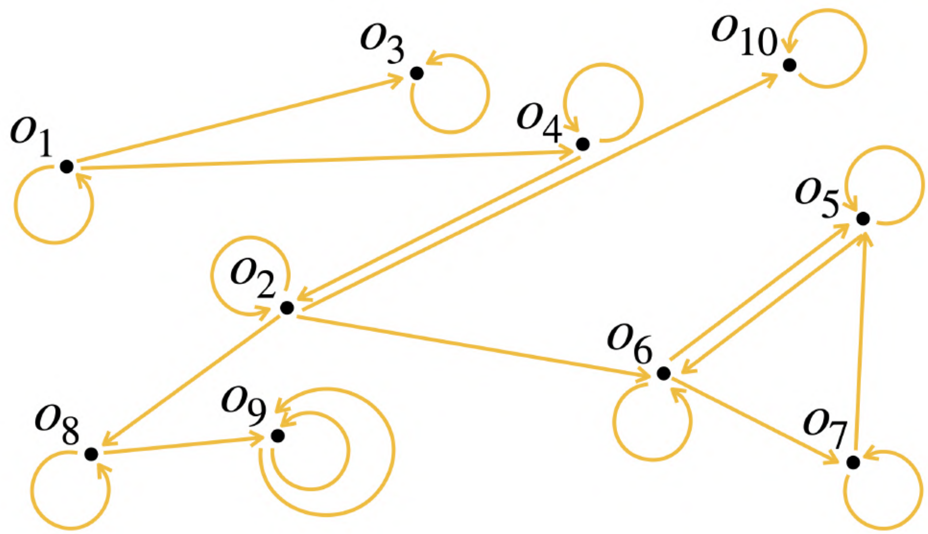
The Most Abstract Area of Pure Mathematics

by DiBeos

"Category theory is abstract nonsense." – **Saunders Mac Lane (one of the co-founders of category theory)**

Introduction

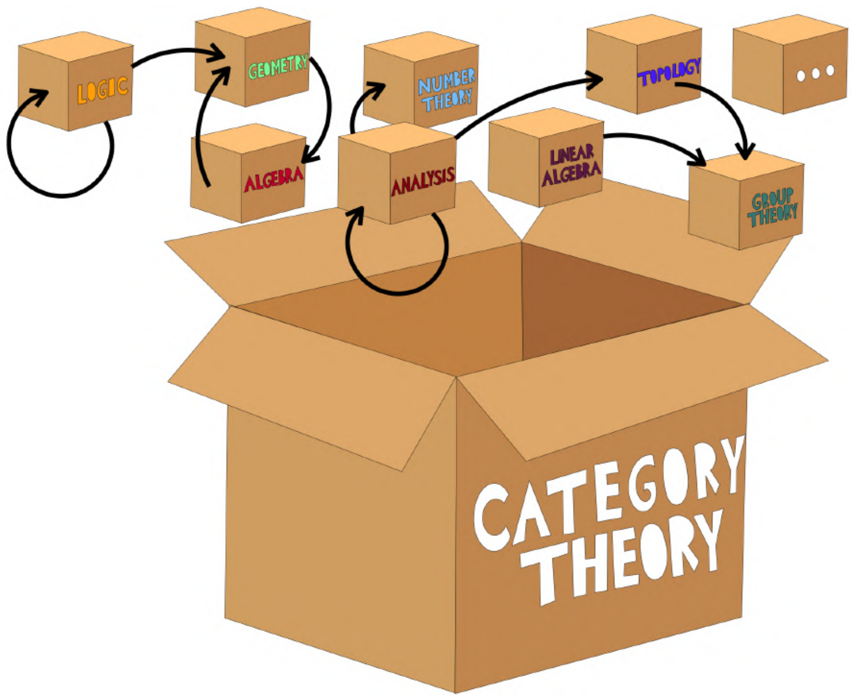
This area doesn't study mathematical objects directly, like numbers, functions, or spaces. It focuses on the relationships (or *morphisms*) between structures and the patterns that arise from those relationships. It is considered a *"theory of theories"*.



Most mathematical areas have concrete starting points (e.g., integers in number theory, points and lines in geometry, and so on), but the area that we will talk about today operates one level higher: it abstracts away the nature of the objects themselves and considers only how they interact. This extreme generality is what makes it the most abstract field of mathematics.

With that said, "abstract" doesn't always mean difficult. In mathematics, "abstraction" is the process of stripping away specific details to focus on the underlying structure, or the most essential ideas. The area that removes the most details in order to study the pure core of a concept could be said to be the most abstract one.

Category theory is powerful because it does precisely that. It reveals deep structural similarities between branches of mathematics that at first seem to be completely unrelated.



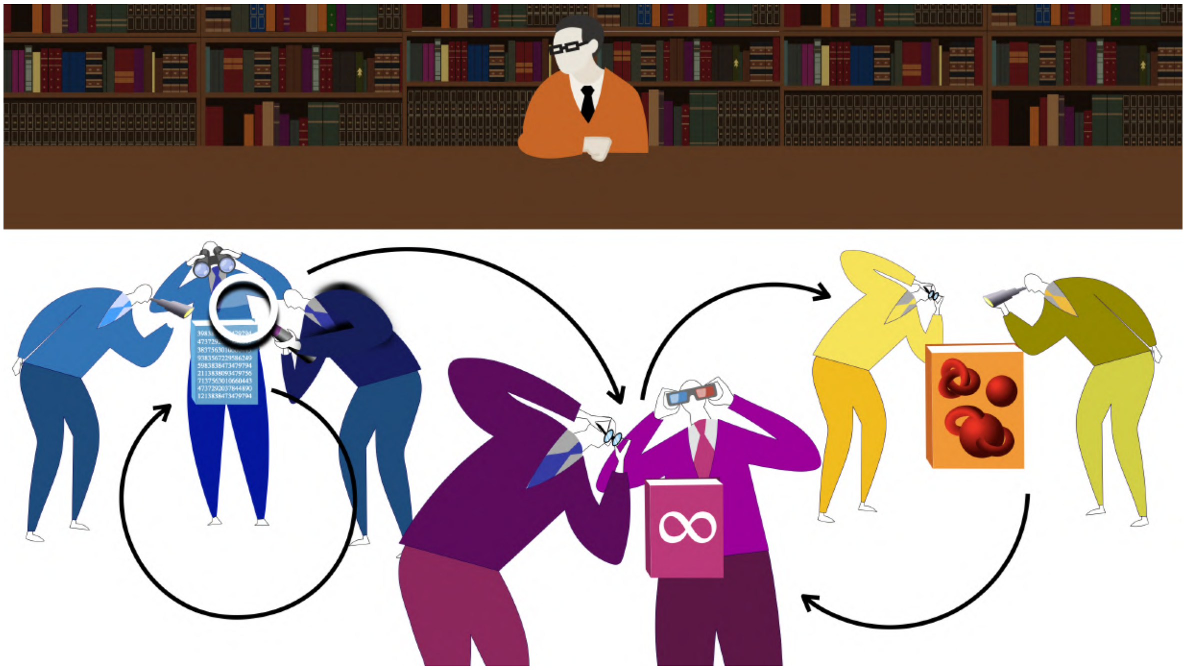
I like to think in terms of analogies.

Let's say you're walking into a massive library, but every book in this library is written in a different "language", and describes different worlds: one about *numbers*, another about *shapes*, a third concerns itself with *logic*, the fourth just focuses on *infinity* and so on.

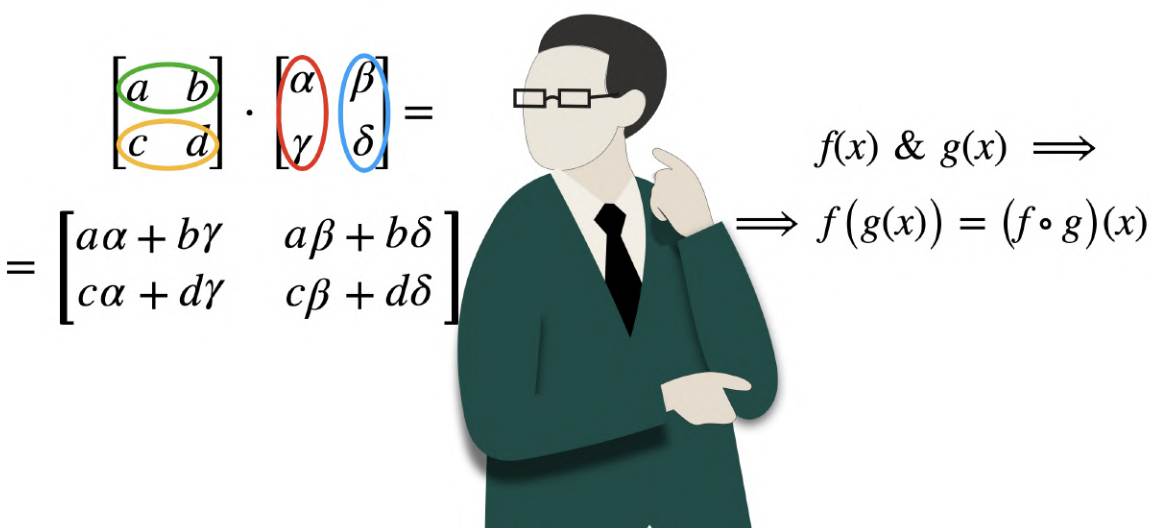
Most mathematicians who study in this library spend their entire lives exploring just one of these books.

But what about category theorists?

Well, they're the ones observing from above. They don't care so much about what the books say, but they do care about how the books relate.

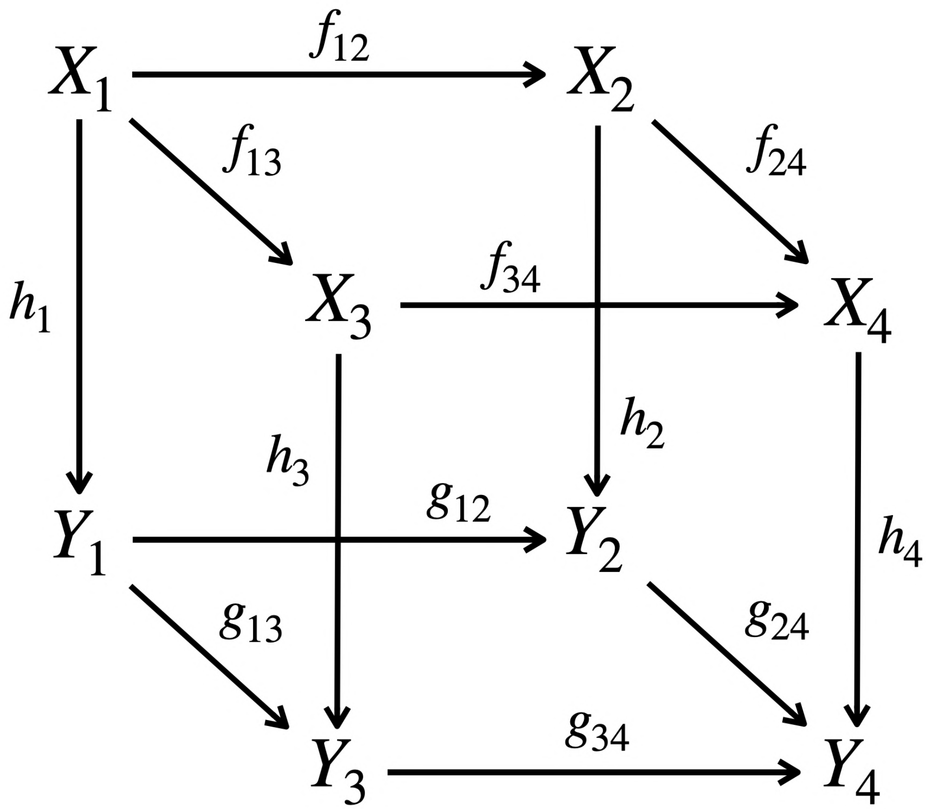


They notice that something about how you multiply matrices is weirdly similar to how you compose functions, which somehow reflects how you glue topological spaces together.



And they ask: *"what if those similarities aren't coincidences? What if they're clues of something deeper?"* This is category theory. It's a theory about relationships and because of that, it feels almost philosophical. In fact, it's often called the *"mathematics of mathematics"*.

Its language is very abstract as well. It doesn't give you pictures. It gives you diagrams made of arrows.



Of course, not every mathematician would agree with the fact that category theory is the most abstract of all fields of pure mathematics. Let us know your opinion in the comment section... but first read through this document, and open your mind to this possibility.

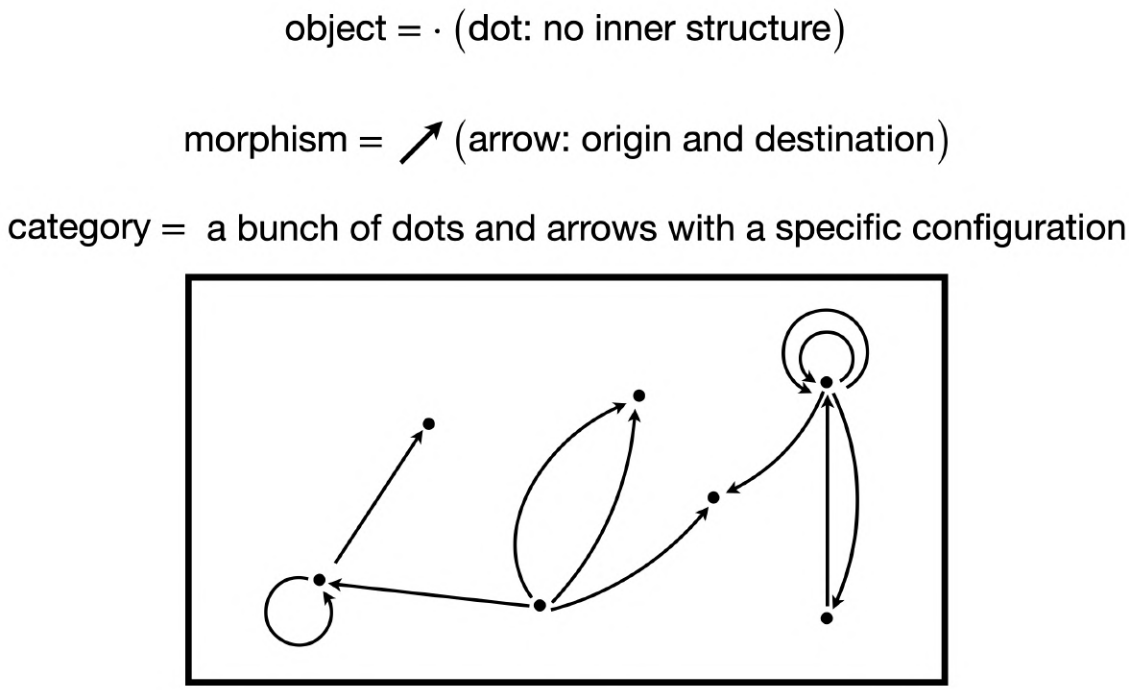
A Few (Loose) Definitions

Let's define a few terms (loosely, though):

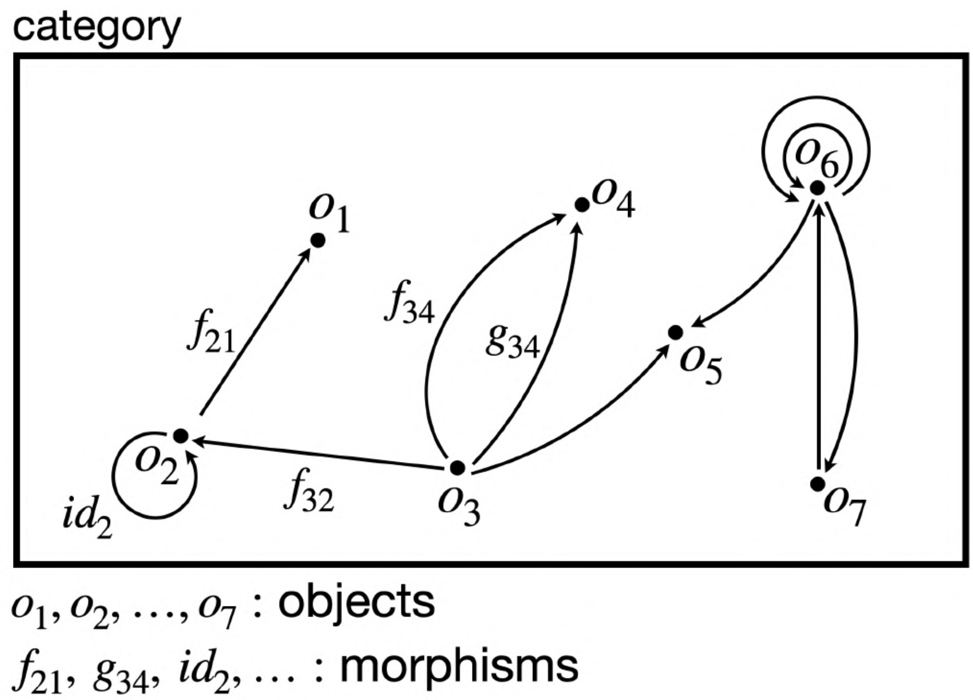
An **object** is a dot. It has no inner-structure.

Morphisms are arrows with an origin and a destination.

A **category** is a bunch of dots and arrows with a specific configuration.



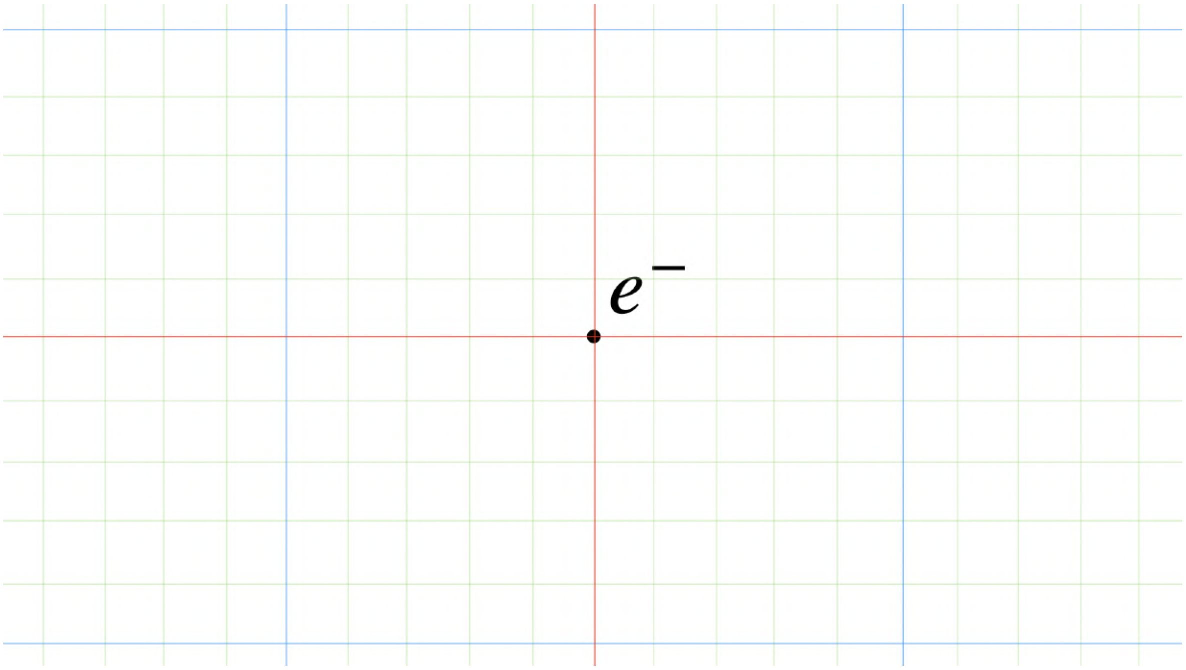
For example, this (below) is a category with objects o_1 , o_2 , . . . , o_7, and morphisms f_{21} , g_{34} , id_2, ...



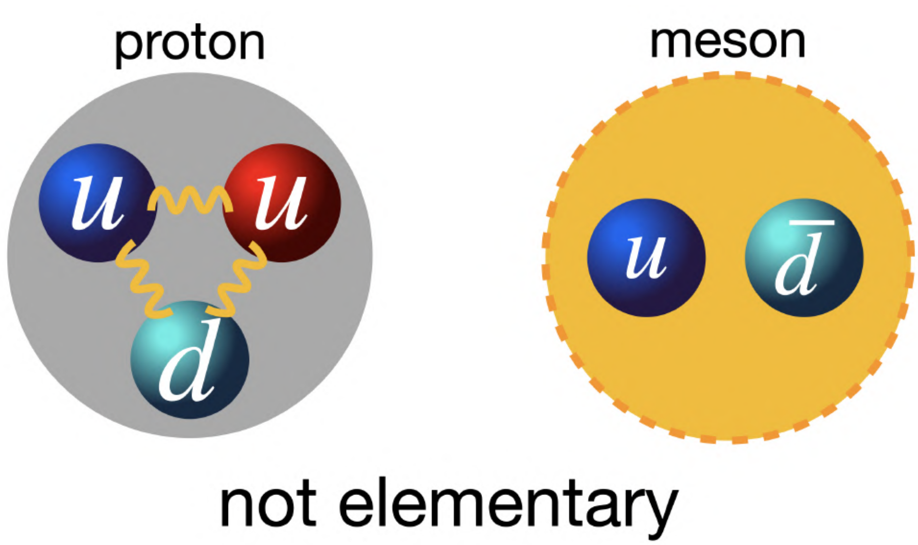
Now, if you are curious about what exactly is the definition of an object, you'll quickly run into some problems. So let's ask the question: *"What is the precise definition of an* ***object*** *in category theory?"*

First of all, many textbooks don't even define it… but you may encounter something like: *"an object is an abstract entity that serves as the domain or codomain of morphisms."* This is not very clarifying, but for a good reason.

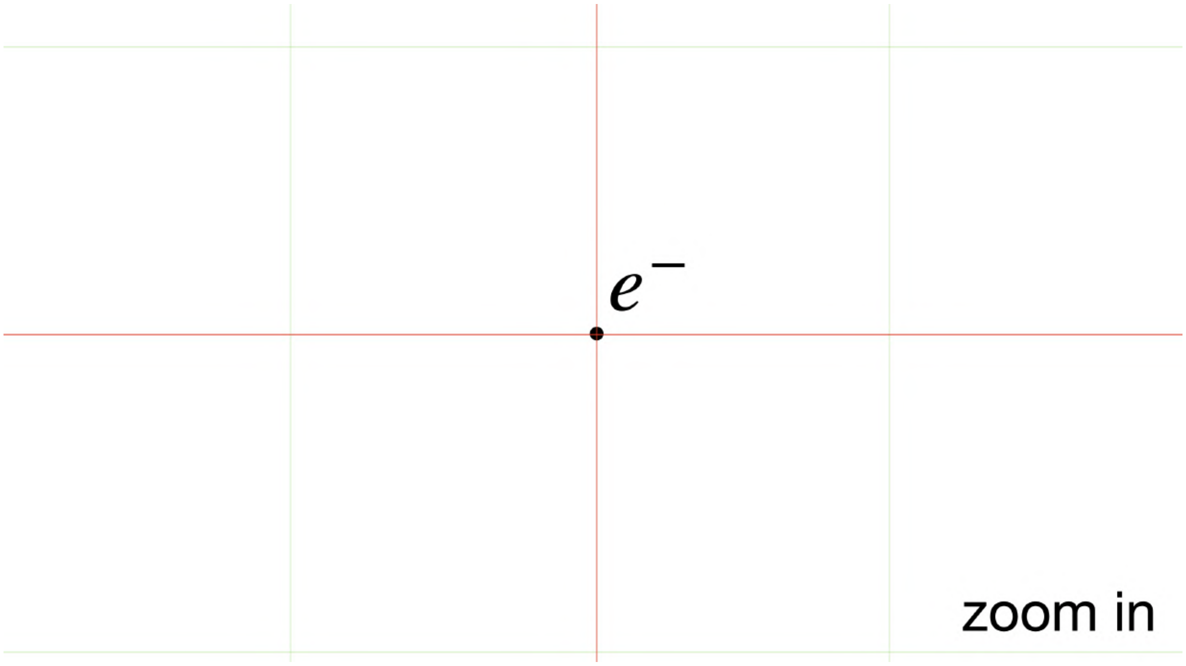
Defining an object is an awkward task because of how fundamental it is. I like to illustrate it with an analogy from physics. The analogy is not perfect, but it will do:



In nature, we have elementary particles, like the electron. They are called *elementary* because they have no internal structure. For example, a proton is composed of 3 quarks. A meson is composed of one quark and one antiquark.



These are not elementary particles. They are made of smaller pieces. But if you could zoom in on an electron, you'd still see just a little dot. That's why it's elementary.



Objects in category theory behave in a similar way. You can describe what an elementary particle does, how it behaves under electric and magnetic fields, how it interacts with other particles. But you can't open it up and look inside. A categorical object is just like that. It can only be described by its relationships with other objects, i.e. the web of arrows (morphisms) that go in and out of it.

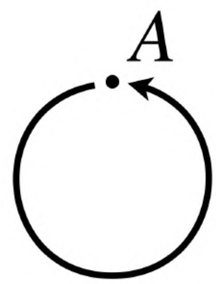
That's why and how category theory is said to be so abstract: it strips away the details of things and keeps only the structure between them.

