$|A|^2 = \frac{1}{2\pi}$$

$$k = k'$$

$$\int_{-\infty}^{\infty} \left|\psi_k(x)\right|^2 dx = \int_{-\infty}^{\infty} \psi_k^*(x)\, \psi_{k'}(x)\, dx$$

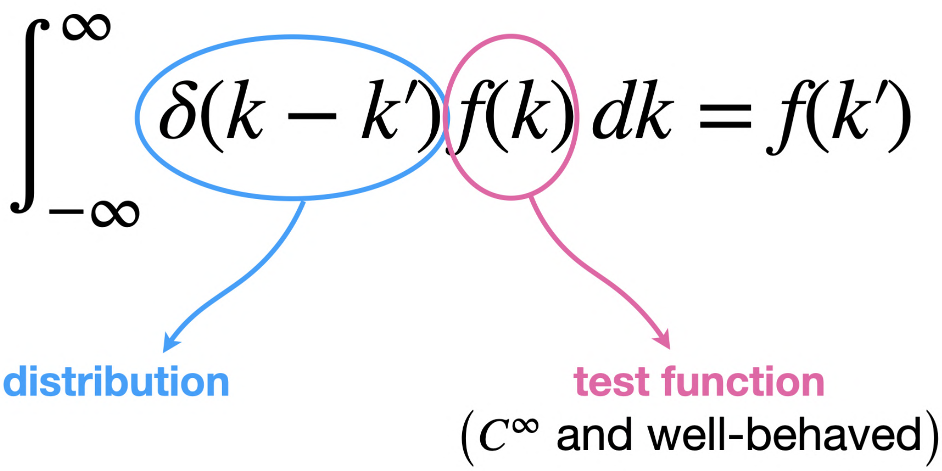
Basically, we just normalized the same wavefunction that we said earlier was **not normalizable**! This is just wrong. And that is precisely the danger of being sloppy...

The Dirac delta is not a function, but a distribution. We cannot divide a number by a **distribution**.

Intuitively, a distribution is a rule that tells you what the result of an integral should be, not the value a "function" takes at a point. Instead of asking *"what's the value of* $\delta(x)$ *at* $x = 0$", you gotta ask:

> *"What happens when I integrate* $\delta(x)$ *against another function?"*

A distribution is an object that only makes sense inside an integral, acting on a test function (which is a smooth and well-behaved function).



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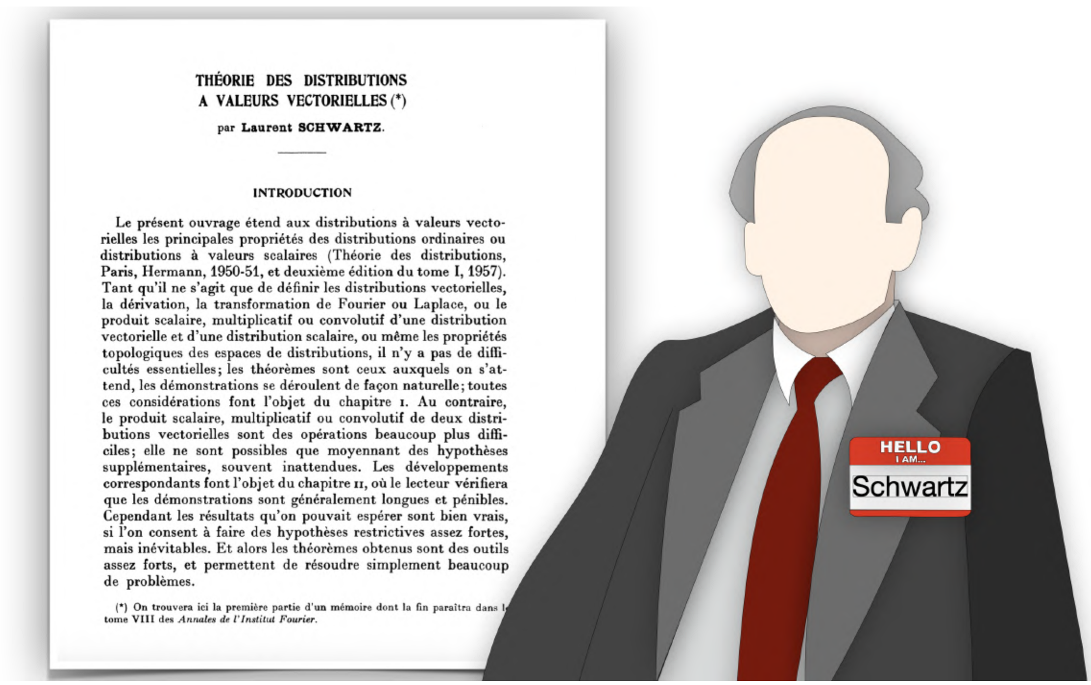
Ok, now that I've made my point about the dangers of sloppiness, let's turn to mathematicians. Let's be honest, if physicists can sometimes be too loose, mathematicians can sometimes be way too strict.