Simple Examples

Enough of intuition, time to see a concrete example:

Category: Monoid

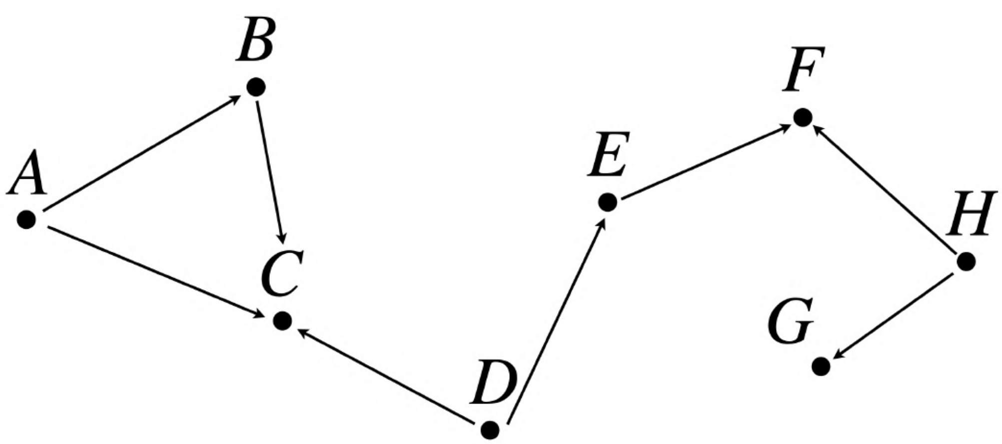
Just one object and its identity. This is the only morphism in here. Pretty simple.



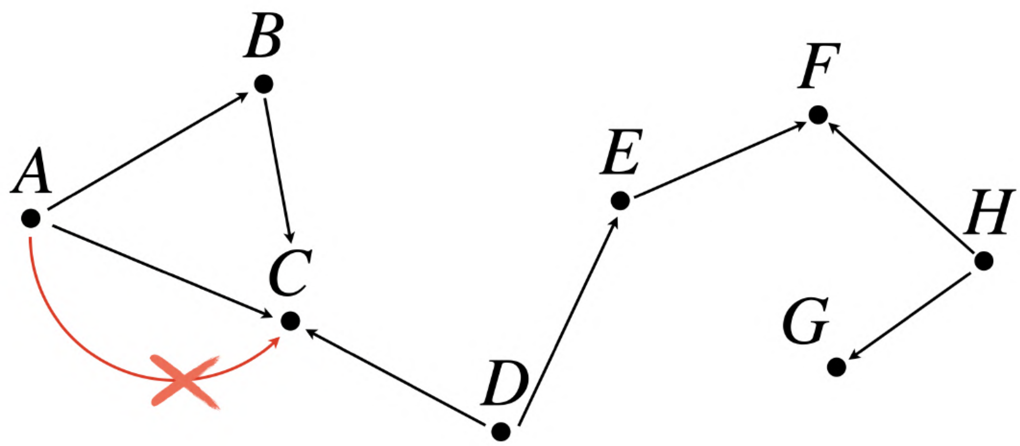
$$id_A : A \longrightarrow A$$

Poset category

There is at most one morphism between two objects in this category.

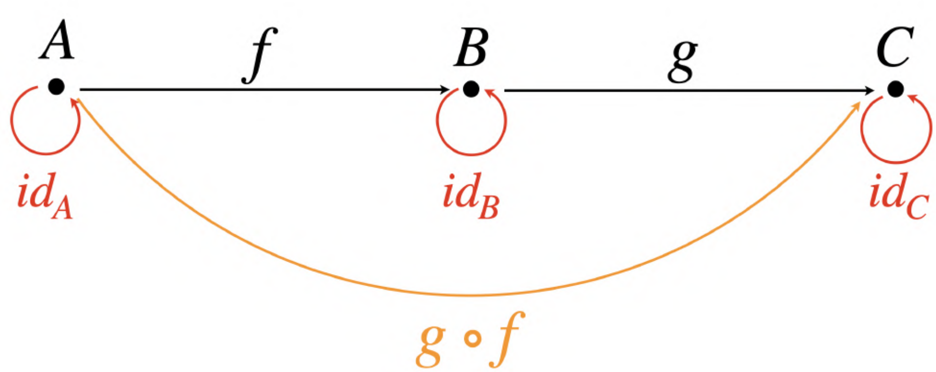


So this red arrow (shown below) would not be allowed here.

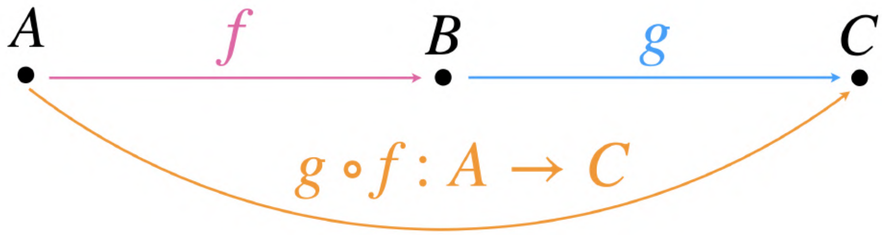


Poset category (again)

Another example of poset category (a simple one) would be a category with just 3 objects (A , B , C) and morphisms (f , g , id_A , id_B , id_C , $g \circ f$).

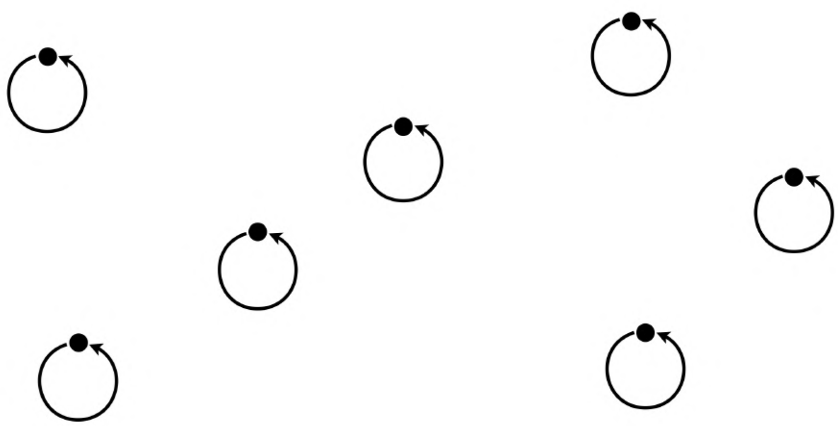


In fact, every time we have a situation like that, where two morphisms can be composed, their composition always exists. This is part of the definition of a category.

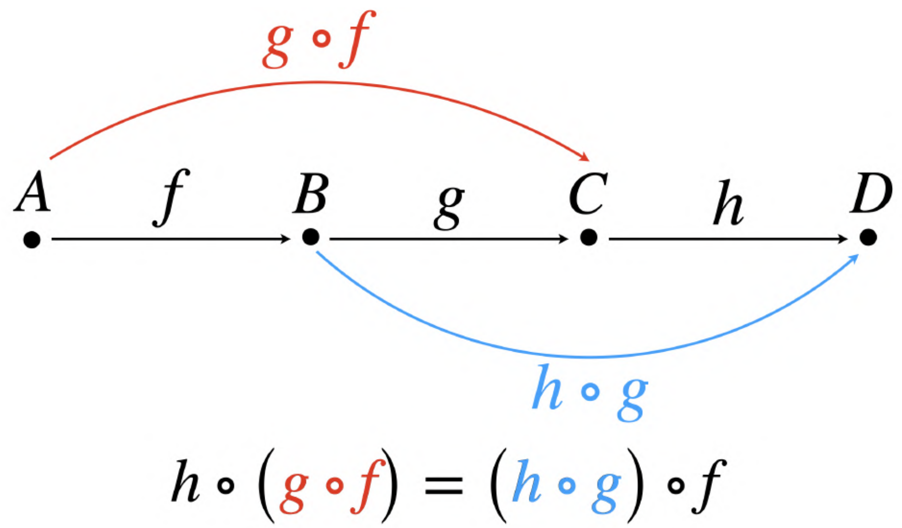


$$f : A \to B$$
$$g : B \to C$$
$$g \circ f : A \to C$$

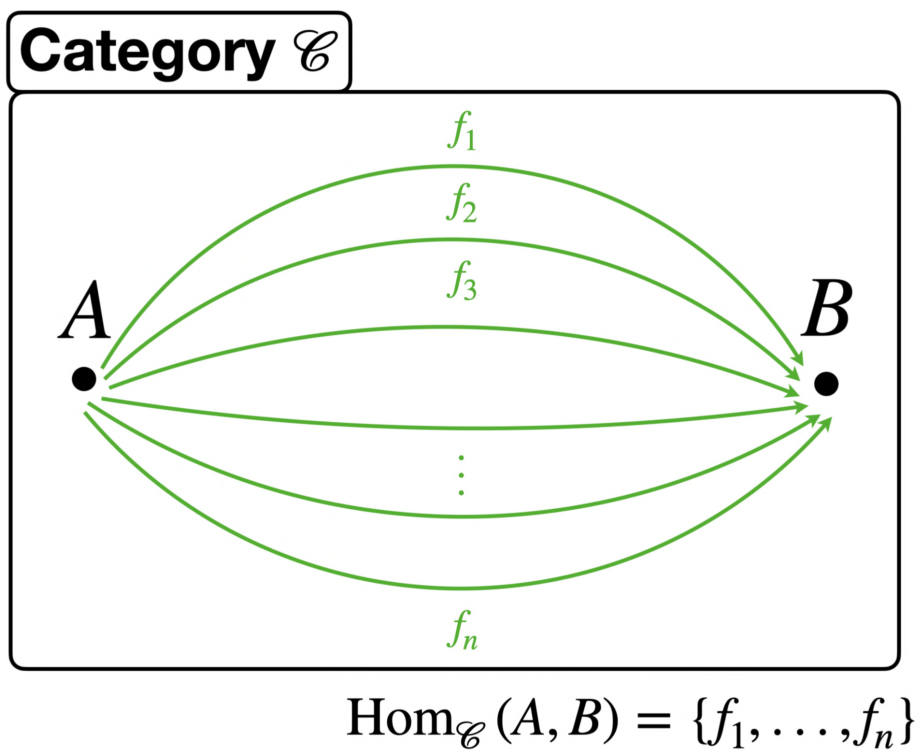
Another thing that's part of the definition of a category, is that every object always has an *identity morphism*.



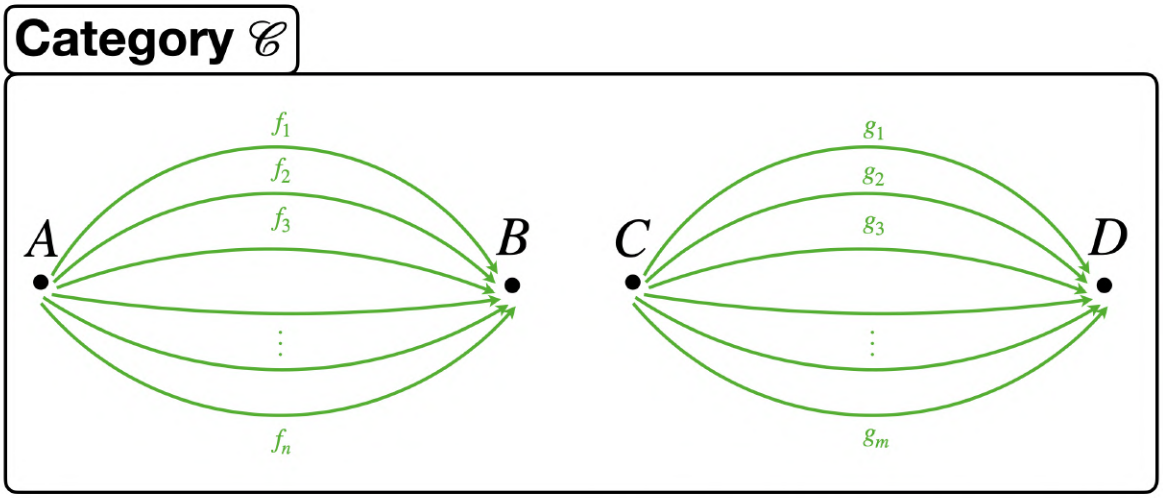
Every composition of morphisms is associative in a category.



The set of all morphisms $f_1, f_2, \ldots, f_n$ between two objects A and B in a category $\mathcal{C}$ is denoted this way $\mathrm{Hom}_{\mathcal{C}}(A, B) = \{f_1, f_2, \ldots, f_n\}$.



In a similar way, for all morphisms $g_1, g_2, \ldots, g_m$ between two other objects C and D in the same category, we write this $\mathrm{Hom}_{\mathcal{C}}(C, D) = \{g_1, g_2, \ldots, g_m\}$.

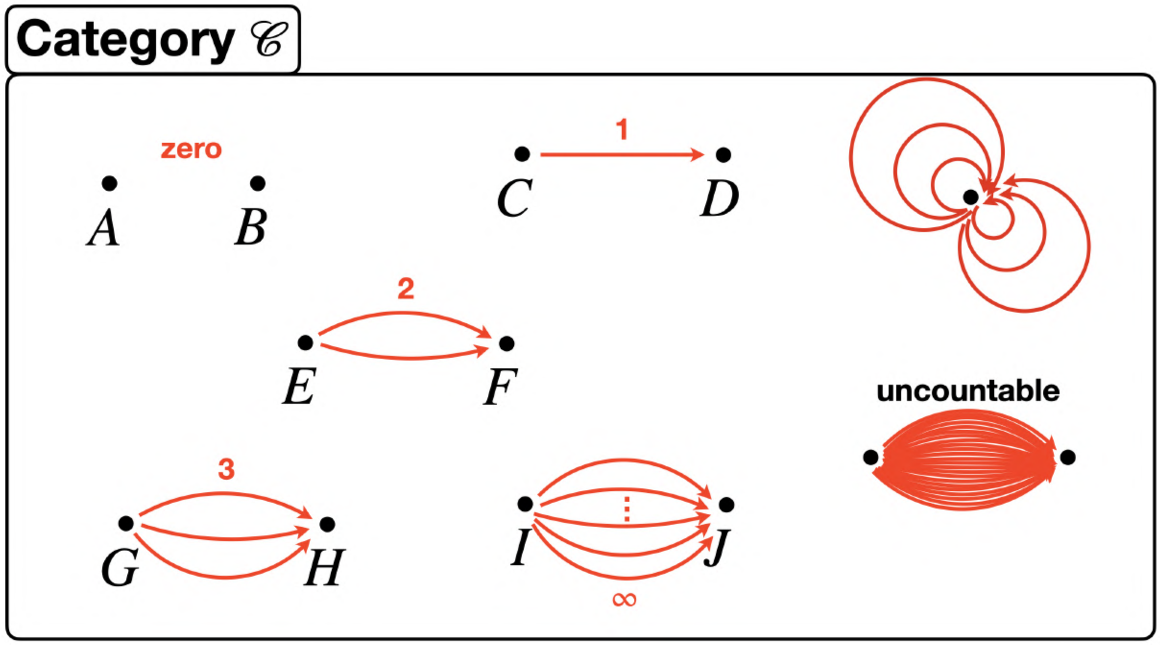


$$\mathrm{Hom}_{\mathscr{C}}(A, B) = \{f_1, \ldots, f_n\}$$

$$\mathrm{Hom}_{\mathscr{C}}(C, D) = \{g_1, \ldots, g_m\}$$

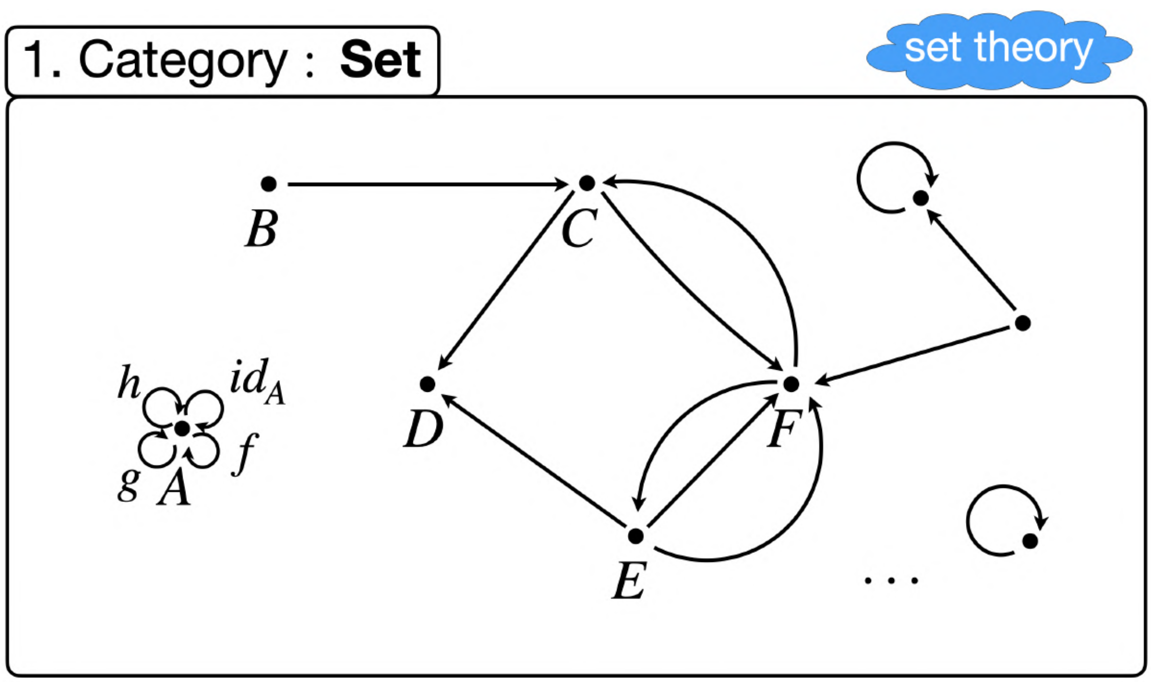
These are called *"Hom-sets"* for each pair of objects. The term *"Hom"* historically comes from *"homomorphism"*, which is a mapping that preserves structures in algebraic settings. We will explain what a homomorphism is later on. So don't worry about it yet. However, not all morphisms are homomorphisms in category theory, but the name *Hom-set* remained as a historical convention.

Another important thing to say is that there can be $0, 1, 2, 3$, or even *infinitely* many morphisms between objects. Indeed, we can go even further: there can be *uncountably* many morphisms between two objects. There can also be multiple morphisms from an object to itself.

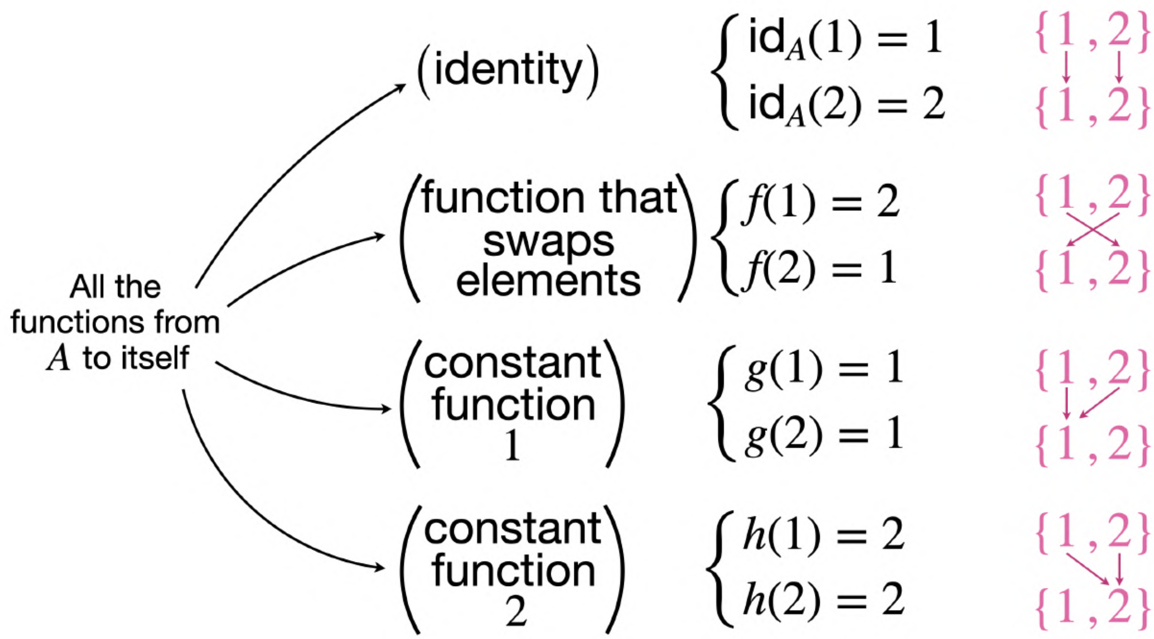


Let's see a few examples to illustrate these concepts:

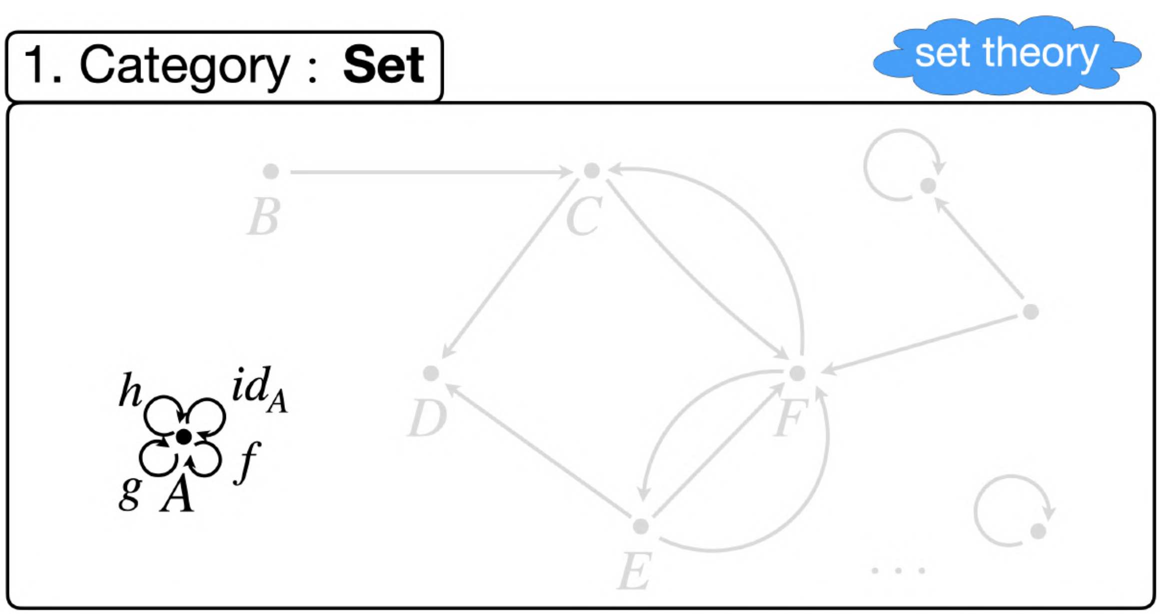
Category: Set (Set Theory)



This is the category called **Set**, where objects are all possible sets and morphisms are all possible functions between them. Let $A = \{1, 2\}$, then the $\text{Hom}_{\textbf{Set}}(A, A) = \{id_A,\ f, g, h\}$ is this set of functions:

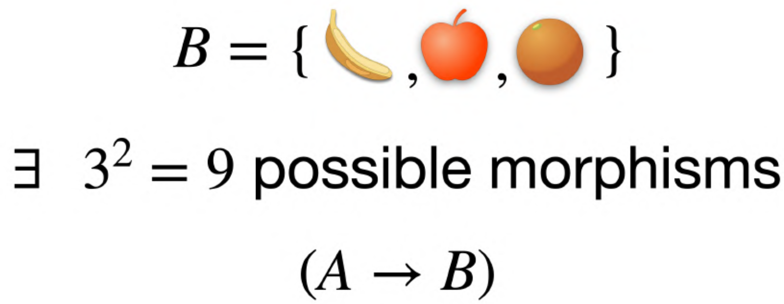


These are all possible functions from A to itself.

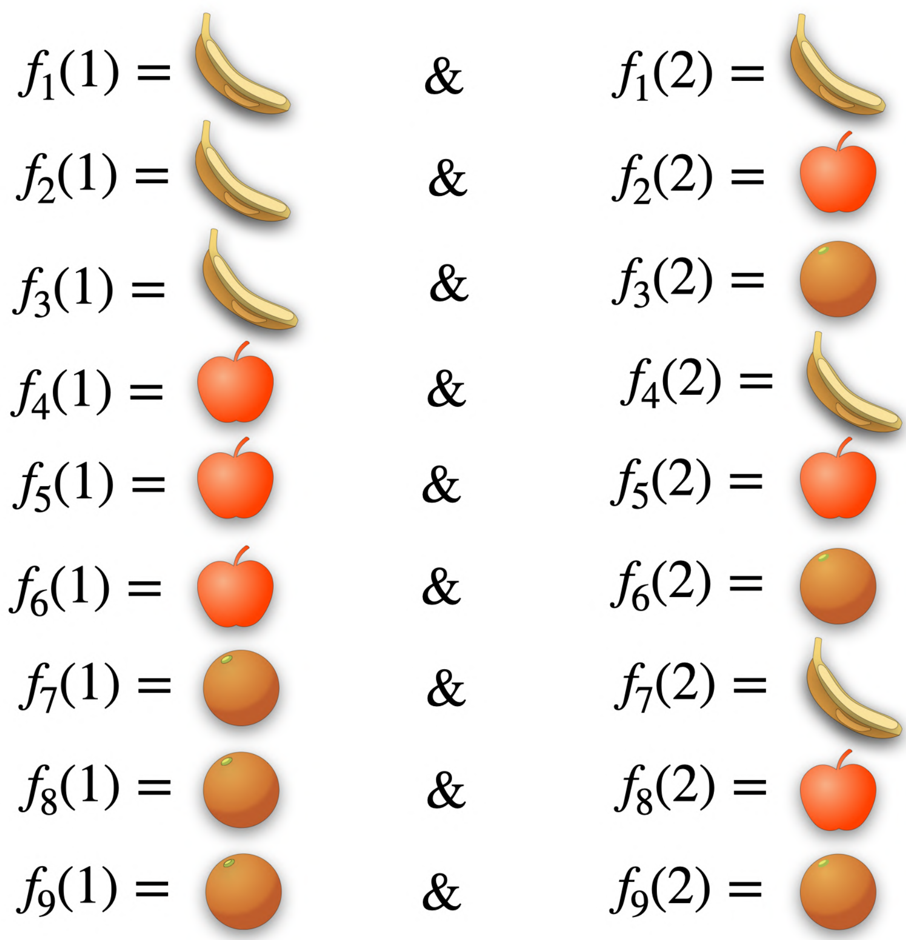


$\text{Hom}_{\textbf{set}}(A, A) = \{id_A, f, g, h\}$

If $B = \{\text{banana}, \text{apple}, \text{orange}\}$, then there are $3^2 = 9$ possible morphisms from A to B.



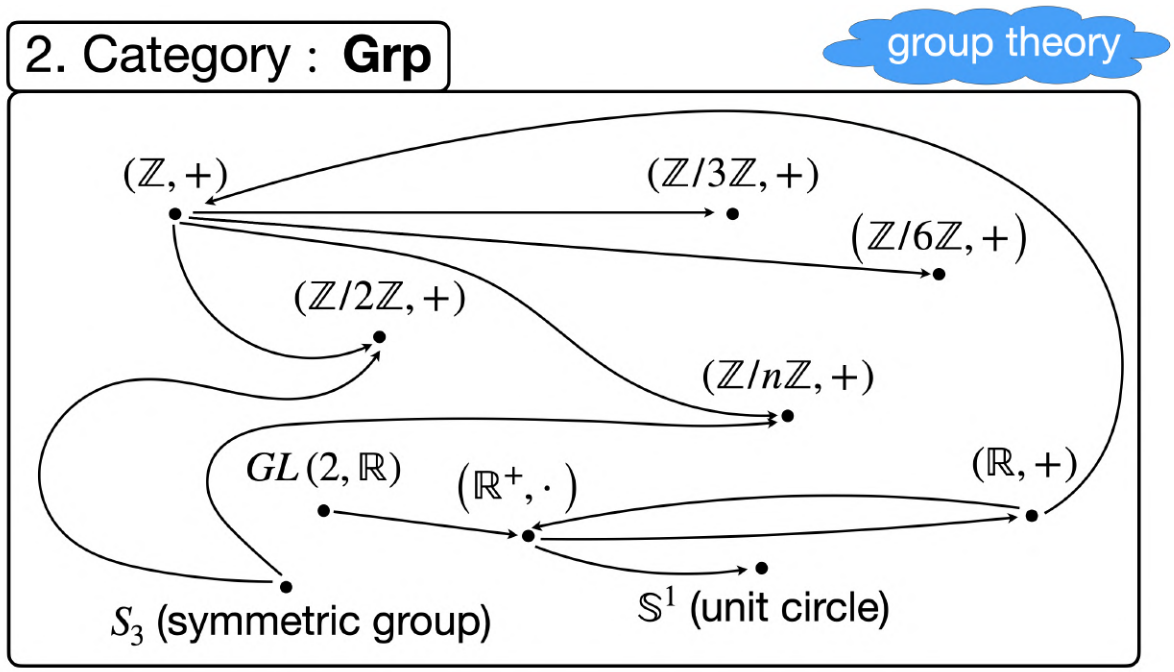
This is the list of all of them:



The *Hom-Set*, then, is expressed this way: (all functions from A to B. There are 9 of them)

$$\text{Hom}_{\textbf{Set}}(A,B) = \left\{ f : A \to B \mid f_n(m) \in B \ , \ n \in \mathbb{N} \cap [1,9] \ , \ m \in \{1,2\} \right\}$$
$$= \{ f_1, f_2, \ \dots \ , f_9 \}$$

Category: Grp (Group Theory)



This is the category called **Grp**, where objects are groups and morphisms are homomorphisms between them.

Let's recall that a group is a set of elements that represent "moves" you can do (like rotating a shape or walking steps forwards and backwards), such that you can combine them, or even "do nothing". Rigorously speaking:

Definition: A *group* is a set G with a binary operation $\cdot : G \times G \to G$ satisfying:

Associativity: $(ab)c = a(bc) \ , \ \forall a, b, c \in G;$

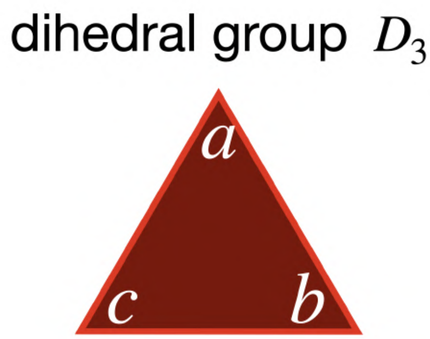
Identity: $\exists e \in G \, : \, ae = ea = a \, , \ \ \forall a \in G \, ;$

Inverse: $\forall a \in G \, , \ \ \exists a^{-1} \in G \, : \, aa^{-1} = a^{-1}a = e \, .$

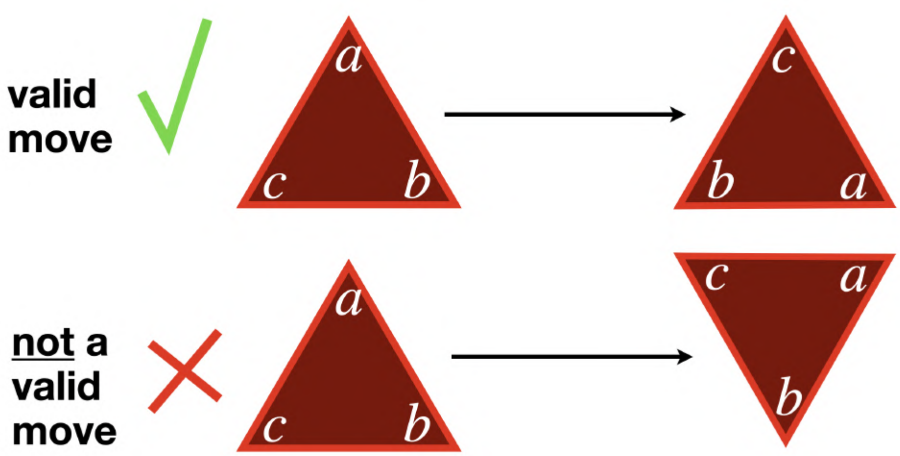
And what about **homomorphisms**?

Let's start with a simple example:

Consider the group of symmetries of an equilateral triangle. This is called the **dihedral group** and it's denoted as D_3.



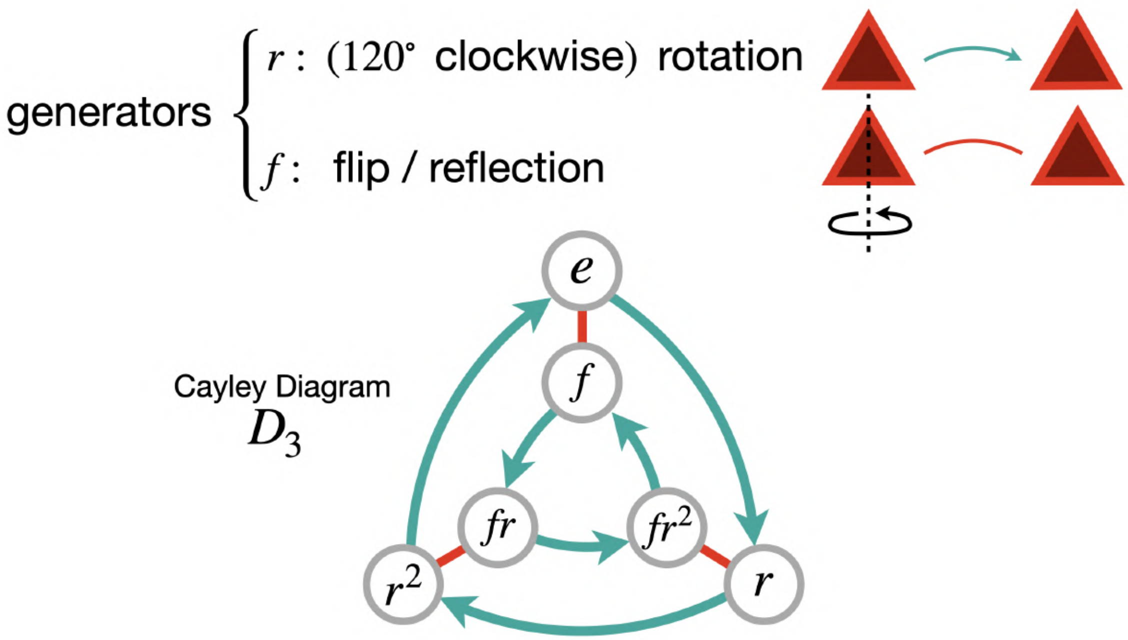
Each of its elements is a "move", or "action", that you can perform without changing the symmetry of the group. So, for example:



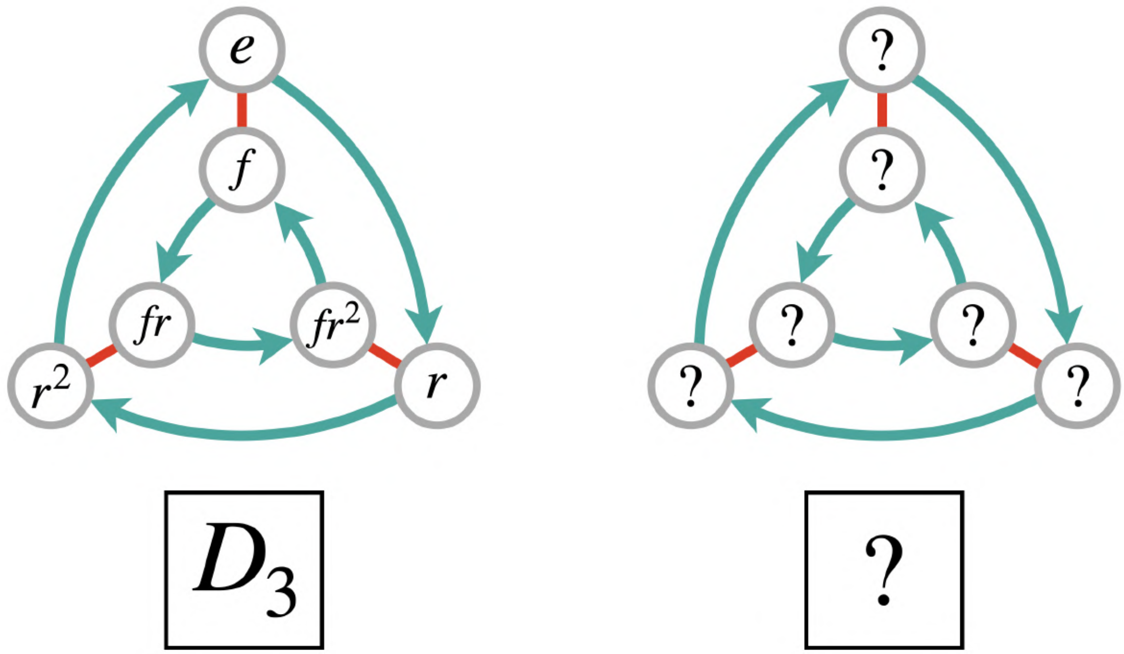
The symmetry is not preserved in the second case.

There are 2 generators of all possible moves in this group, i.e. any action you can think of is just a combination of those:

$$\begin{cases} r: \ (120^\circ \text{ clockwise}) \text{ rotation} \\ f: \ \text{flip / reflection} \end{cases}$$



It turns out that there is another group that is incredibly similar to this one. And these similarities can be best noticed when drawing their Cayley diagrams:

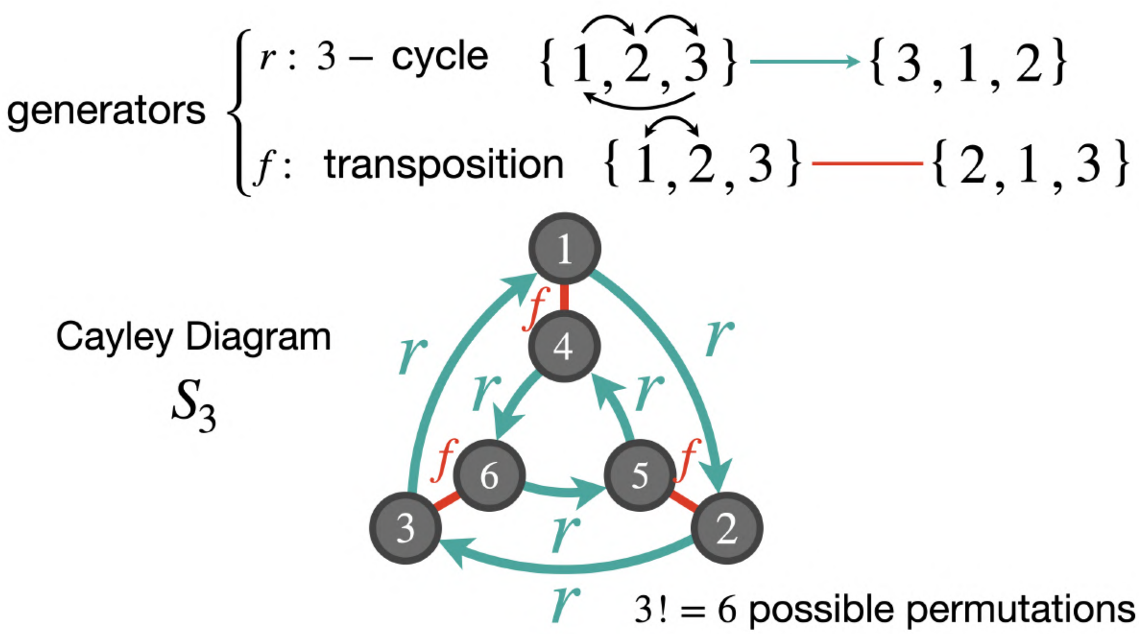


This is the group of symmetries of a list of 3 elements $\{1, 2, 3\}$, where the order matters. It's called *symmetric group on 3 elements* S_3. Each of its elements is a "move", or "action", that you can perform without changing the symmetry of the group, i.e. without changing the fact that there are 3 listed elements in a specific order. So, for example:

The symmetry is not preserved in the second case.

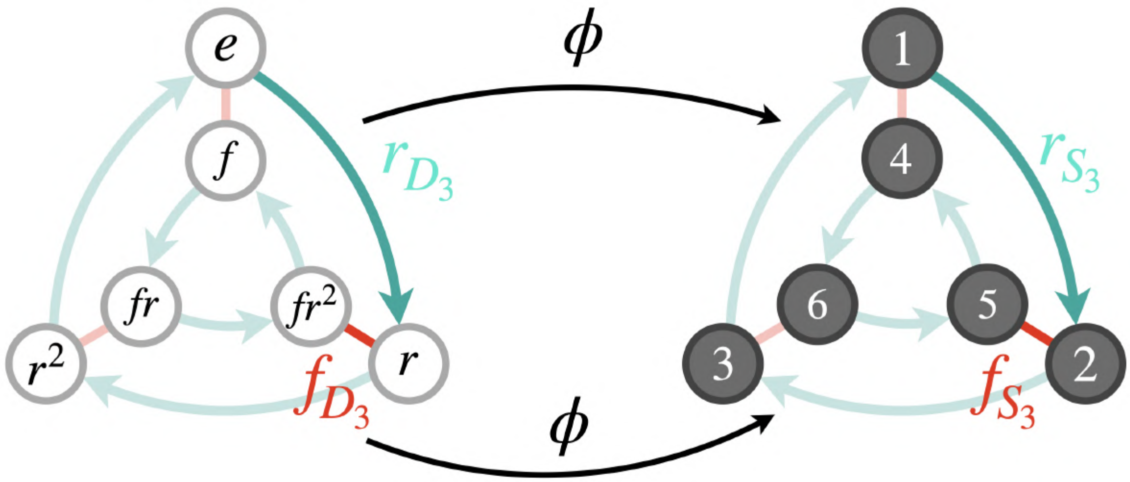
There are 2 generators of all possible moves in this group, i.e. any action you can think of is just a combination of those:

$$\begin{cases} r: \text{ 3-cycle} \\ f: \text{ transposition} \end{cases}$$



All of this to say that this intuitive similarity between them can be formalized with the concept of a homomorphism, which is a way of "translating" moves from one group to another.

In our case, there is a homomorphism ϕ that "translates" rotations and flips in D_3 into 3-cycles and transpositions in S_3, respectively.



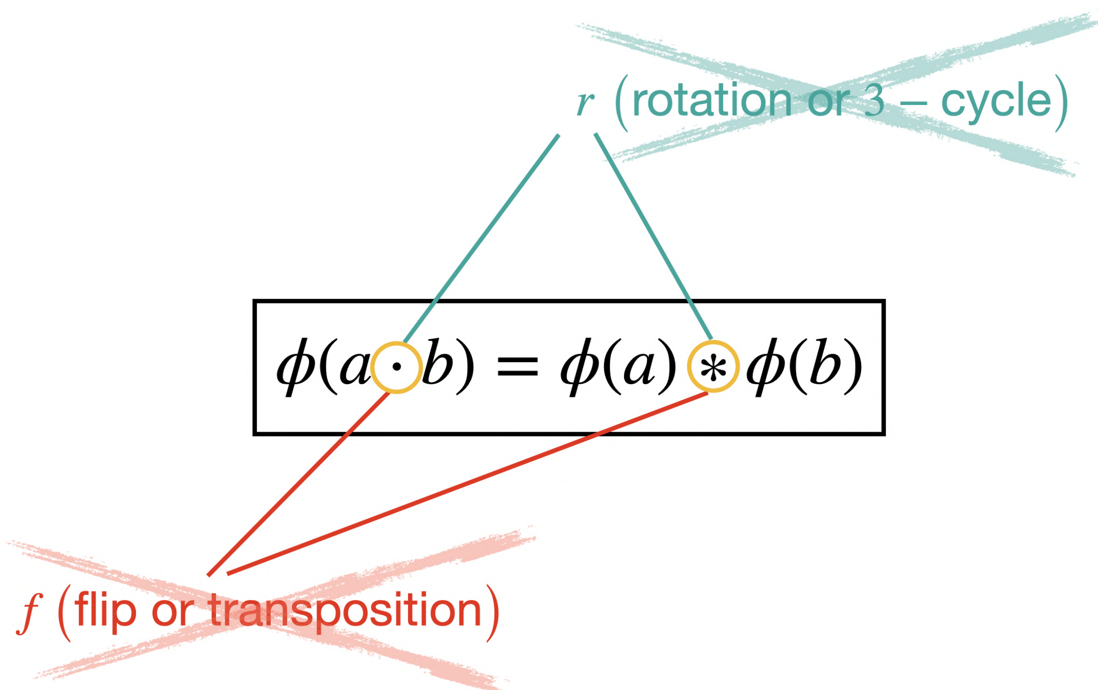
Therefore, here's the rigorous definition of a homomorphism:

Let $(G\,,\,\cdot)$ and $(H\,,\,*)$ be groups G and H with operations $\cdot$ and $*$, respectively. A group homomorphism is a function $\phi : G \to H$ such that, $\forall a, b \in G$:

$$\boxed{\phi(a \cdot b) = \phi(a) * \phi(b)}$$

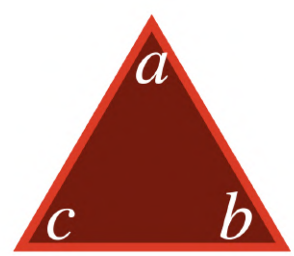
In our previous example, the homomorphism ϕ translates the moves r and f from D_3 to S_3.

A common confusion for people when learning these things for the very first time is that they look at the equation $\phi(a \cdot b) = \phi(a) * \phi(b)$ and may think that $\cdot$ and $*$ are the moves r (rotation or 3-cycle) and f (flip or transposition) that we've seen before.



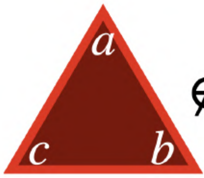
This is not the case, because the elements of a group are not the “thing” about which we study all the symmetries (i.e. not the triangle or the list themselves).

$$a, b \neq$$



$$\phi(a), \ \phi(b) \ \neq \ \{1, 2, 3\}$$

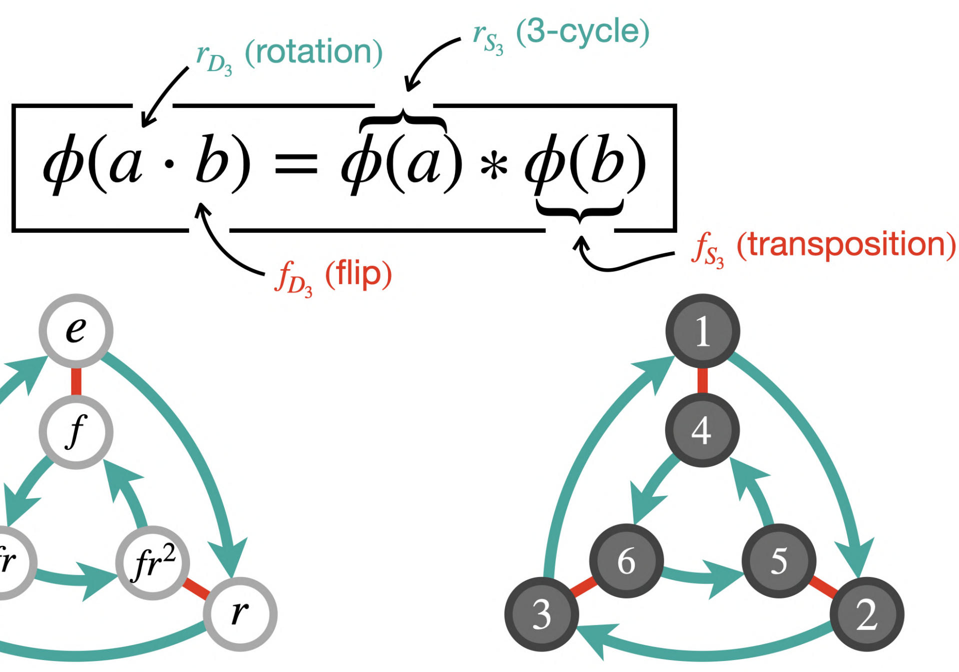
The elements of a group are the moves one applies on these “things”.



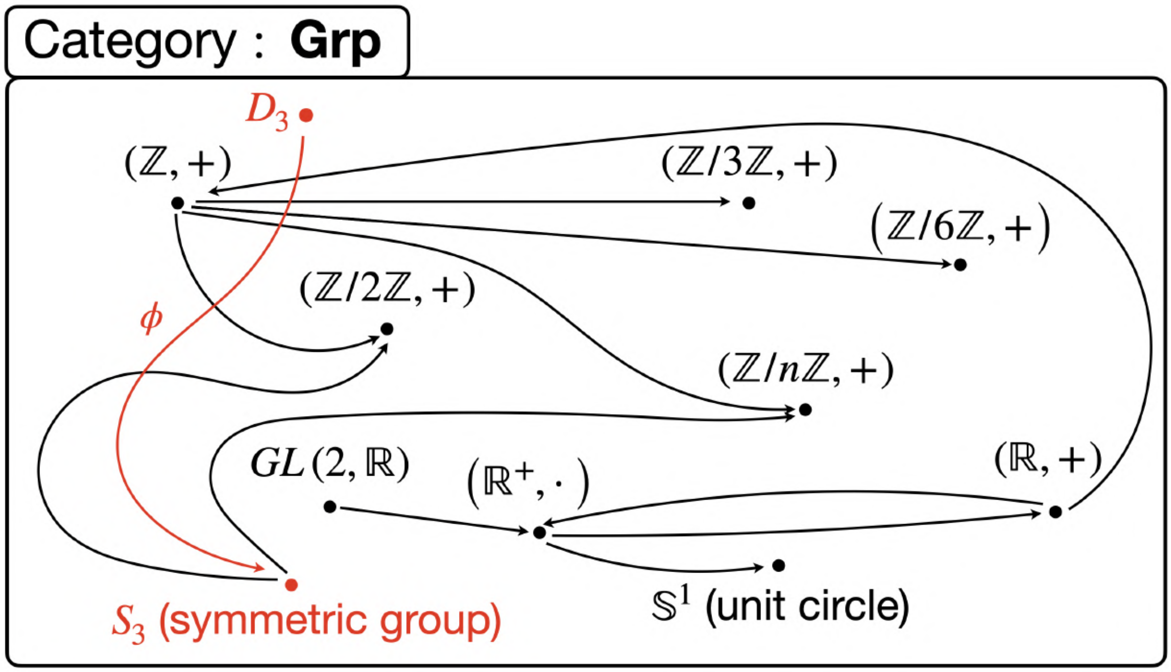
$\notin D_3$, but $\overbrace{r_{D_3}, f_{D_3}, (rf)_{D_3}, (r^{-1})_{D_3} \ldots}^{a,\, b} \in D_3$

$$\{1, 2, 3\} \notin S_3, \text{ but } \overbrace{r_{S_3}, f_{S_3}, (rf)_{S_3}, (r^{-1})_{S_3} \ldots}^{\phi(a),\, \phi(b)} \in S_3$$

So, if in the equation $\phi(a \cdot b) = \phi(a) * \phi(b)$ the left-hand side represents D_3 and the right-hand side represents S_3, we then have, for example that a could be rotation, b could be flip, $\phi(a)$ 3-cycle and $\phi(b)$ transposition.