Practicality

Some mathematicians are so obsessed with *definitions*, *axioms*, and *precision* that they forget why they were using this math in the first place. These are the people who, when a new idea is presented, interrupt with *"wait wait wait, but is this fully well-defined?"* or *"what about this obscure teeny little edge case (that nobody really cares about, but for some reason this person does)?"*, and all of this pickiness even before understanding the main idea. This kind of hyper-rigidity can prevent us from exploring, which is something that is extremely important to do especially at the beginning.

Many times, physicists arrive at new mathematics exactly because they don't focus on the rigor, but just try to *brain-storm* ideas that are only partially defined, and they do it just using intuition and practical necessity. And yeah, later, rigorous mathematics needs to come in to "clean up the mess", so rigor is absolutely necessary, but at the right moment. We have to take initial *risks*, because without them many areas of mathematics might never have existed at all. So rigor is essential, but sometimes it can also become an obstacle if it's applied too early and too harshly, and I have just the right example to show that.



THÉORIE DES DISTRIBUTIONS
A VALEURS VECTORIELLES (*)

par **Laurent SCHWARTZ**.

INTRODUCTION

Le présent ouvrage étend aux distributions à valeurs vectorielles les principales propriétés des distributions ordinaires ou distributions à valeurs scalaires (Théorie des distributions, Paris, Hermann, 1950-51, et deuxième édition du tome I, 1957). Tant qu'il ne s'agit que de définir les distributions vectorielles, la dérivation, la transformation de Fourier ou Laplace, ou le produit scalaire, multiplicatif ou convolutif d'une distribution vectorielle et d'une distribution scalaire, ou même les propriétés topologiques des espaces de distributions, il n'y a pas de difficultés essentielles; les théorèmes sont ceux auxquels on s'attend, les démonstrations se déroulent de façon naturelle; toutes ces considérations font l'objet du chapitre I. Au contraire, le produit scalaire, multiplicatif ou convolutif de deux distributions vectorielles sont des opérations beaucoup plus difficiles; elle ne sont possibles que moyennant des hypothèses supplémentaires, souvent inattendues. Les développements correspondants font l'objet du chapitre II, où le lecteur vérifiera que les démonstrations sont généralement longues et pénibles. Cependant les résultats qu'on pouvait espérer sont bien vrais, si l'on consent à faire des hypothèses restrictives assez fortes, mais inévitables. Et alors les théorèmes obtenus sont des outils assez forts, et permettent de résoudre simplement beaucoup de problèmes.

(*) On trouvera ici la première partie d'un mémoire dont la fin paraîtra dans le tome VIII des *Annales de l'Institut Fourier*.

A proper (and rigorous) framework for the delta function came only in the 1950s, when French mathematician Laurent Schwartz developed his theory of distributions. But physicists had already spent about two decades using this mathematically undefined object to make correct physical predictions in quantum mechanics. So in hindsight, if physicists had listened to the mathematicians' objections and waited for the delta function to be made rigorous first, and used it only afterwards, the advancement of quantum mechanics would've been much slower than it has been. Who knows if we would even understand as much about nature as we do today? All of this is thanks to the risk-taking spirit of physicists. They didn't really care about what the delta function was, they just cared that it worked.

Conclusion

Therefore, where does this leave us? As with most things in life, the answer is in the ability of finding the right **balance**. Rigor is necessary, but its importance depends on the context you find yourself in. You should understand rigor enough to know where the gaps are, and need to be aware of their existence, but at the same time you shouldn't let rigor *paralyze* you. Most times, the intuitive insight comes first, and the rigor comes later. This is what we tend to see in history: physicists, and even applied mathematicians, push with bold leaps of intuition, and pure mathematicians solidify the foundation, to make sure that those initial leaps can be trusted.

I honestly always thought that this conflict between mathematicians and physicists was so silly. I started my path as a physicist, and then eventually completely changed it to pure mathematics, but it never prevented me from appreciating both areas of knowledge. Anyway, let me know what you guys think about it in the comments.

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