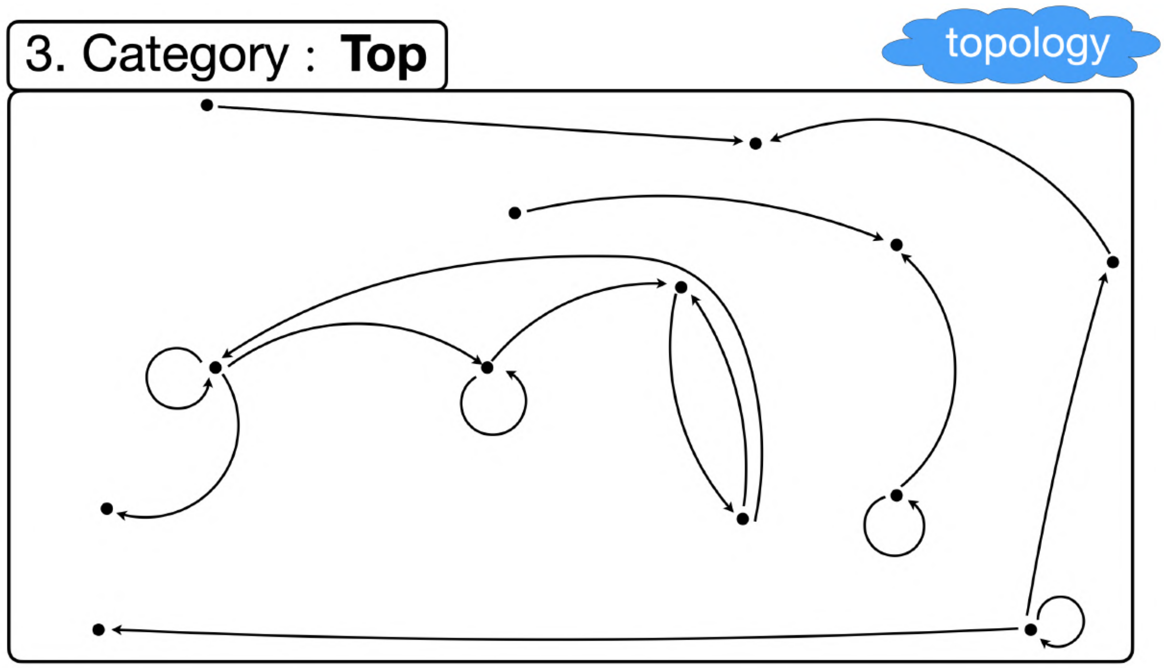


Going back to the category *Grp*, all this internal structure of D_3 and S_3 is ignored under the lens of category theory, and in the diagram representation here they become just dots (since they are objects) with an arrow ϕ (which is the morphism between them, in this case also a homomorphism):



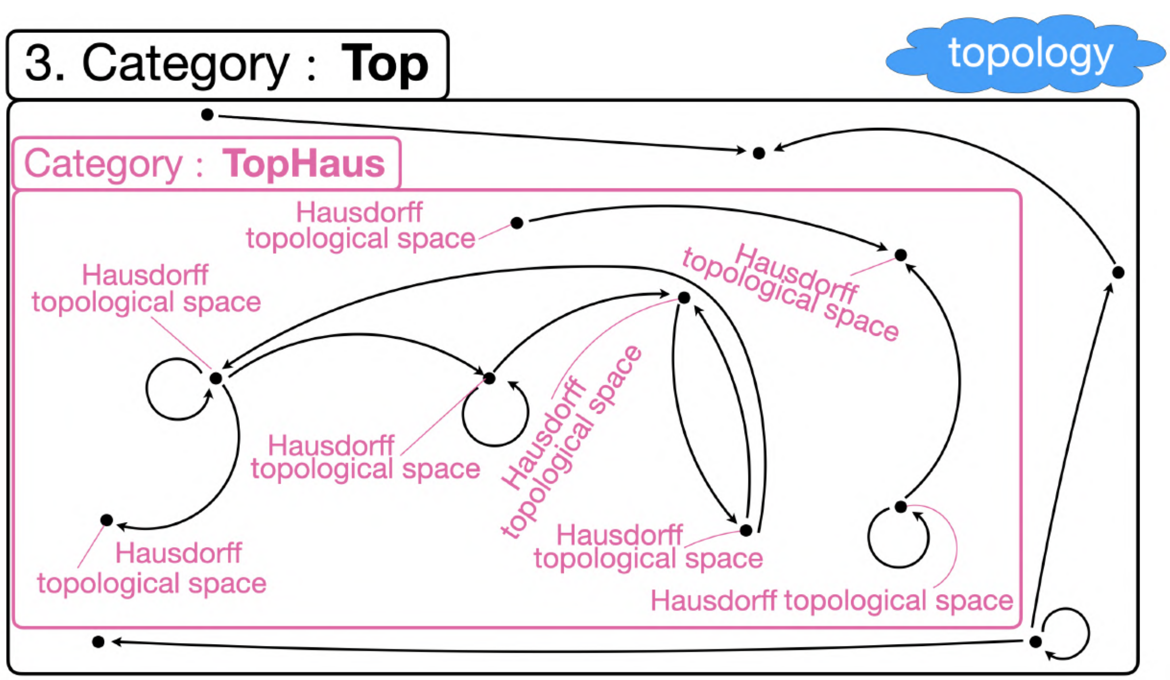
There are, of course, many other examples of objects (i.e. groups) in this category that are homomorphic to each other.

Category: Top (Topology)

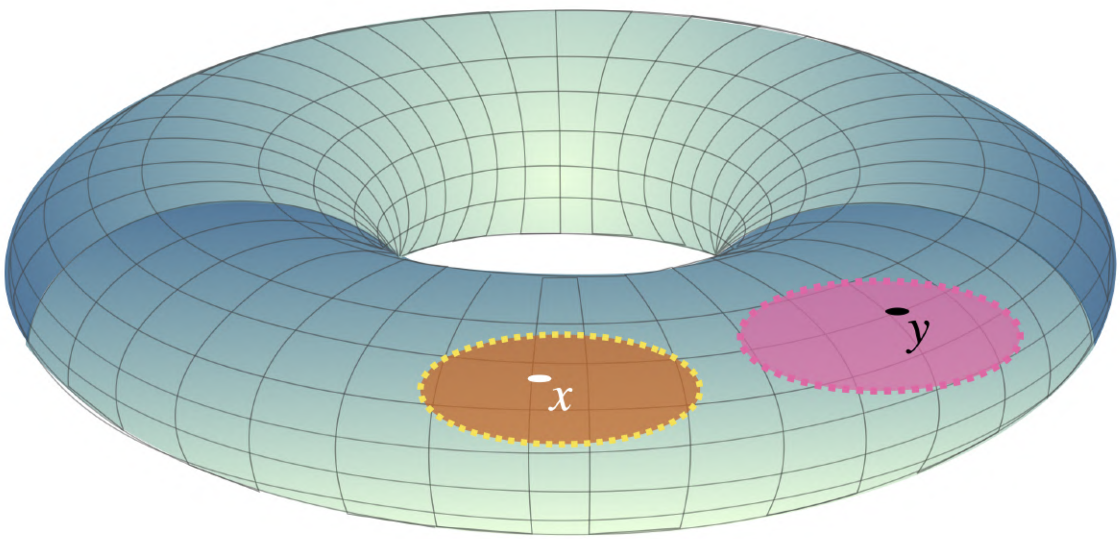


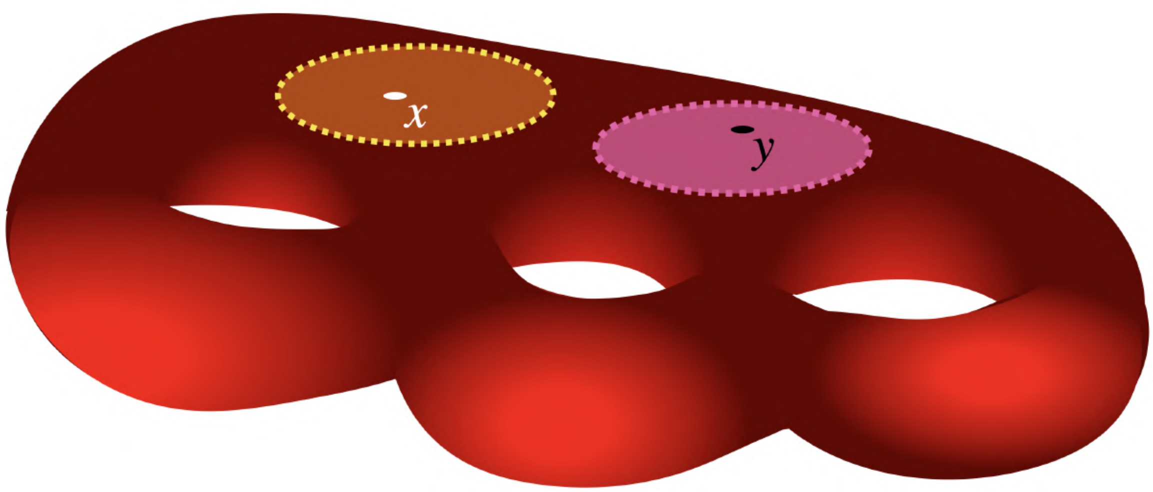
In this category, objects are topological spaces and morphisms are continuous maps between them.

Now, this category has a subcategory called **TopHaus**, which is a category in its own right.



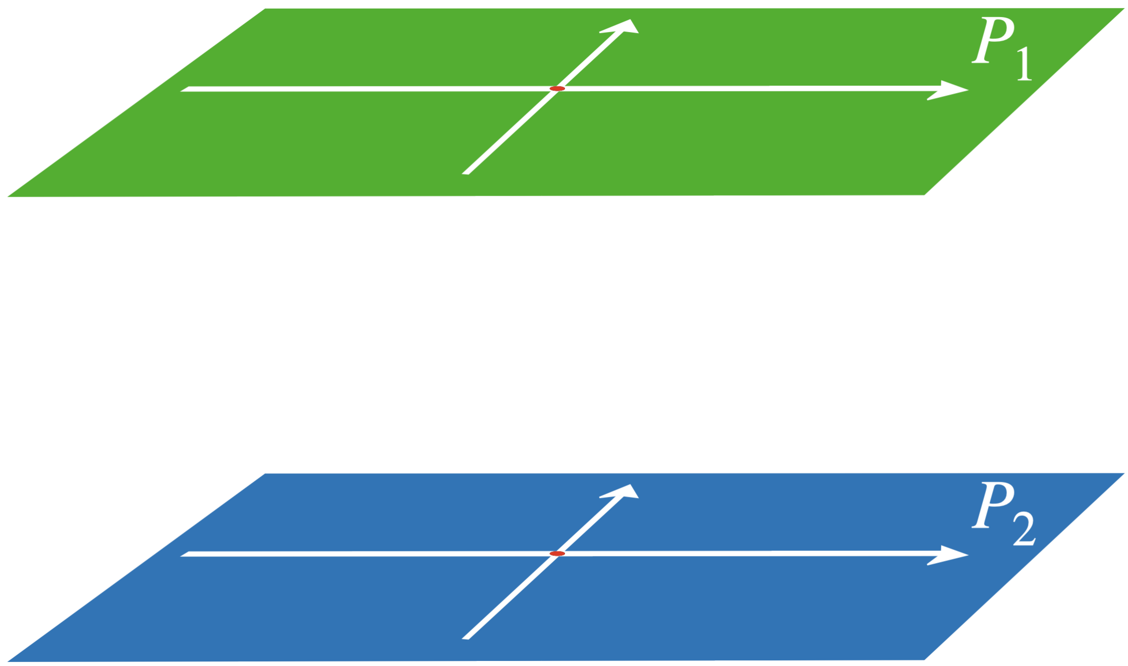
It is the category in which the objects are topological spaces that are also *Hausdorff*. We will not get into the details of point-set topology here, but a Hausdorff space is a topological space where any two distinct points can be separated by disjoint open sets.



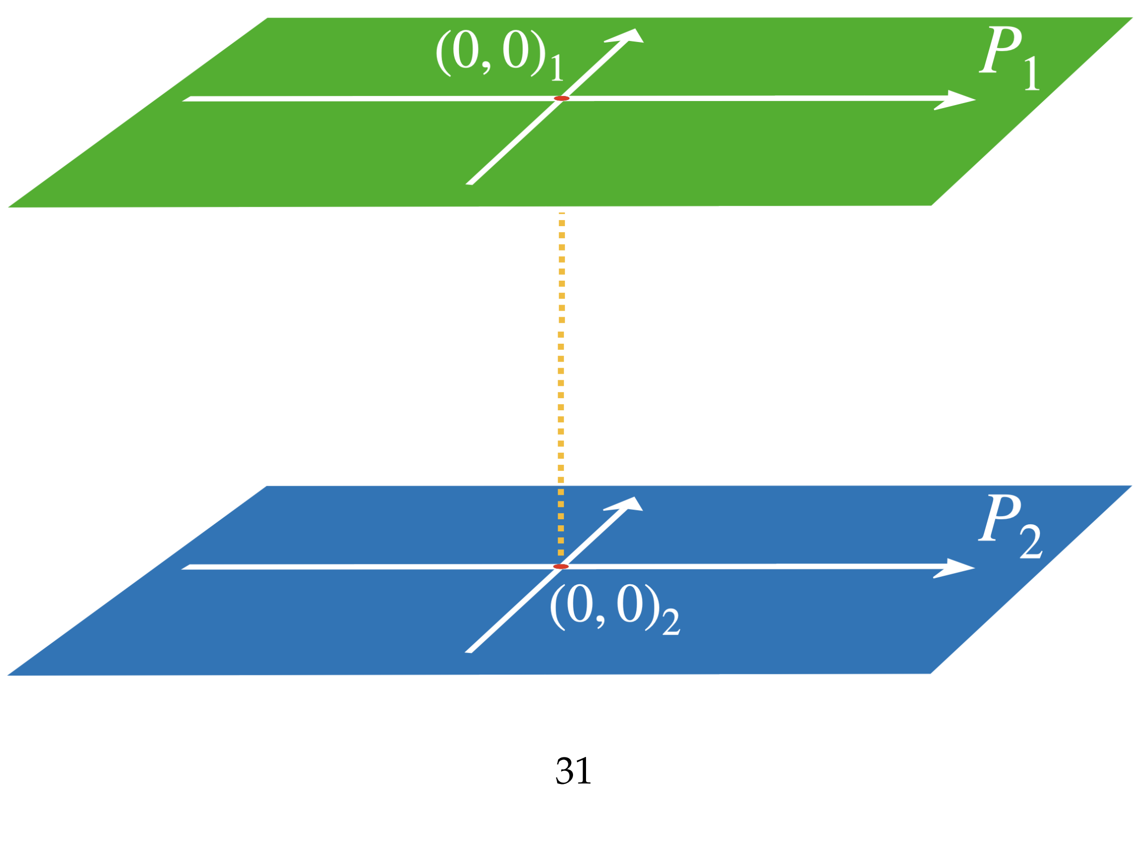


I found out that people can better understand (and appreciate) the fact that a space is Hausdorff by first grasping the idea of a *non-Hausdorff* space. Let's see an example:

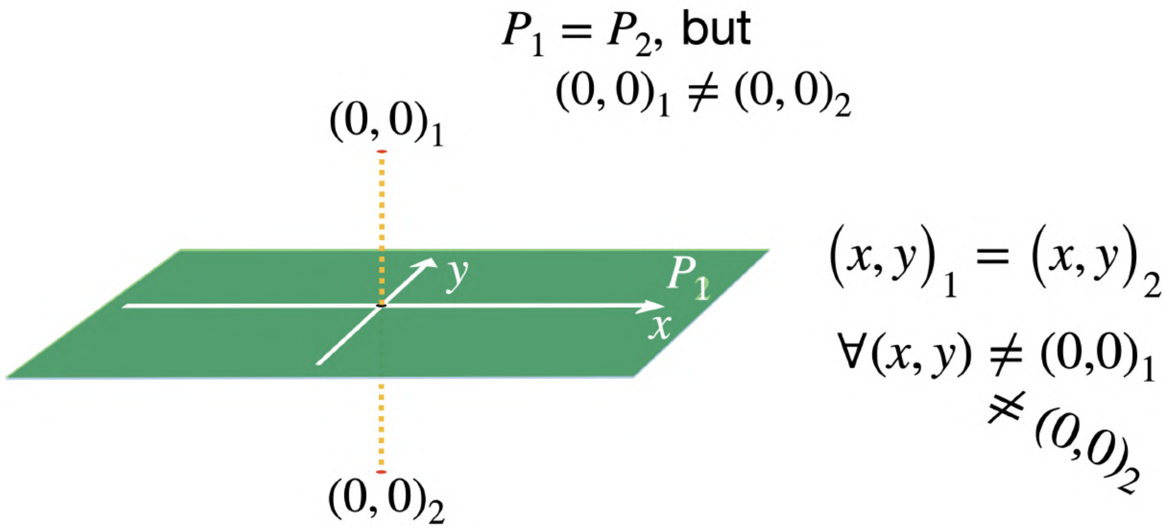
Imagine two copies of the plane $\mathbb{R}^2$, say P_1 and P_2.



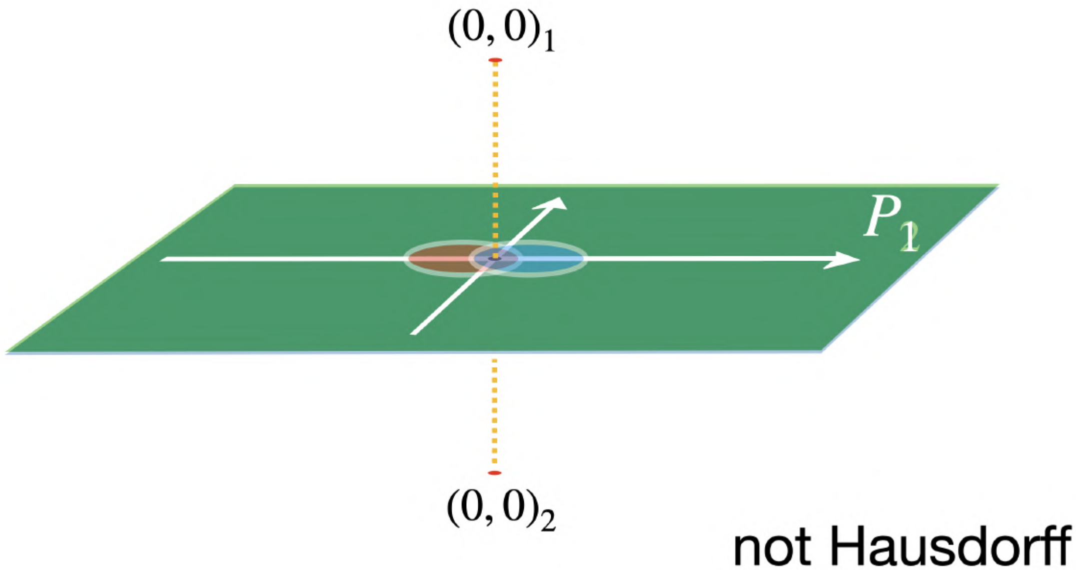
Now, glue them together along every point except the origins (let's call them $(0\,,\,0)_1$ and $(0\,,\,0)_2$).



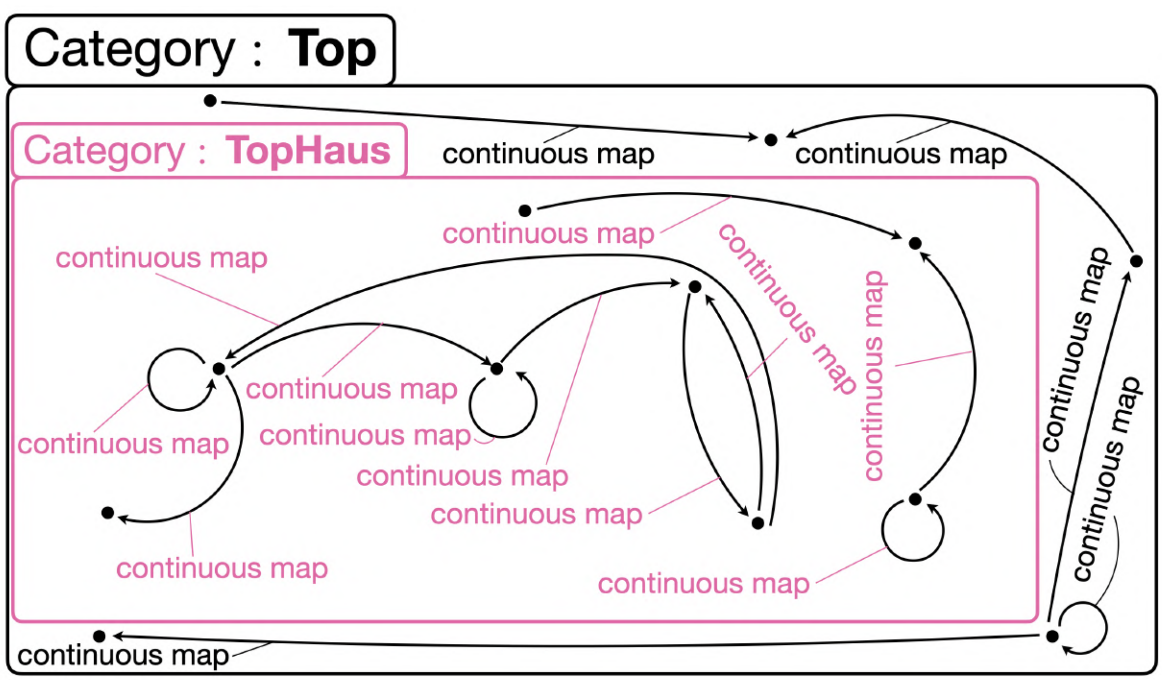
That means all points $(x, y) \neq (0, 0)$ in P_1 and P_2 are identified (i.e. they're the same now). But the origins are not: $(0, 0)_1 \neq (0, 0)_2$. They are kept as separate points.



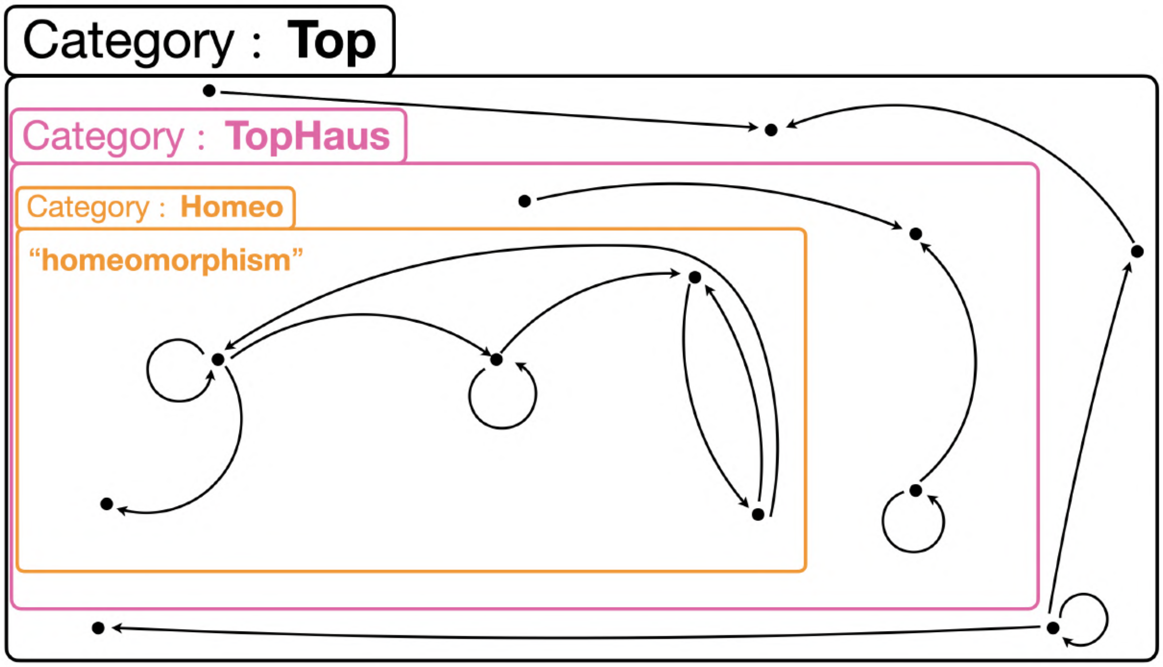
What happens if we try to separate $(0, 0)_1$ from $(0, 0)_2$? We can create two open disks (or neighborhoods): one around $(0, 0)_1$ and another one around $(0, 0)_2$. The problem is, no matter how small the disks are, they will always share all the same points except the origins. So, the disks always overlap! This makes it impossible to separate $(0, 0)_1$ and $(0, 0)_2$ with non-overlapping open sets. I.e. the space is not Hausdorff.



The morphisms of the category **TopHaus**, connecting Hausdorff spaces, are continuous maps just as the morphisms in **Top**. Literally, the only difference is that this time these morphisms are restricted to associating only objects that are Hausdorff spaces.

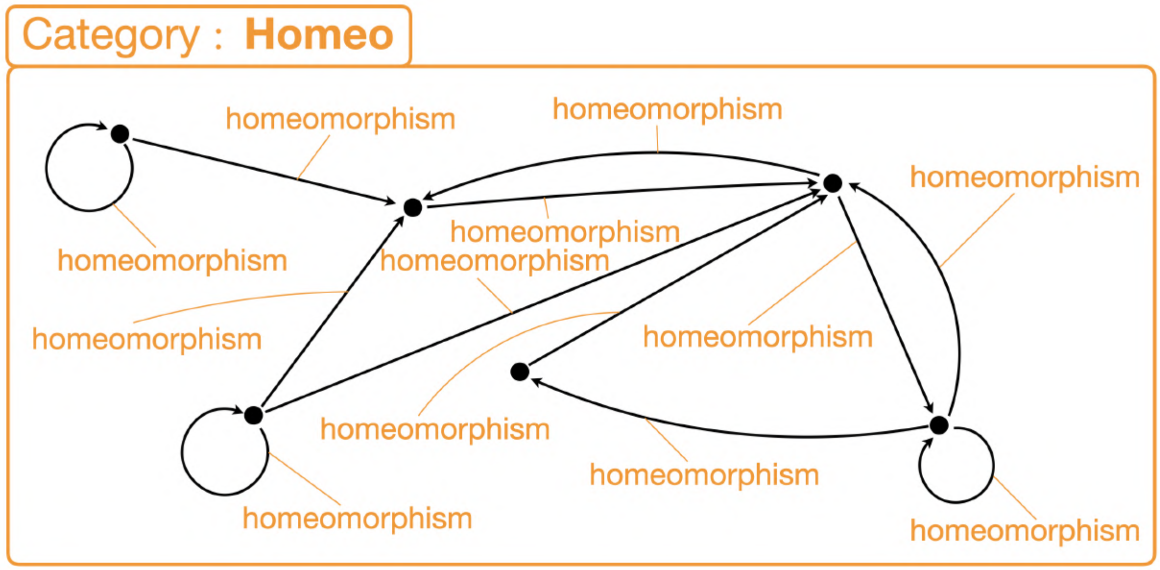


Interestingly, the category **TopHaus** has a subcategory as well, called **Homeo**, which abbreviates the word *"homeomorphism"* and is a category of its own.



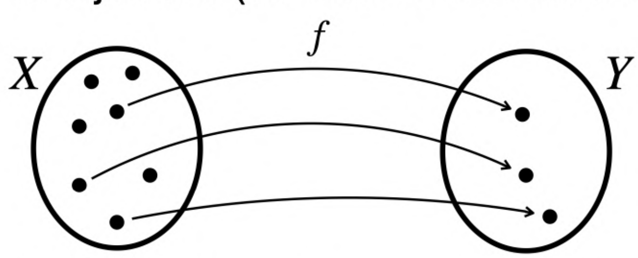
In this category, objects are also Hausdorff topological spaces (like spheres, tori (with genus $1, 2, \ldots, n$), Möbius strips, and so on...).

Morphisms though are homeomorphisms now.

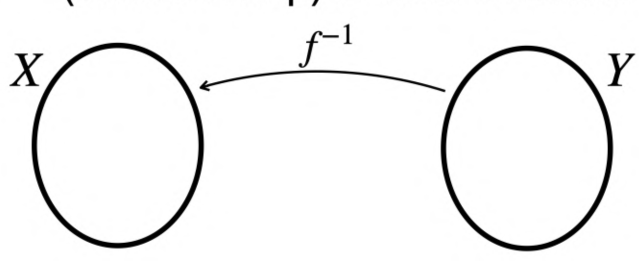


Definition : A homeomorphism is a continuous map $f: X \to Y$ between topological spaces $(X \text{ and } Y)$ such that

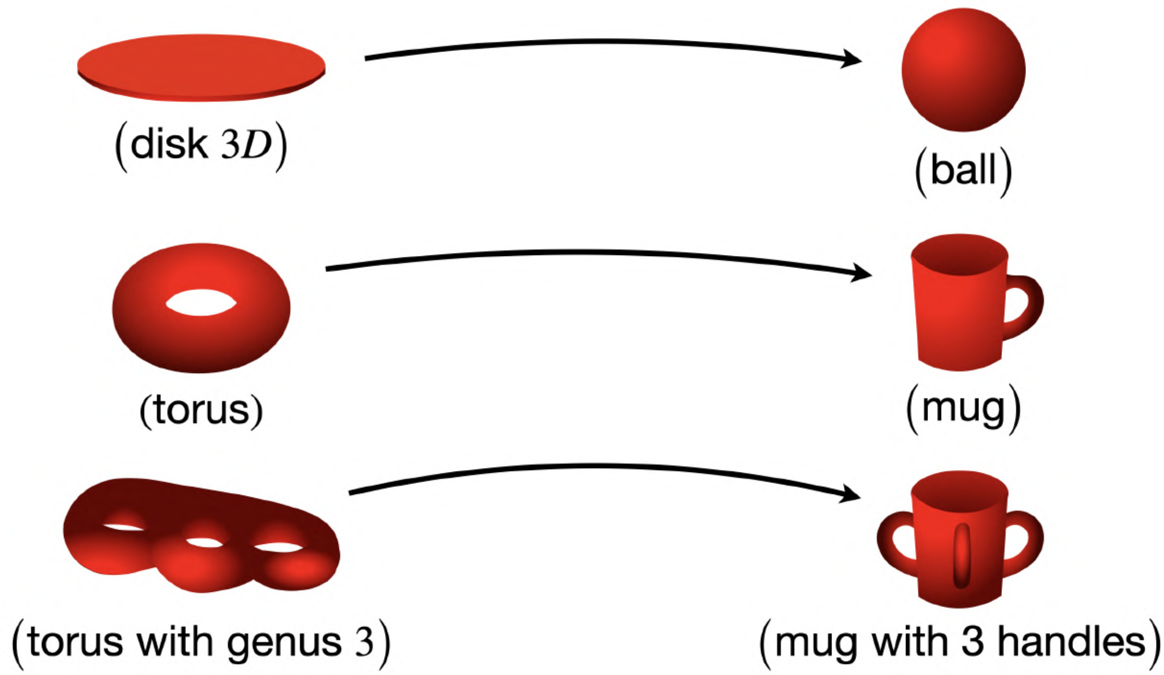
★ f is bijective (so one-to-one and onto)



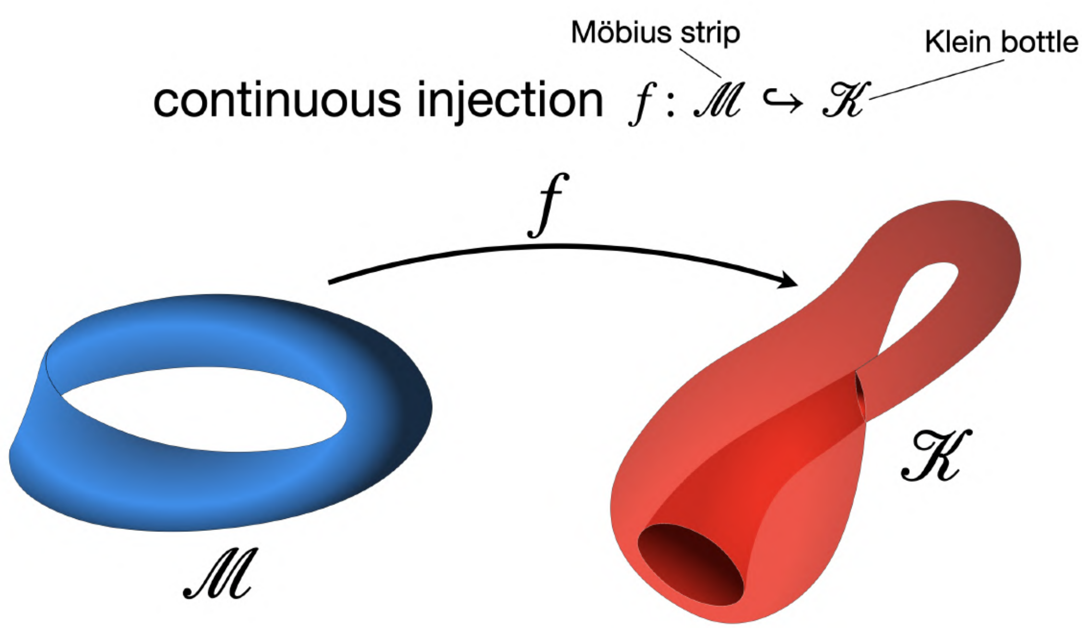
★ f^{-1} (inverse map) is also continuous



Intuitively, a homeomorphism is a way to deform one shape into another by stretching, bending, or twisting (but without tearing, cutting or glueing), so that the two spaces remain topologically the same:

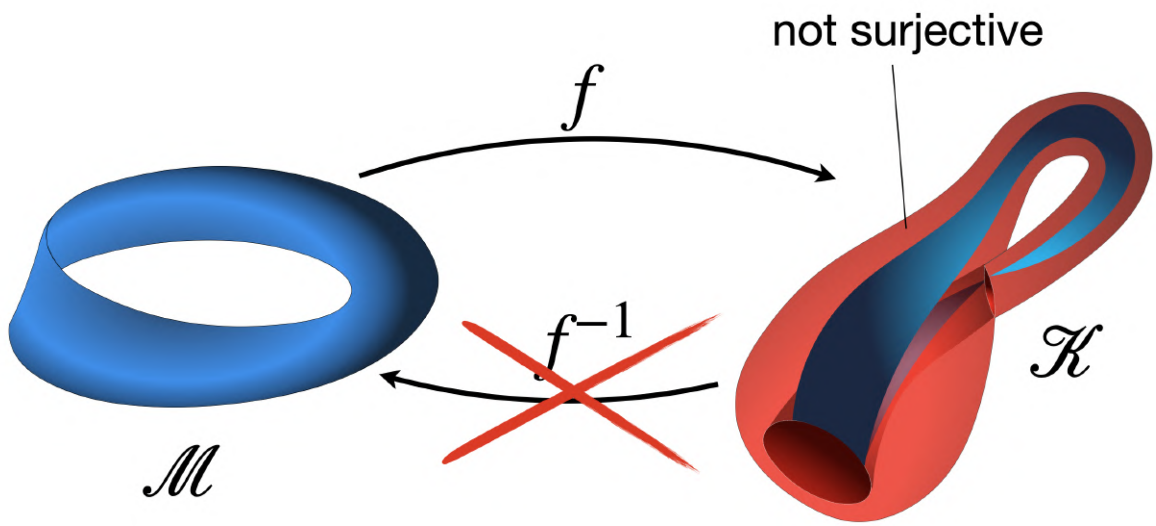


An example of a pair of objects with a morphism connecting them in the category **TopHaus**, but not in **Homeo** is the continuous injection $f : \mathcal{M} \hookrightarrow \mathcal{K}$.

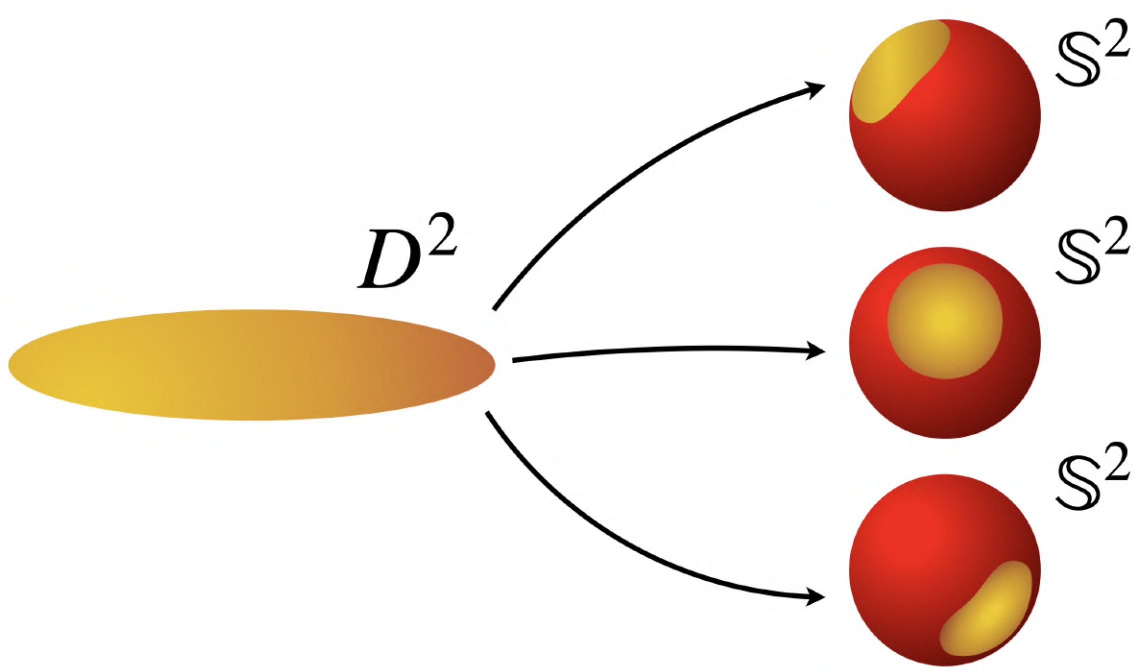


I.e., the Möbius strip can be embedded in the Klein bottle. However, f is not surjective, and furthermore there is no continuous inverse map

$\mathcal{K} \to \mathcal{M}$. Which means that the Klein bottle is not embeddable into the Möbius strip. Therefore, there is no homeomorphism between them.



Another example would be the objects D^2 (disk) and $\mathbb{S}^2$ (2-sphere). There are many continuous injections from D^2 to $\mathbb{S}^2$:

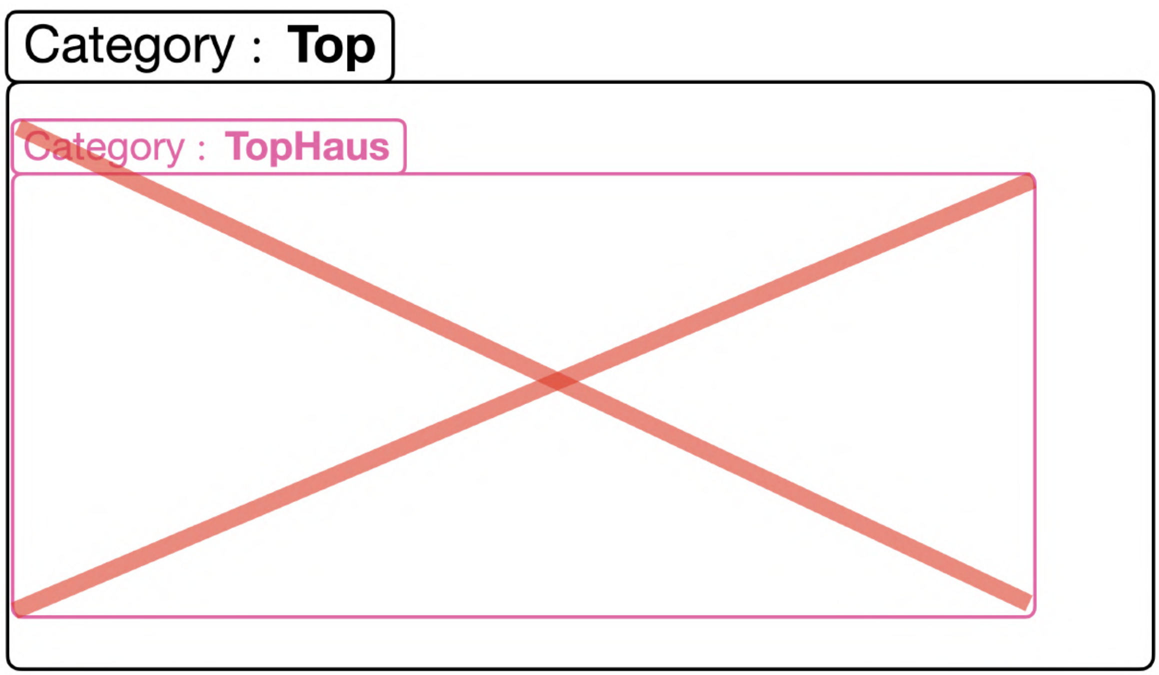


But none of them are surjective!

Likewise, there are many continuous surjections from $\mathbb{S}^2$ to D^2. But none of them are injective!

Therefore, no continuous bijection exists between them whose inverse is also continuous. They are definitely in the category **TopHaus**, since D^2 and $\mathbb{S}^2$ are Hausdorff spaces with many continuous maps between them, but they're not in the category **Homeo**, because there is no homeomorphism between them.

Now, let's see examples of pairs of objects, with their morphisms, in the category **Top**, but not in **TopHaus**.

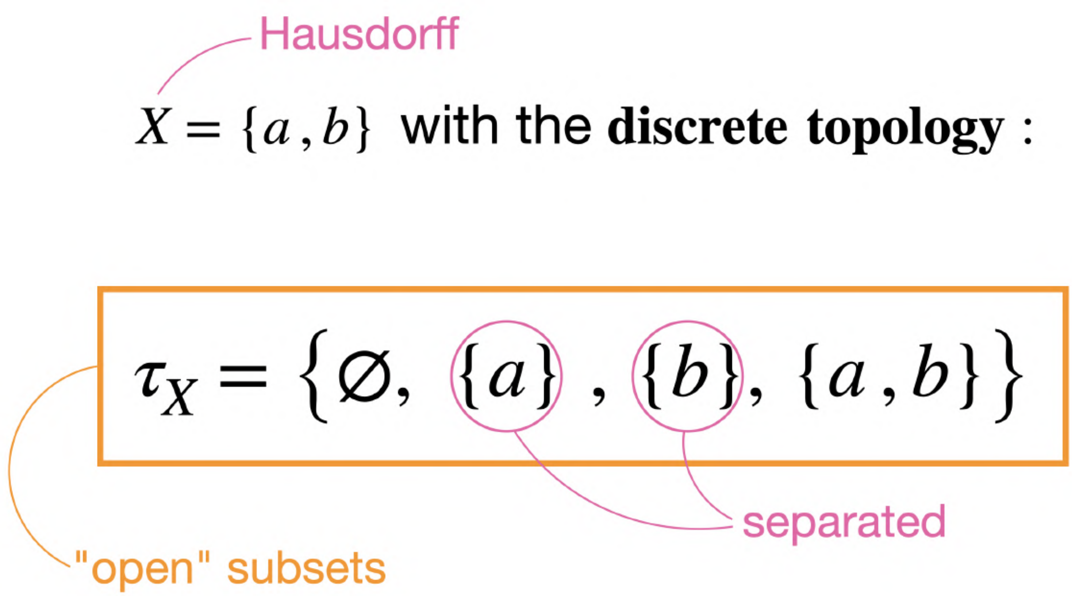


$X = \{a, b\}$ with the *discrete topology*:

$$\tau_X = \{\emptyset, \{a\}, \{b\}\}, \{a, b\}$$

This set defines all the subsets of X that can be called *"open"*.

This is clearly a Hausdorff space, since we can separate its elements. Indeed, we just did it here!



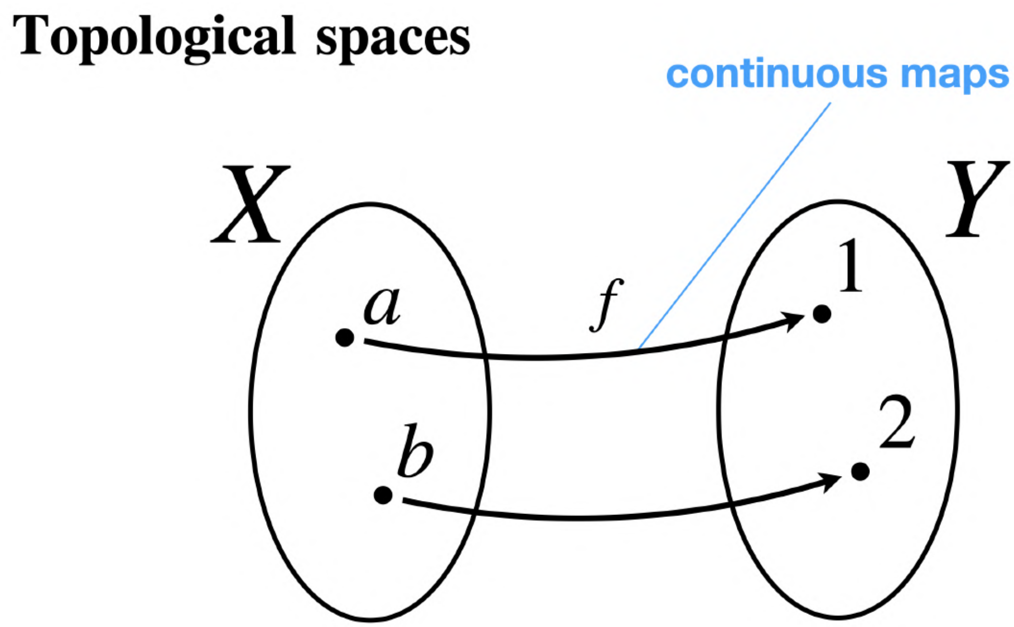
$Y = \{1\,,\,2\}$ with the *trivial* (aka *indiscrete*, or *coarse*) *topology*:

$$\tau_Y = \{\varnothing\,,\ \{1\,,\,2\}\}$$

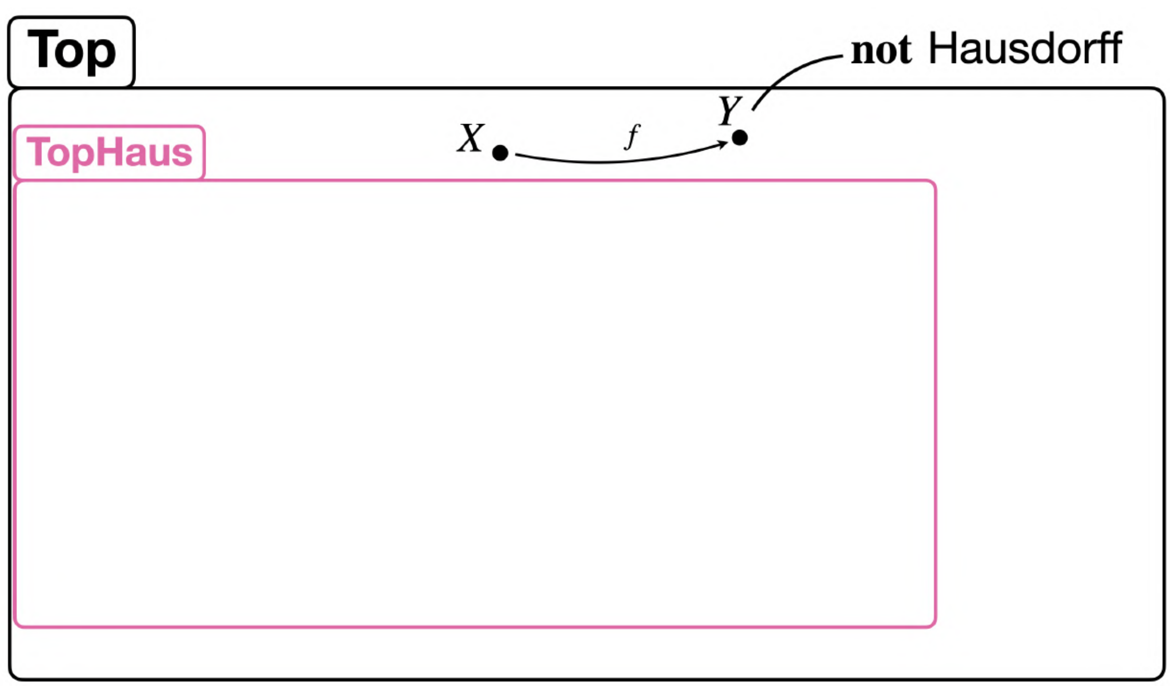
This is not Hausdorff because there are no open sets that separate 1 and 2:

$$\{1\}\ ,\ \{2\} \notin \tau_Y$$

X and Y are definitely topological spaces, and there are certainly continuous maps between them, like this one:

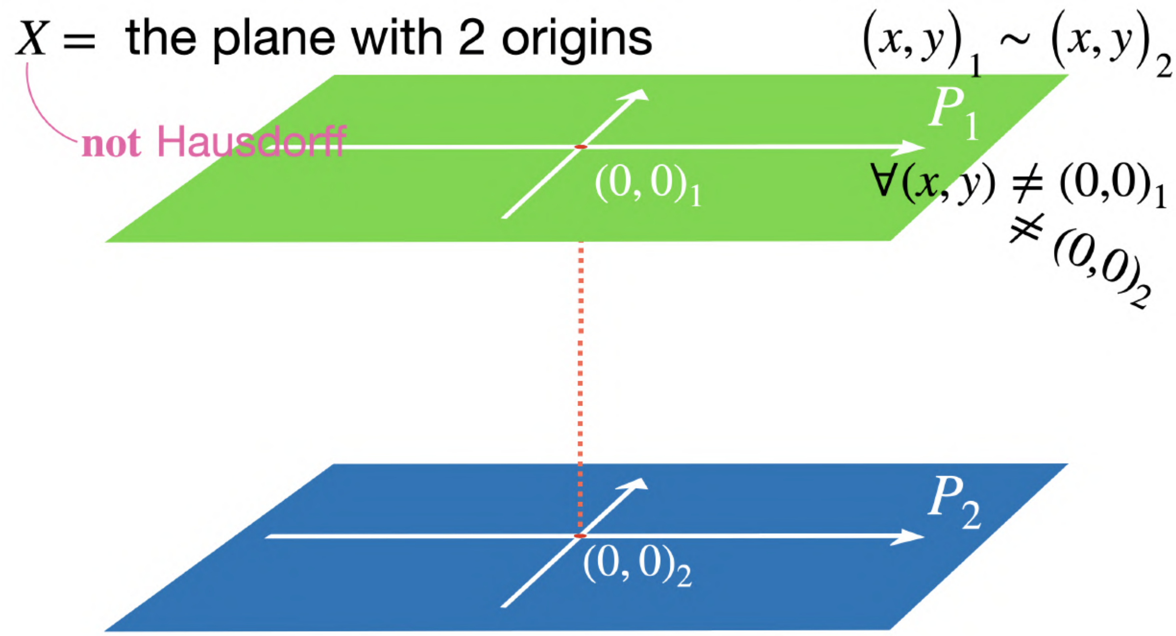


But since Y is not Hausdorff, this pair of objects with morphisms do not belong to **TopHaus**, even though they do belong to **Top**.



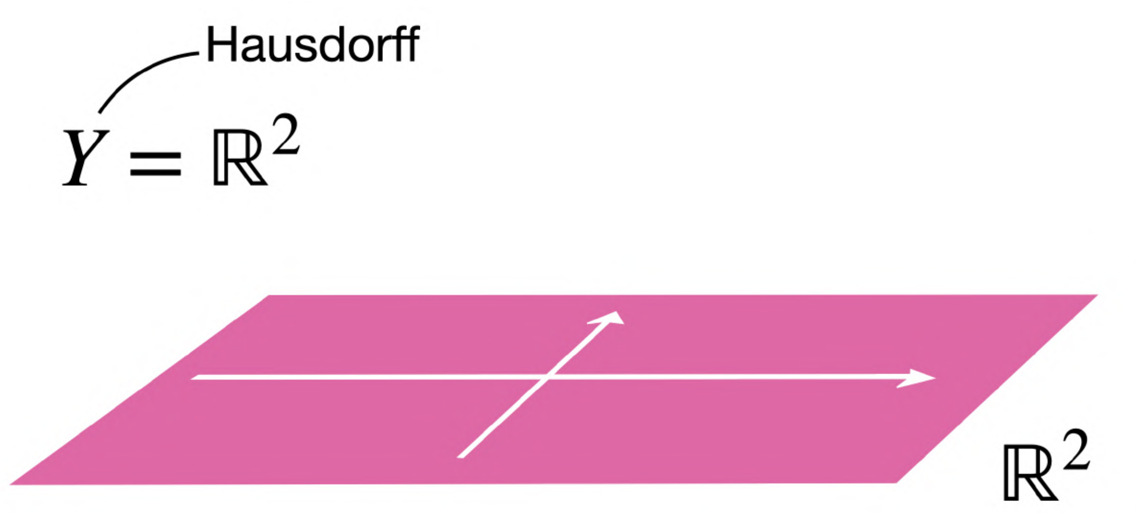
Another example is:

$X =$ the plane with two origins with the standard topology on $\mathbb{R}^2$.

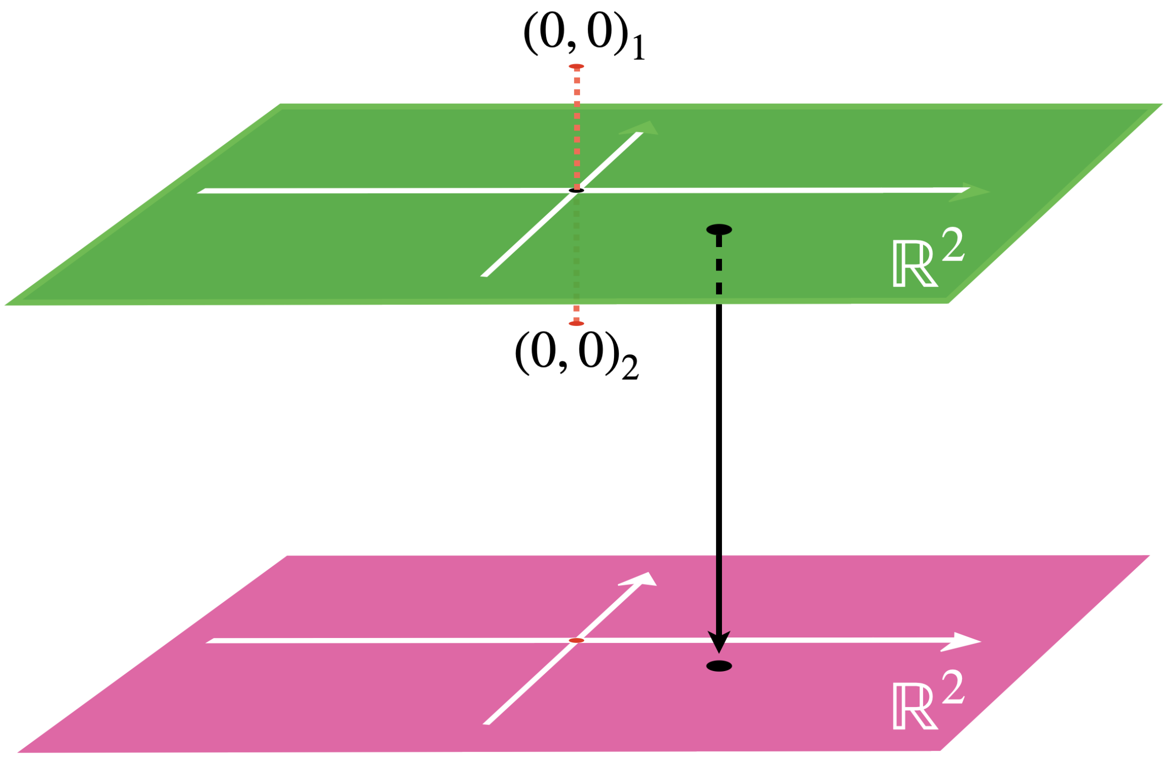


This is a non-Hausdorff space.

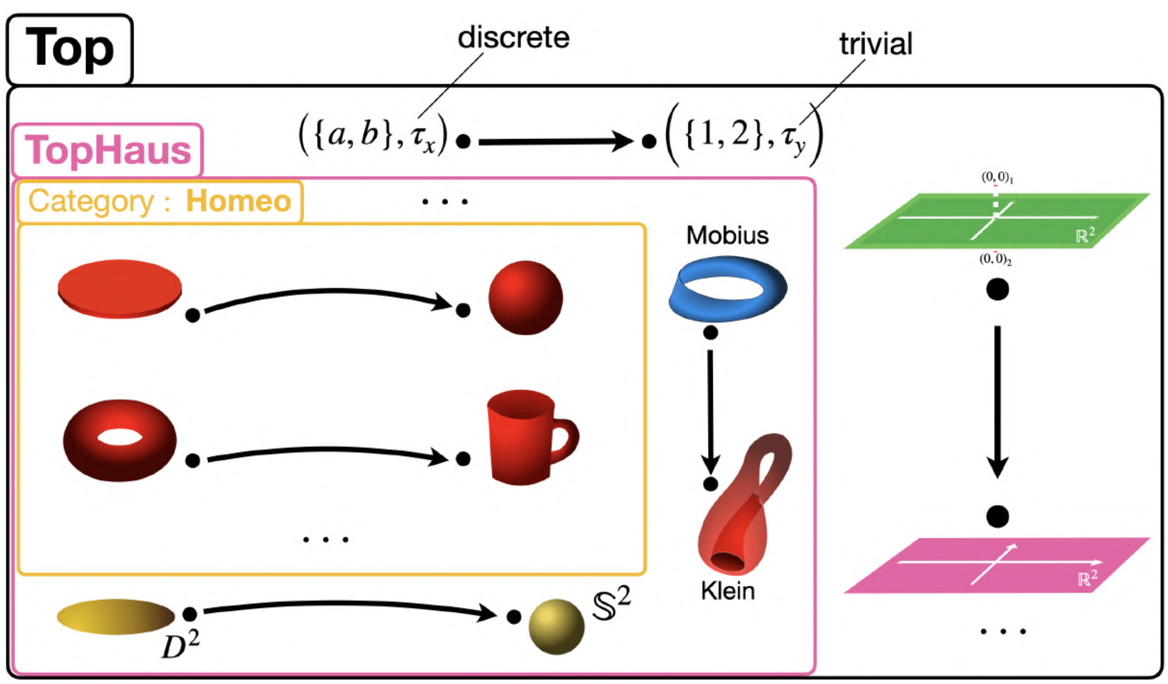
Let $Y = \mathbb{R}^2$ with the same usual topology. This space is Hausdorff.



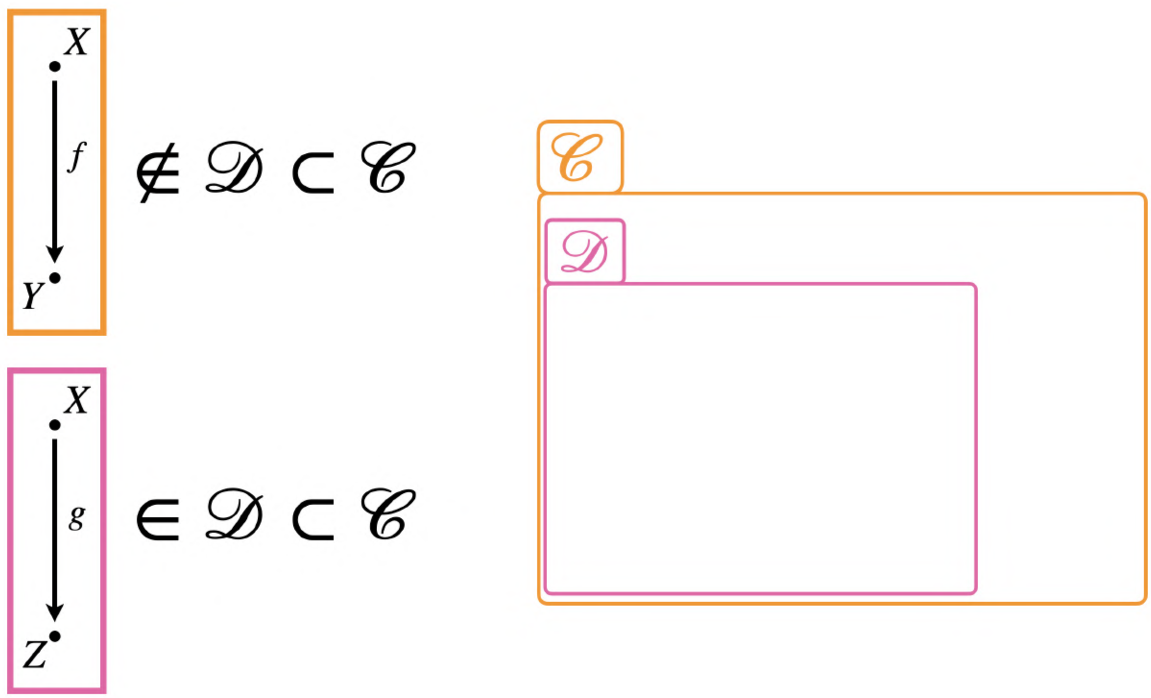
Now, we can create a continuous map that identifies each pair of points X and Y just as you might've expected, including the origins $(0, 0)_1$, $(0, 0)_2 \in X$ that are identified to the unique origin $(0, 0) \in Y$.



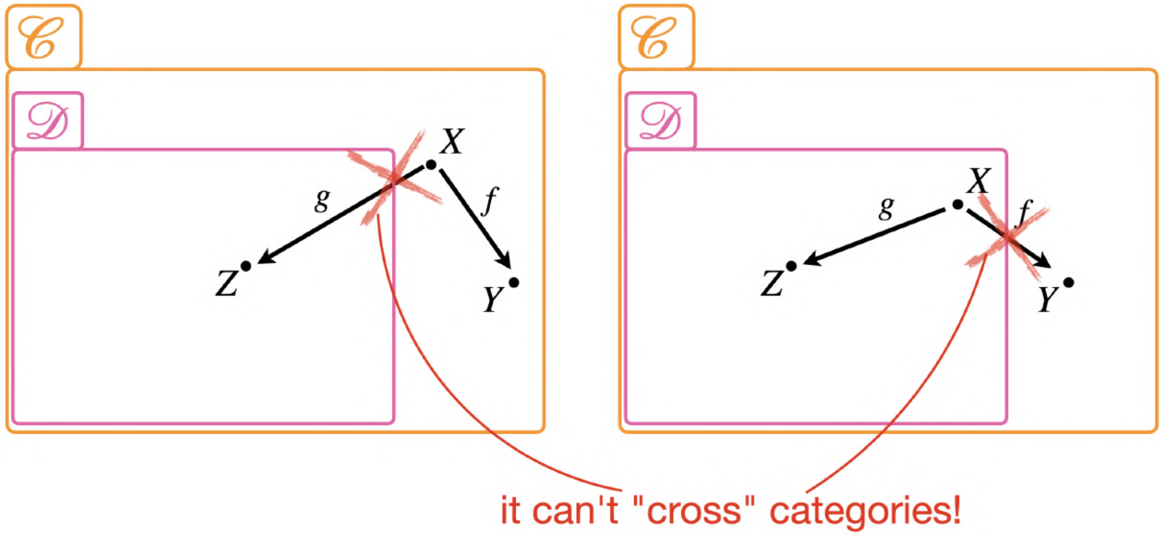
This is a continuous map, but since X is not Hausdorff, the pair of objects with all its possible morphisms belong to the category **Top**, but not to **TopHaus**.



Of course, this representation is not perfect, and you can see how it becomes problematic when we talk about a pair of objects, and a morphism, $\left(X \xrightarrow{f} Y\right)$ that belongs to the category $\mathcal{C}$, but not to its subcategory $\mathcal{D}$. And at the same time the pair of objects, and a morphism, $\left(X \xrightarrow{g} Z\right)$ that belong to $\mathcal{D}$.

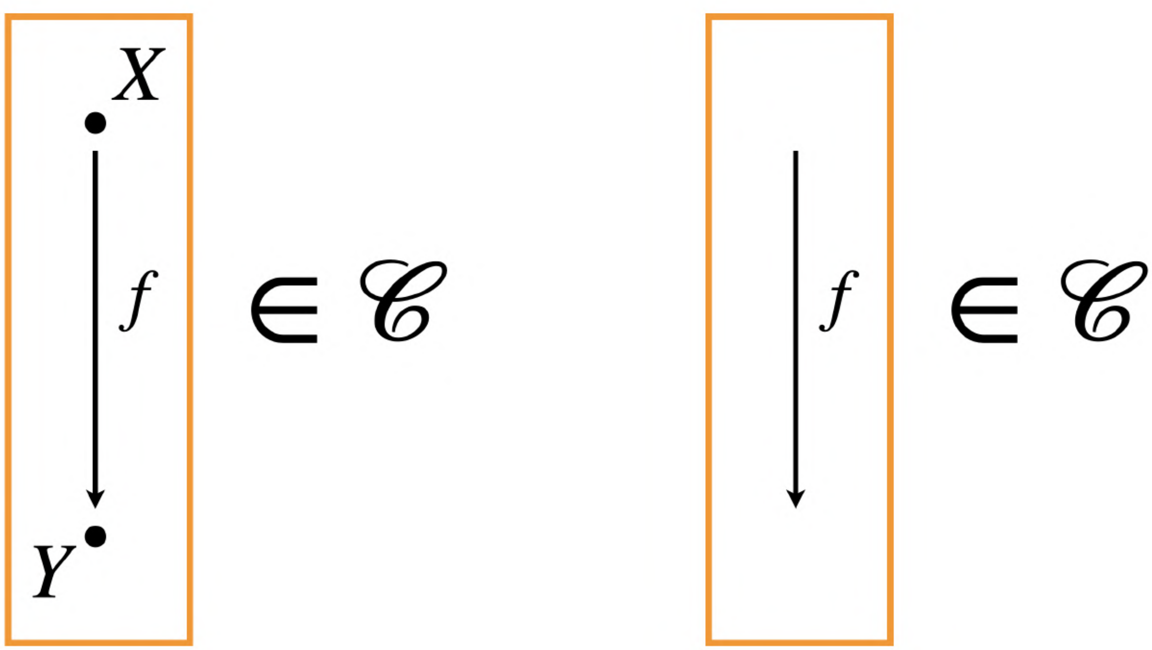


The problem arises when you need to decide where to place the object X:



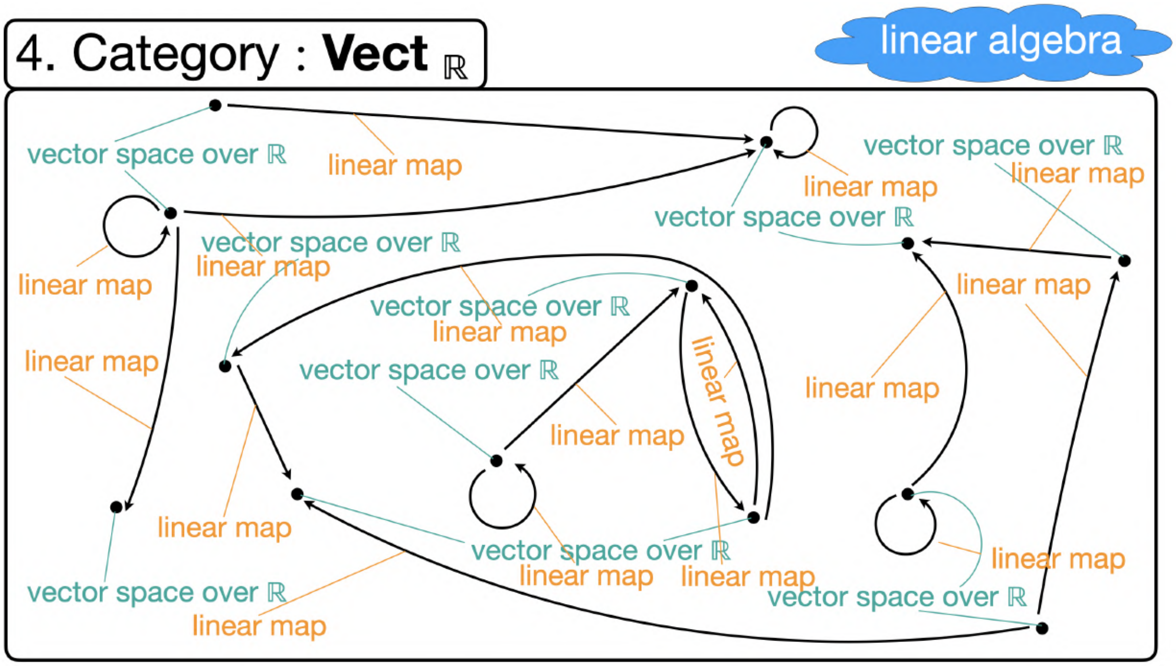
Should we place X in the $\mathcal{C}$, but not in the $\mathcal{D}$ box? Or vice-versa? This problem is just a reflection of the fact that categories are not sets, they are way more abstract than sets. So don't take these drawings too seriously...

Instead of talking about *"pairs of objects with morphisms connecting them"*, in category theory it's preferable to only talk about *"morphisms belonging to categories"*.



Category: $\textbf{Vect}_{\mathbb{R}}$ (Linea Algebra)

This is the category where objects are *vector spaces over the field* $\mathbb{R}$, and morphisms are *linear maps*.

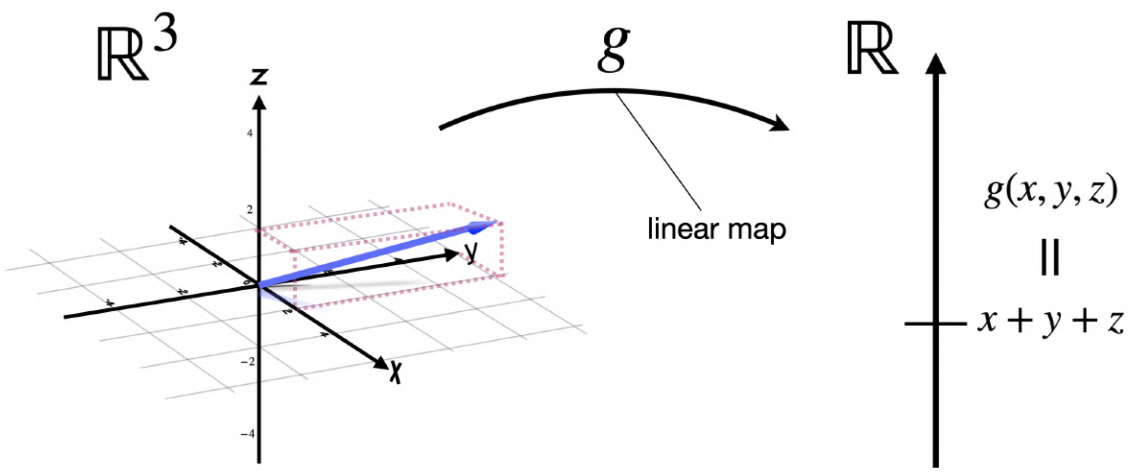


As you might've guessed, we are now studying the abstract structure of Linear Algebra, through the lens of category theory.

Let's see some examples:

Consider the objects $\mathbb{R}^3$ and $\mathbb{R}$ connected by the morphism:

$$\boxed{g(x, y, z) = x + y + z}$$



This is clearly a linear map from one vector space over the field $\mathbb{R}$ to another vector space.

Now, let $M_{2\times 2}(\mathbb{R})$ represent the vector space of real 2×2 matrices. We can, for example, define the following linear transformation:

$$\boxed{f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3}$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d \\ b+c \\ a-b \end{bmatrix} \in \mathbb{R}^3$$

$$f \in \mathbf{Vect}_{\mathbb{R}}$$

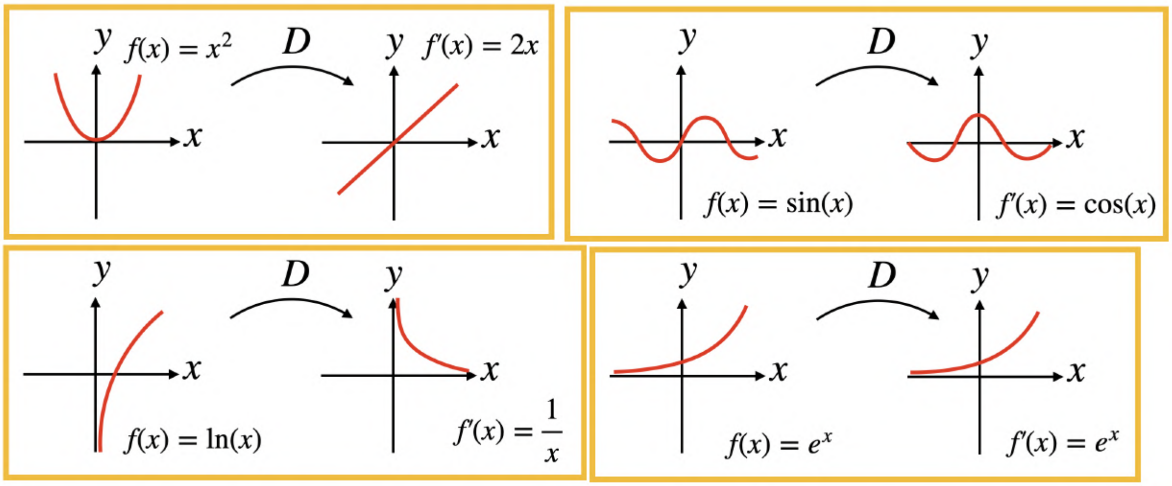
$$\Downarrow \text{ (more precisely)}$$

$$f \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}\left(M_{2\times 2}(\mathbb{R}) \ , \mathbb{R}^3\right)$$

This linear transformation belongs to our category.

Another example is the pair of infinitely differentiable real functions $(C^\infty(\mathbb{R}))$ and continuous functions on the interval $[a\,, b] \subset \mathbb{R}$, with the morphism being the derivative map $D : C^\infty(\mathbb{R}) \to C^\infty([a\,, b])$:

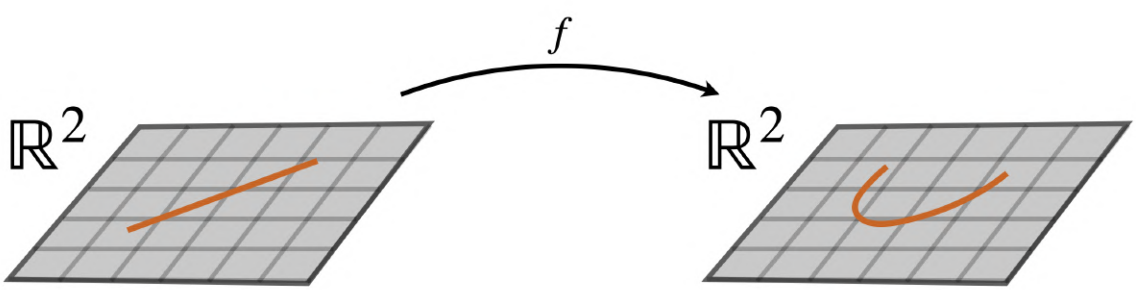
$$D(f) = f'\Big|_{[a,b]} \qquad C^{\infty}(\mathbb{R}) \xrightarrow{D} C\big([a,b]\big)$$



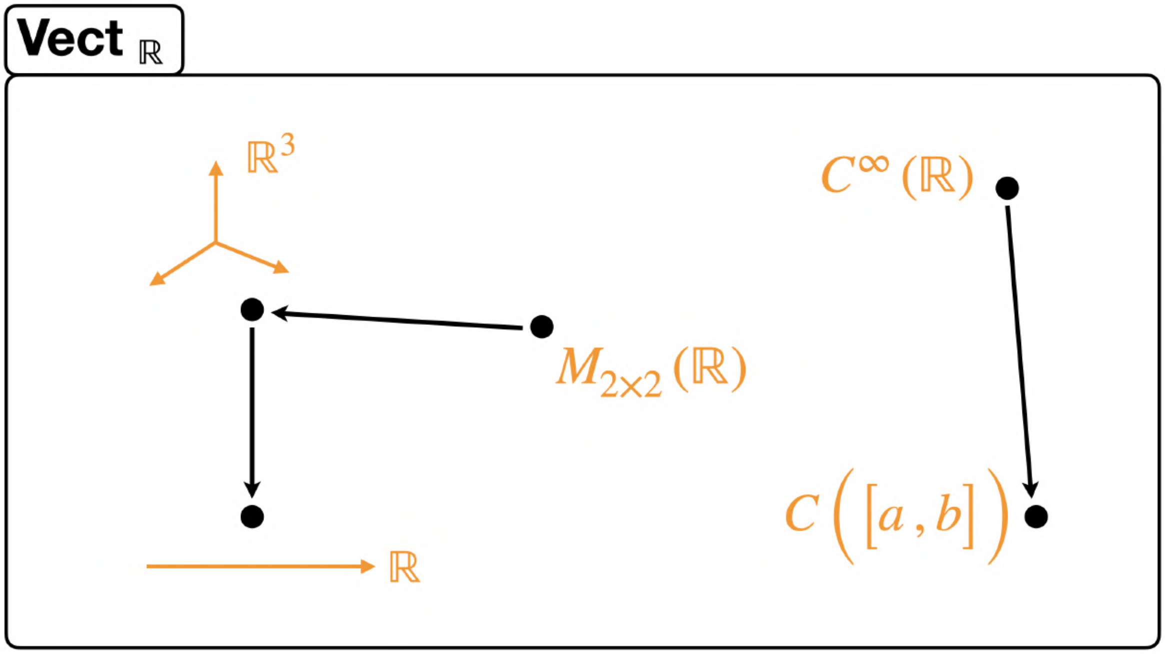
A counterexample would be the non-linear map $f(x, y) = (x^2, y)$ between vector spaces $\mathbb{R}^2$ and $\mathbb{R}^2$:

$$f(x, y) = \left(x^2, y\right)$$

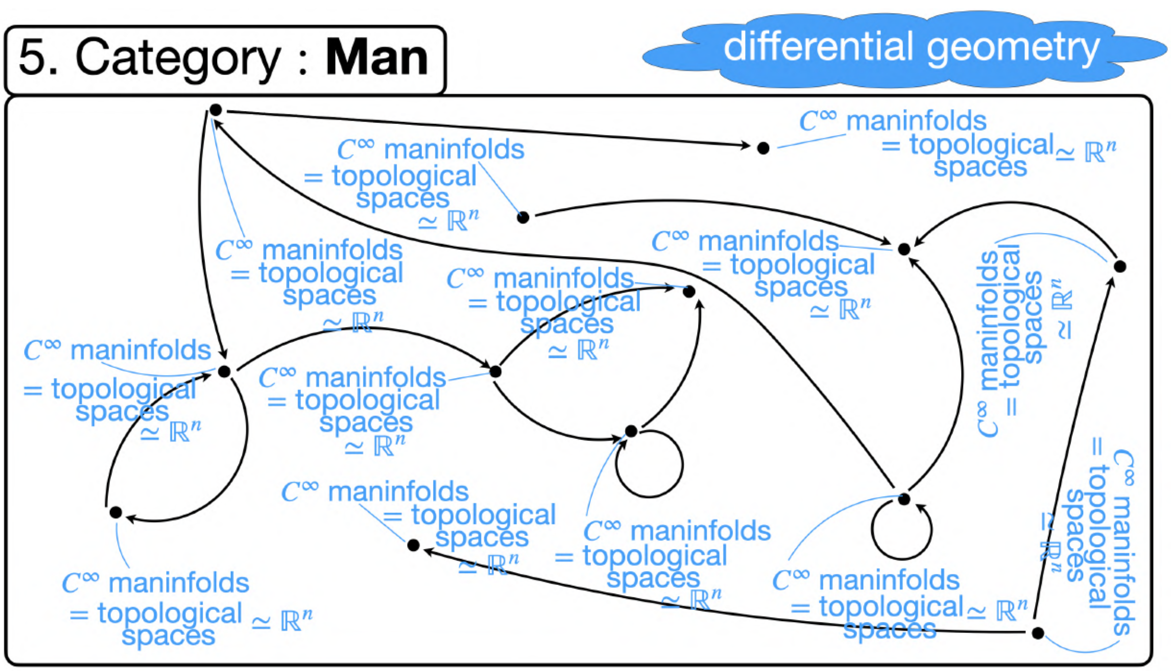
$$f \notin \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{R}}}\left(\mathbb{R}^2, \mathbb{R}^2\right)$$



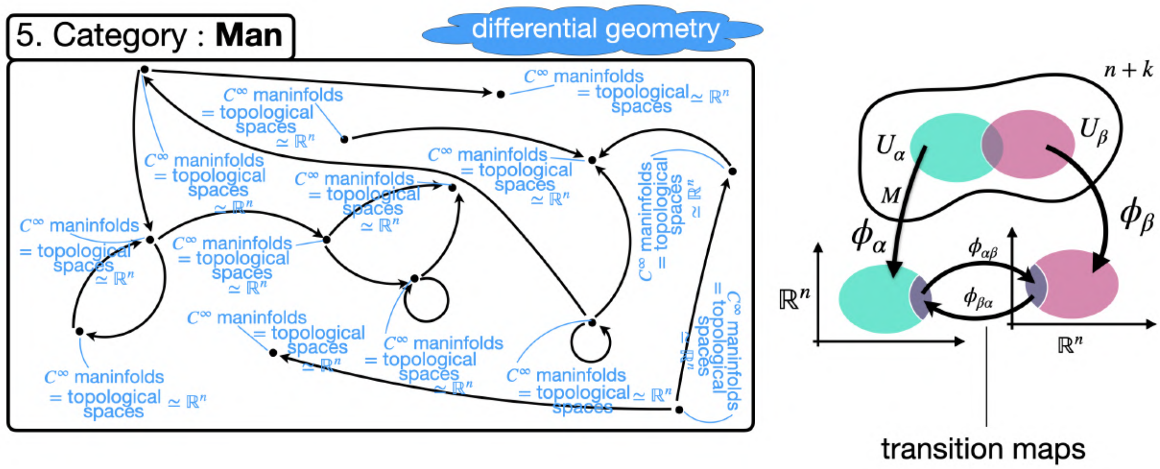
So we can simplistically think of this category this way:



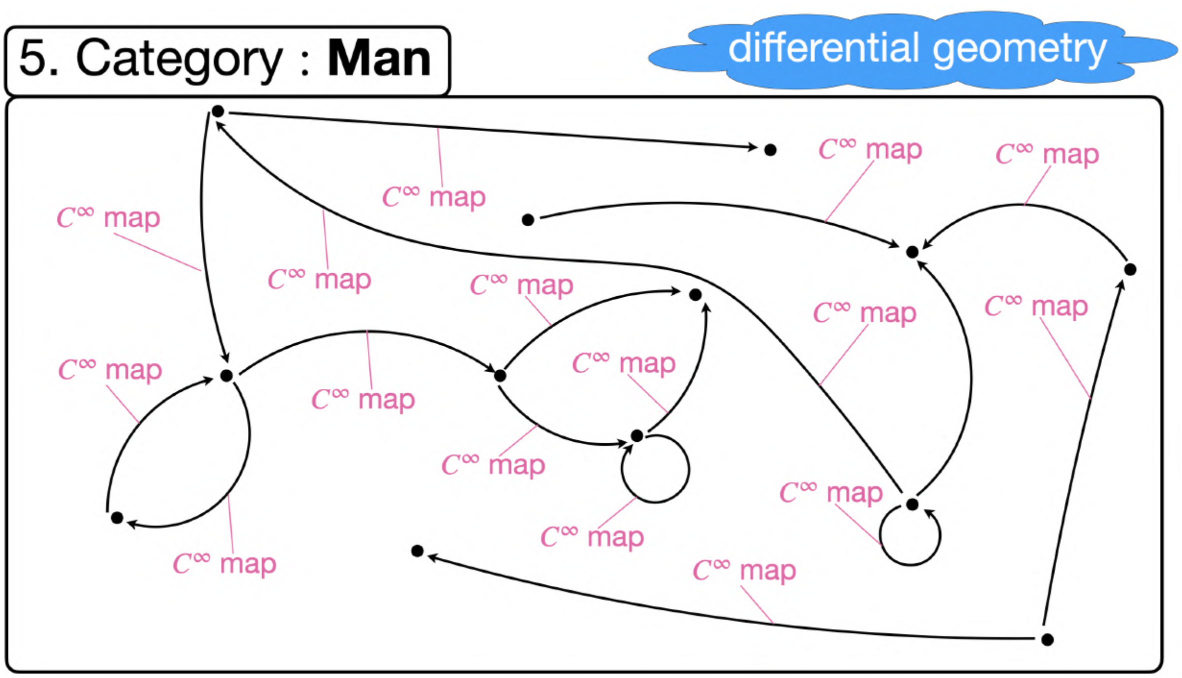
Category: Man (Differential Geometry)



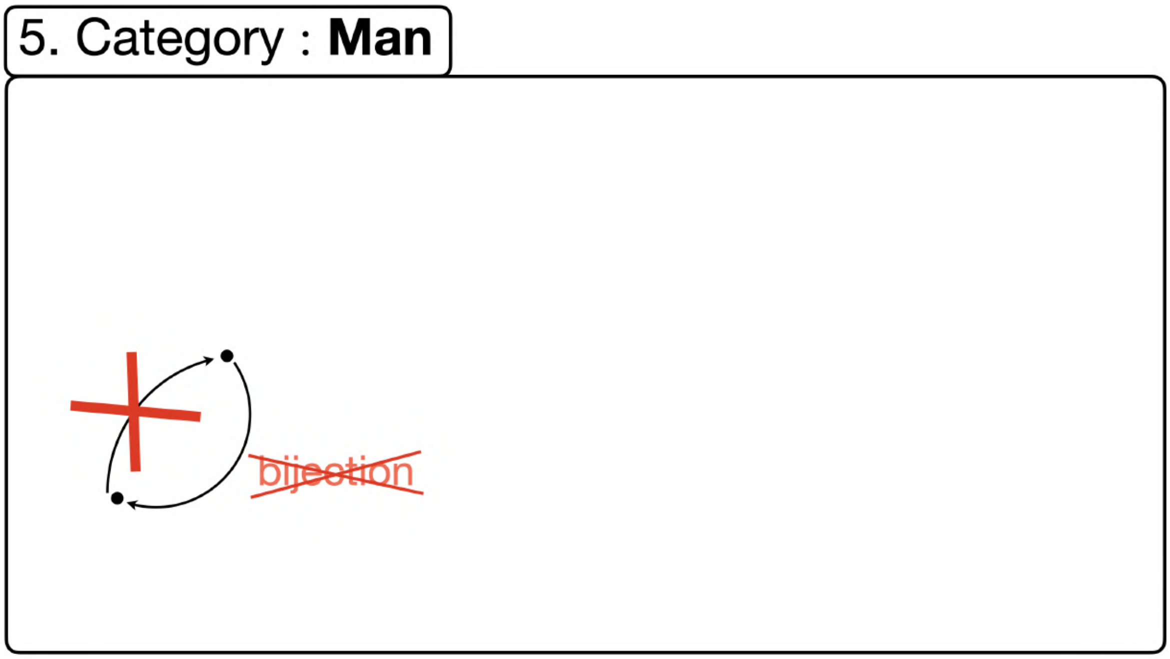
The objects in this category are *differentiable* (or *smooth*, or C^∞) *manifolds*, i.e. topological spaces locally homeomorphic to $\mathbb{R}^n$ that have a smooth structure (so, charts and smooth transition maps).



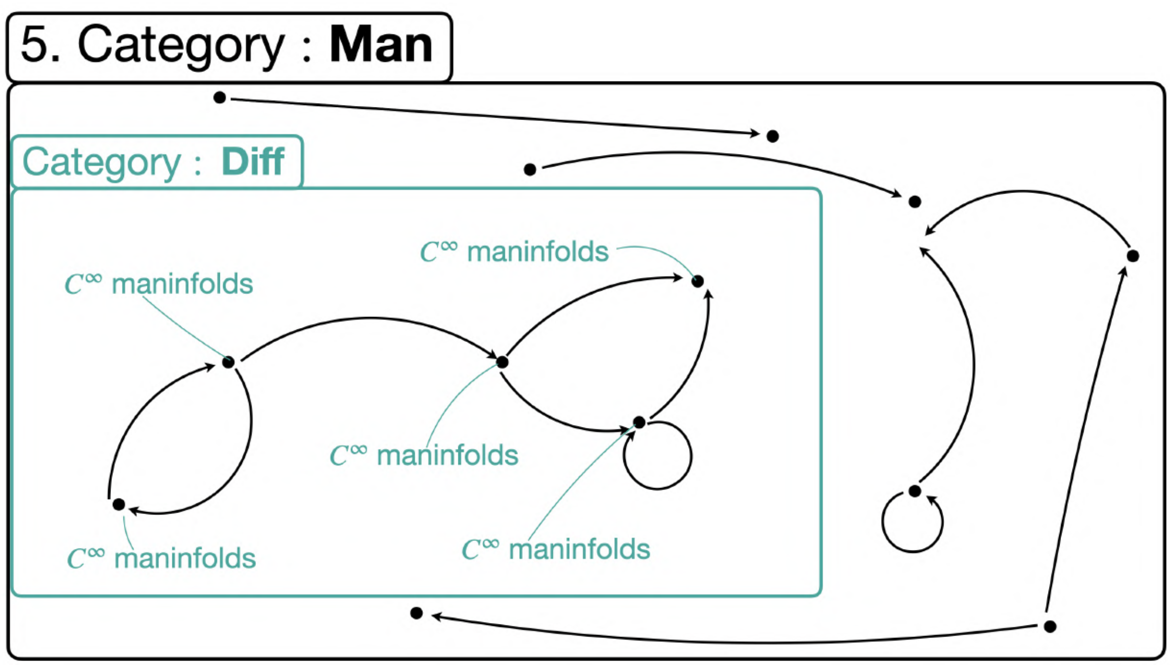
Morphisms are *smooth maps*, i.e. infinitely differentiable (C^∞) functions between manifolds.



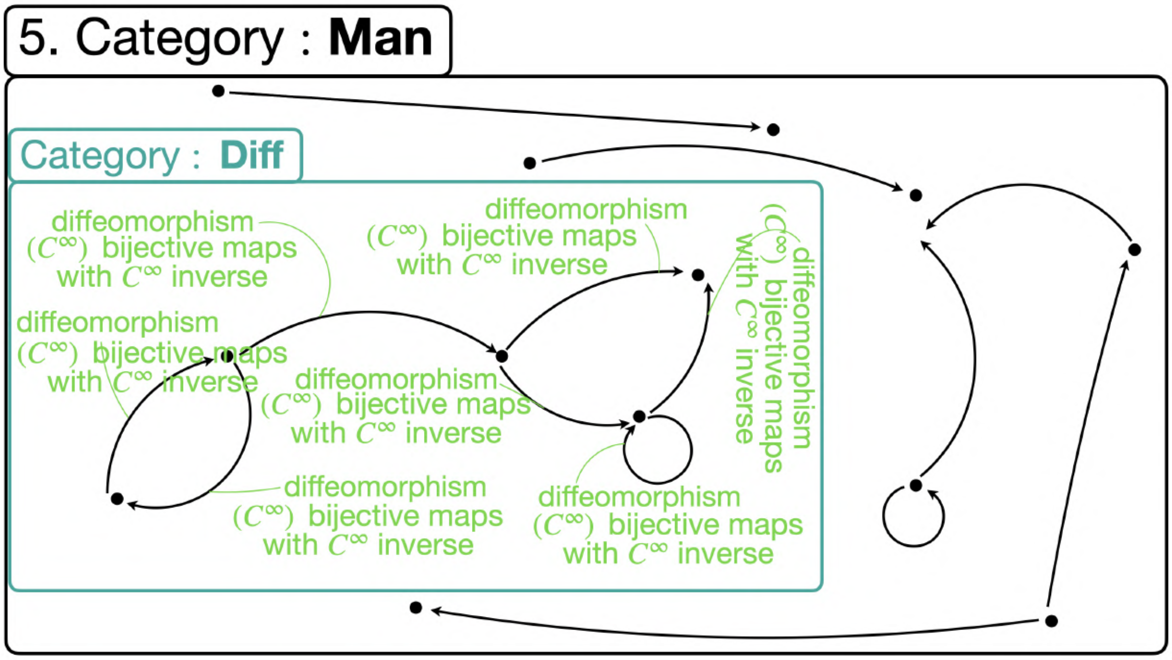
This is a huge category. The morphisms *don't* need to be invertible or bijective, just smooth.



There is also the category **Diff**, which is a subcategory of **Man**. Here, objects are smooth manifolds (just as in the **Man** category).



Morphisms are *diffeomorphisms*, i.e. smooth (C^∞) maps that are bijective and whose inverse is also smooth. So, the morphisms now are more strict than before.



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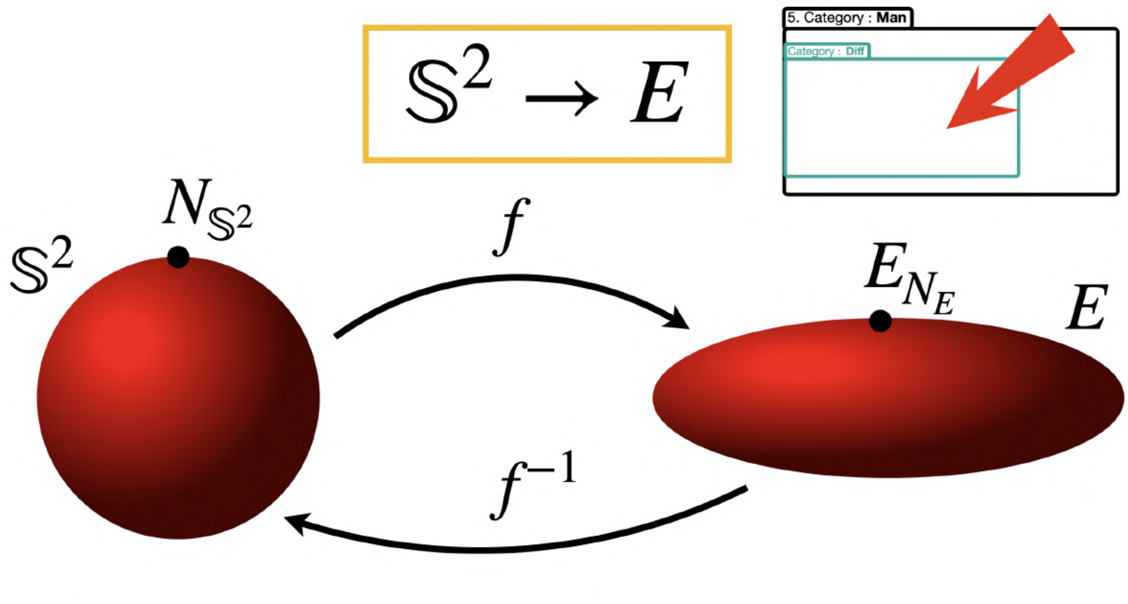
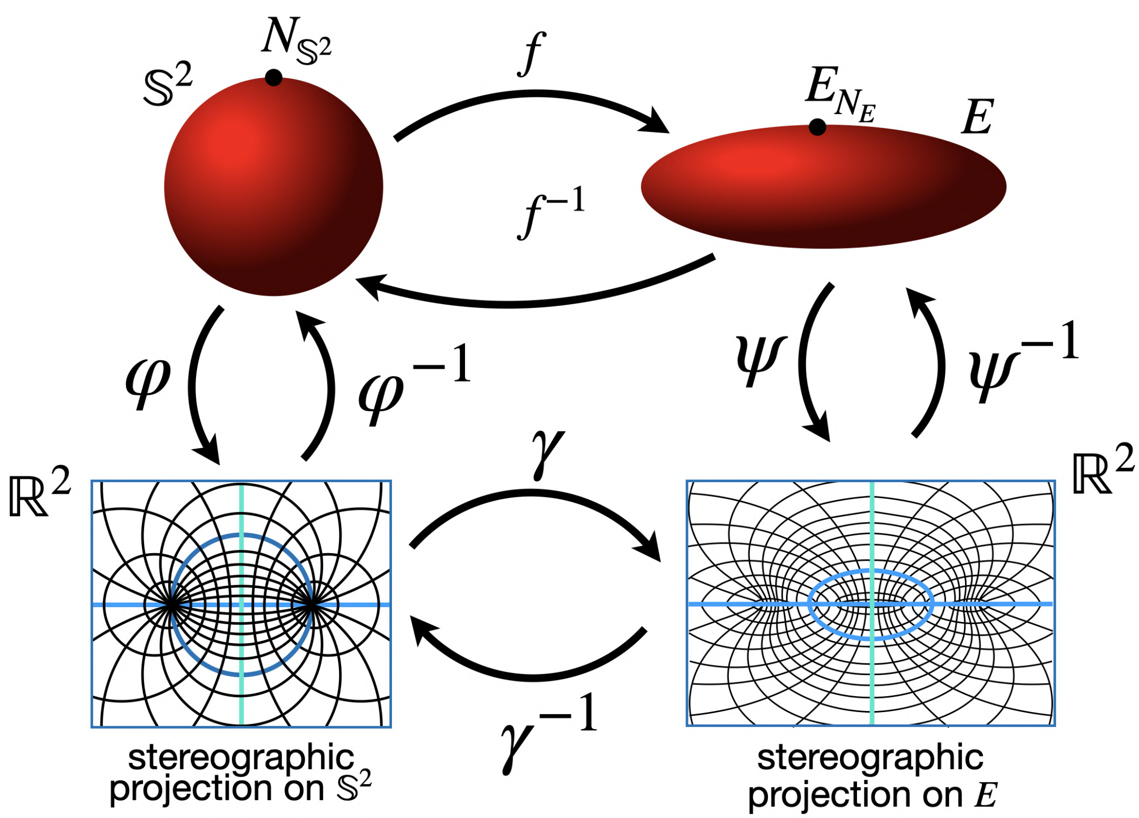
Let's see some concrete examples:

(1) $\mathbb{S}^2 \to E$

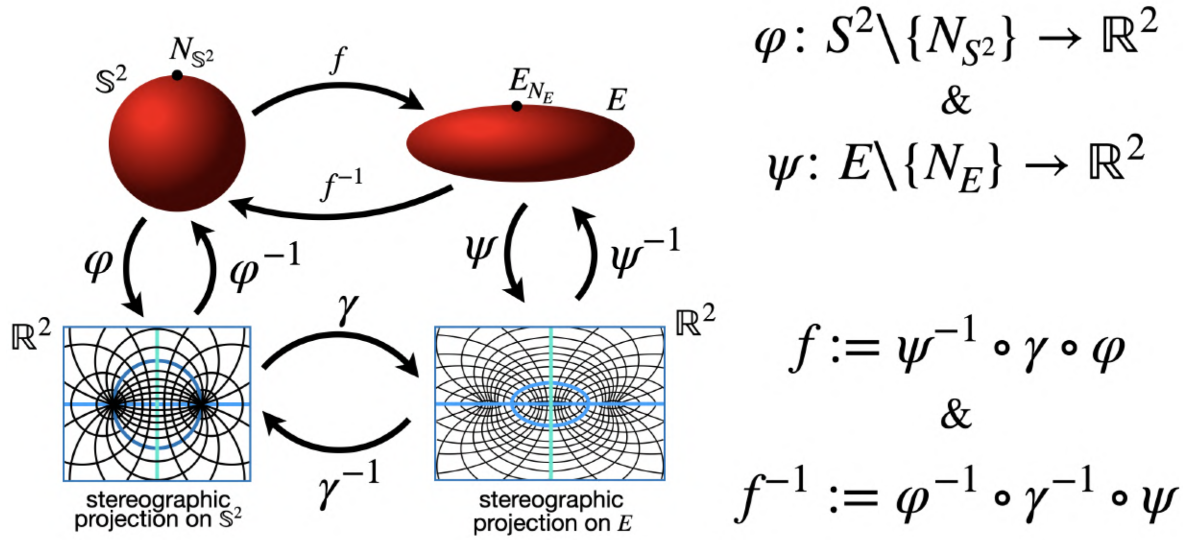
The first example is a morphism between two smooth manifolds that relate them using a smooth bijection with smooth inverse, and therefore this morphism lives in the category **Diff**, but also in **Man**, as a consequence.



We are defining a diffeomorphism f between the 2-sphere $\mathbb{S}^2$ and an ellipsoid E. But, how exactly can we do it?



First we can use *stereographic projections* for each of them. These will be global charts $\varphi : \mathbb{S}^2 \setminus \{N_{\mathbb{S}^2}\} \to \mathbb{R}^2$ and $\psi : E \setminus \{N_E \to \mathbb{R}^2\}$, which map every point in each manifold to their own copies of $\mathbb{R}^2$, minus a point at ∞.



Now, we can compose mappings to define f and f^{-1}:

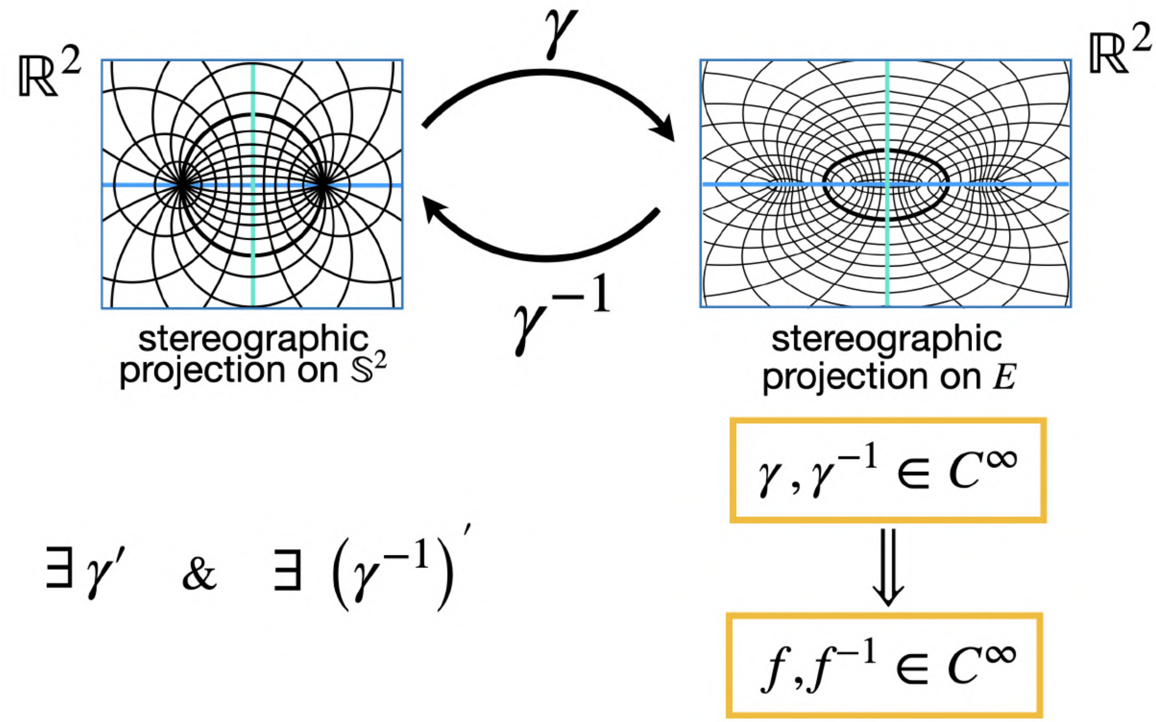
$$f := \psi^{-1} \circ \gamma \circ \varphi \qquad f^{-1} := \varphi^{-1} \circ \gamma^{-1} \circ \psi$$

You might be asking yourself right now: *"why would a person, in their right mind, complicate things so much to the point of using all these charts, and maps, and compositions, just to define the functions f and f^{-1} ? I mean, can't we just associate points on $\mathbb{S}^2$ to points on E ? It would be much easier!"*

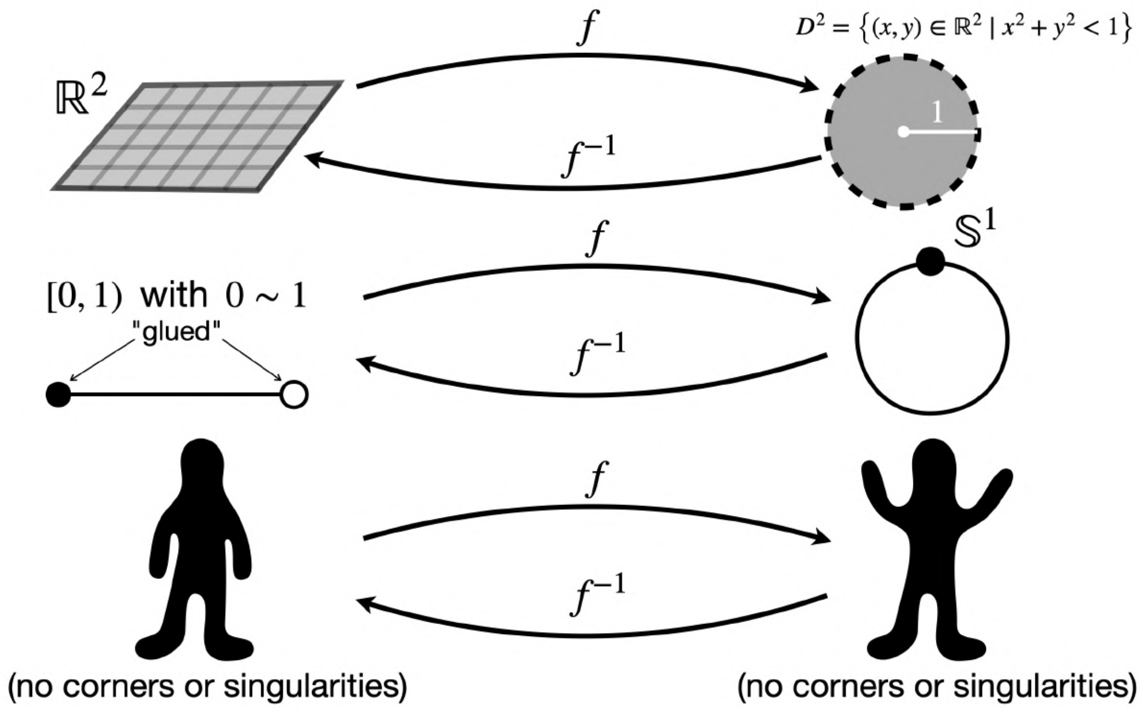
Well, it turns out that there is a deep reason for doing so in differential geometry. The functions f and f^{-1} are maps between curved spaces (between $\mathbb{S}^2$ and E). So we cannot directly apply the tools of multivariable calculus to define, and compute, their derivative, since those tools are formulated in linear (flat) spaces, like $\mathbb{R}^n$.

And that's why we need all these coordinate charts $(\varphi, \varphi^{-1}, \psi, \psi^{-1})$ and intermediate maps (γ, γ^{-1}).

Since γ and γ^{-1} are maps between flat spaces, we can take their derivatives and talk about them being smooth in the classical calculus sense. Then, we define f and f^{-1} to be smooth if γ and γ^{-1} are smooth.



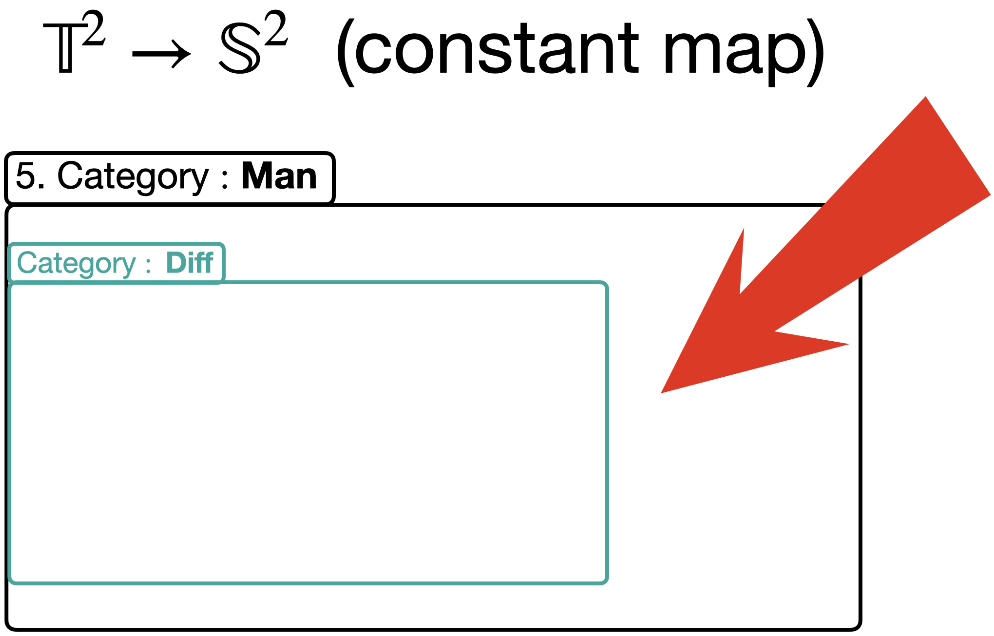
Of course, we could define smooth bijections (in a similar way) for other pairs of smooth manifolds:



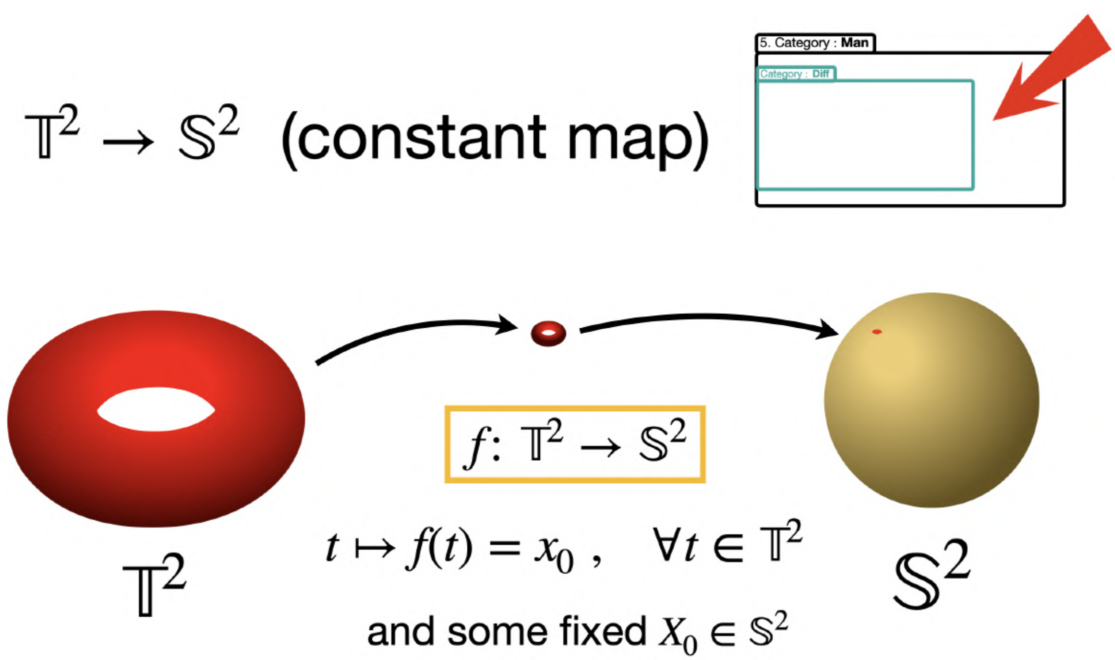
f
$D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$
$\mathbb{R}^2$
1
f^{-1}
f
$\mathbb{S}^1$
$[0, 1)$ with $0 \sim 1$
"glued"
f^{-1}
f
f^{-1}
(no corners or singularities)
(no corners or singularities)

(2) $\mathbb{T}^2 \to \mathbb{S}^2$ (constant map)

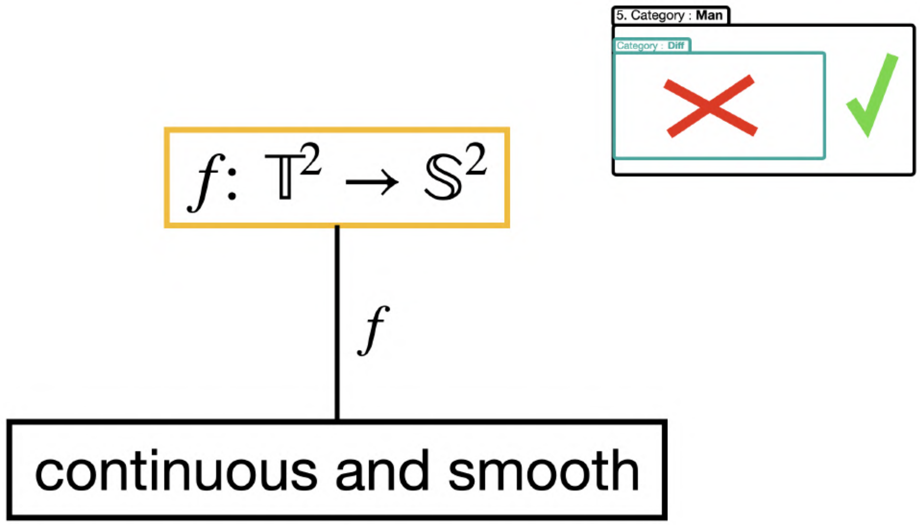
The next example is a morphism in the category **Man**, but not in **Diff**. We will collapse the entire torus onto a single point on the sphere.



Consider the map $f : \mathbb{T}^2 \to \mathbb{S}^2$, with $t \mapsto f(t) = x_0 \quad , \quad \forall t \in \mathbb{T}^2$ and some fixed $x_0 \in \mathbb{S}^2$. This map is continuous and smooth.



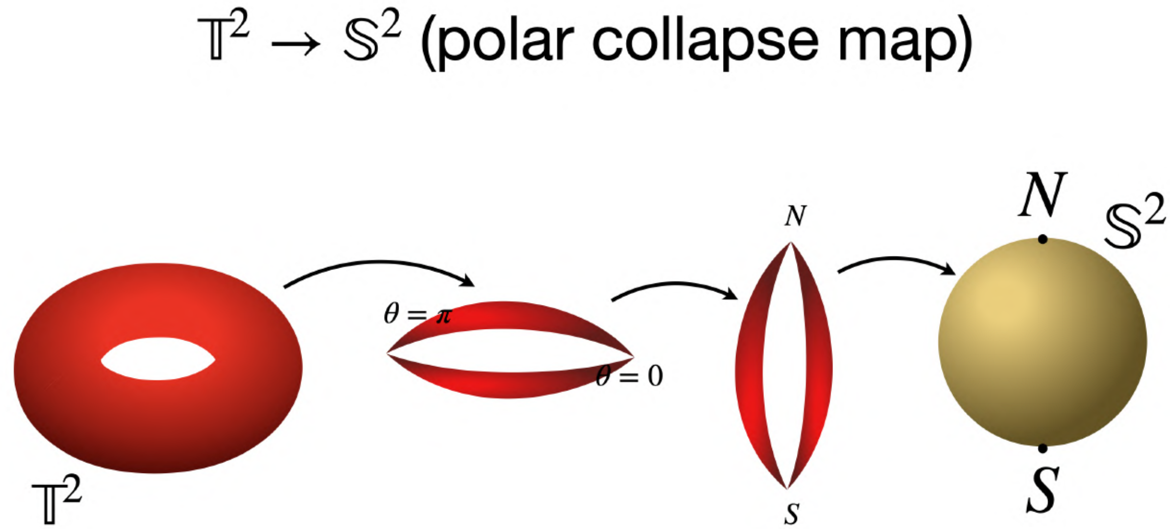
In fact, all its derivatives vanish, since it's constant. Therefore, it's a valid morphism in the category **Man**. However, it's not a diffeomorphism because it is not bijective. As a result, it does not belong to **Diff**.



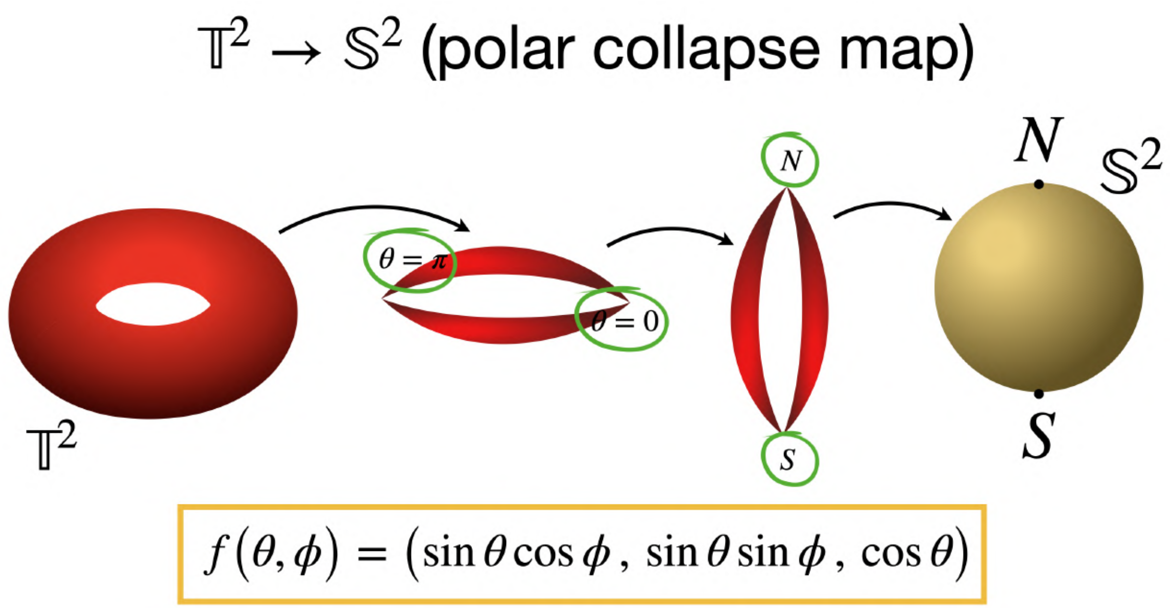
(3) $\mathbb{T}^2 \to \mathbb{S}^2$ (polar collapse map)

We can also consider a more geometric example. The smooth map:

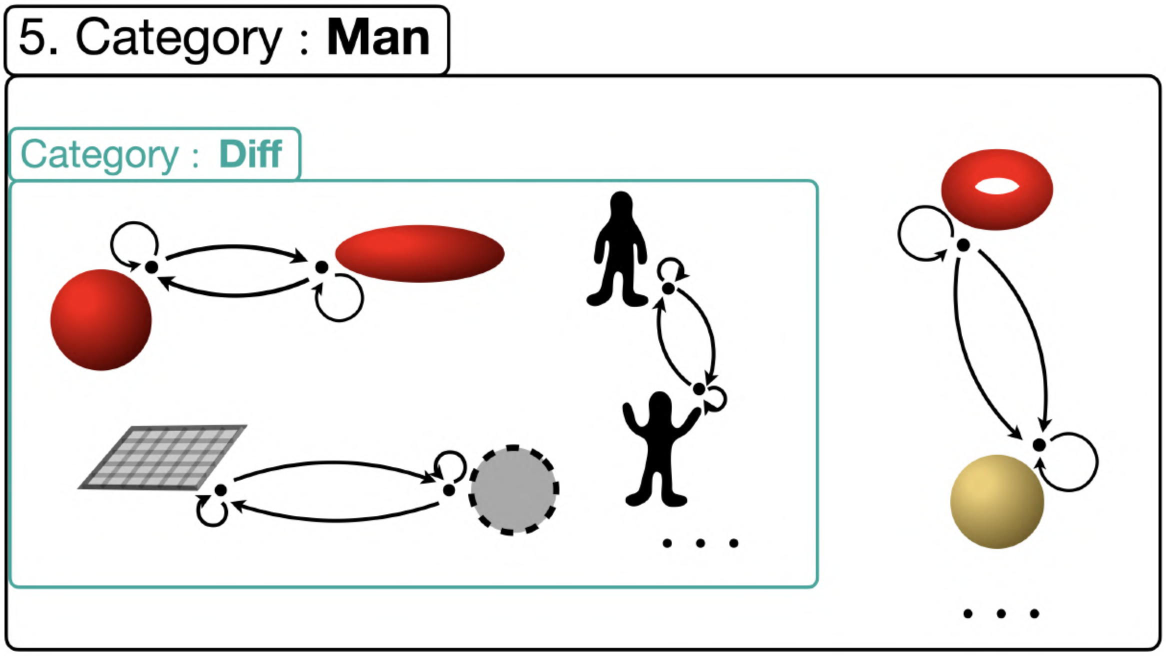
$$f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



This map sends points from the torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ into the 2-sphere $\mathbb{S}^2$. It's smooth and continuous, so it defines a morphism in **Man**. However, it's not injective: the two entire ϕ-circles at $\theta = 0$ and $\theta = \pi$ are both collapsed to the north and south poles of the sphere, respectively.



Since this map is not a bijection, it is not a diffeomorphism either, and therefore not a morphism in **Diff**.



Great! After this marathon of examples and intuitive explanations, we can finally define a category rigorously:

Definition: A **category** $\mathcal{C}$ consists of:

Objects: a collection of entities (denoted $A, B, C, \ldots$).

Morphisms: for every pair of objects A and B, a set of morphisms (also called arrows) can be defined, and denoted as:

$$\mathbf{Hom}_{\mathcal{C}}(A, B)$$

Each morphism $f \in \mathbf{Hom}_{\mathcal{C}}(A, B)$ is written as:

$$f : A \to B$$

Composition law: for any 3 objects A, B, C, and morphisms $f : A \to B$ and $g : B \to C$, there is a composition:

$$g \circ f : A \to C$$

such that $\circ$ is a binary operation on morphisms:

$$\circ : \ \mathbf{Hom}_{\mathcal{C}}(B, C) \times \mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{C}}(A, C)$$

Identity morphism: for every object A, there is an identity morphism $id_A : A \to A$ such that, $\forall f : A \to B$ and $g : C \to A$, we have:

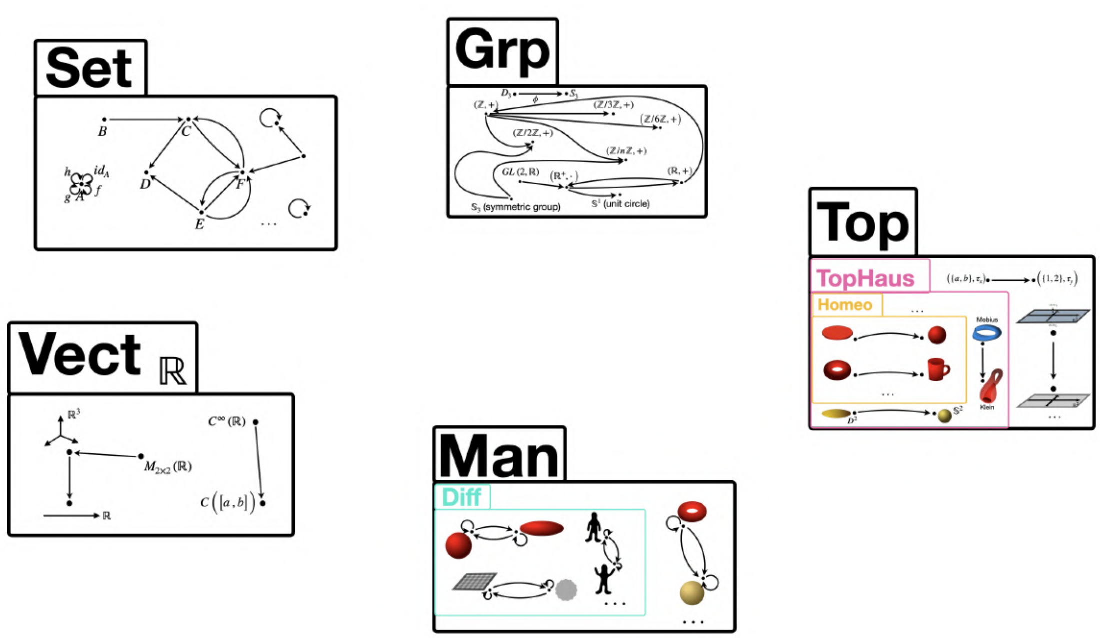
$$\boxed{f \circ id_A = f} \quad \text{and} \quad \boxed{id_A \circ g = g}$$

Associativity: for morphisms $f : A \to B, \ \ g : B \to C, \ \ h : C \to D$, we require that this composition always exists:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

A category, therefore, is a context where we can compose maps and where these compositions behave just as in set theory and algebra: they're associative and with identity elements.

The examples of categories we've seen were: **Set**, **Grp**, **Top** (and its subcategories), **Vect**$\mathbb{R}$, **Man** (and **Diff**).



Now, let's point out something very interesting: If we take *sets* and give them some extra structure, we can turn them into *groups*. So I guess we can represent it with an arrow coming out of the category **Set** and landing on **Grp**.

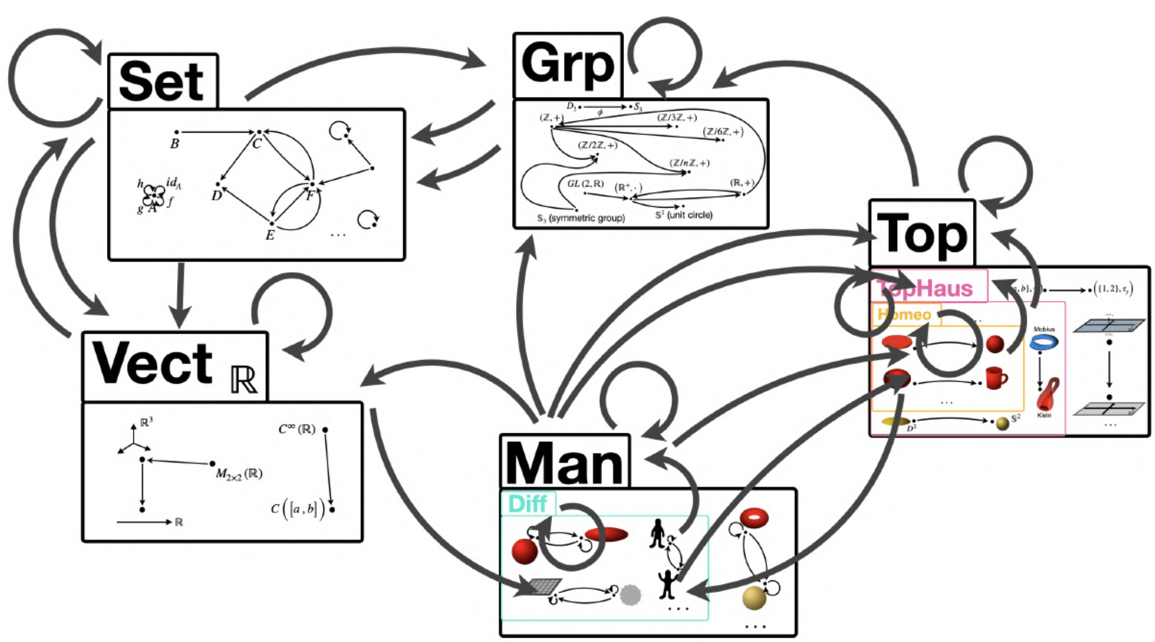
We can also add another collection of extra structure to those *sets*, and then turn them into real *vector spaces*. Let's represent it with an arrow connecting the categories **Set** and $\mathbf{Vect}_{\mathbb{R}}$.

Also, each category should be connected to their subcategories through an *inclusion* arrow.

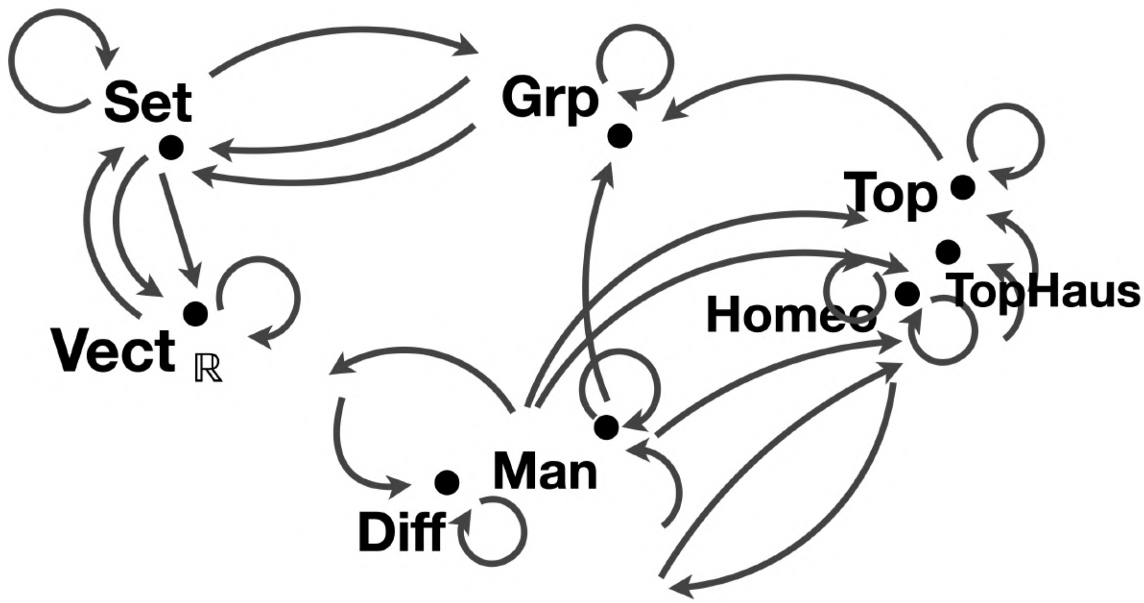
What about *identity* arrows? I guess we can do that...

We can also create some sort of *forgetful* arrow, that instead of adding structure, "forgets" it.

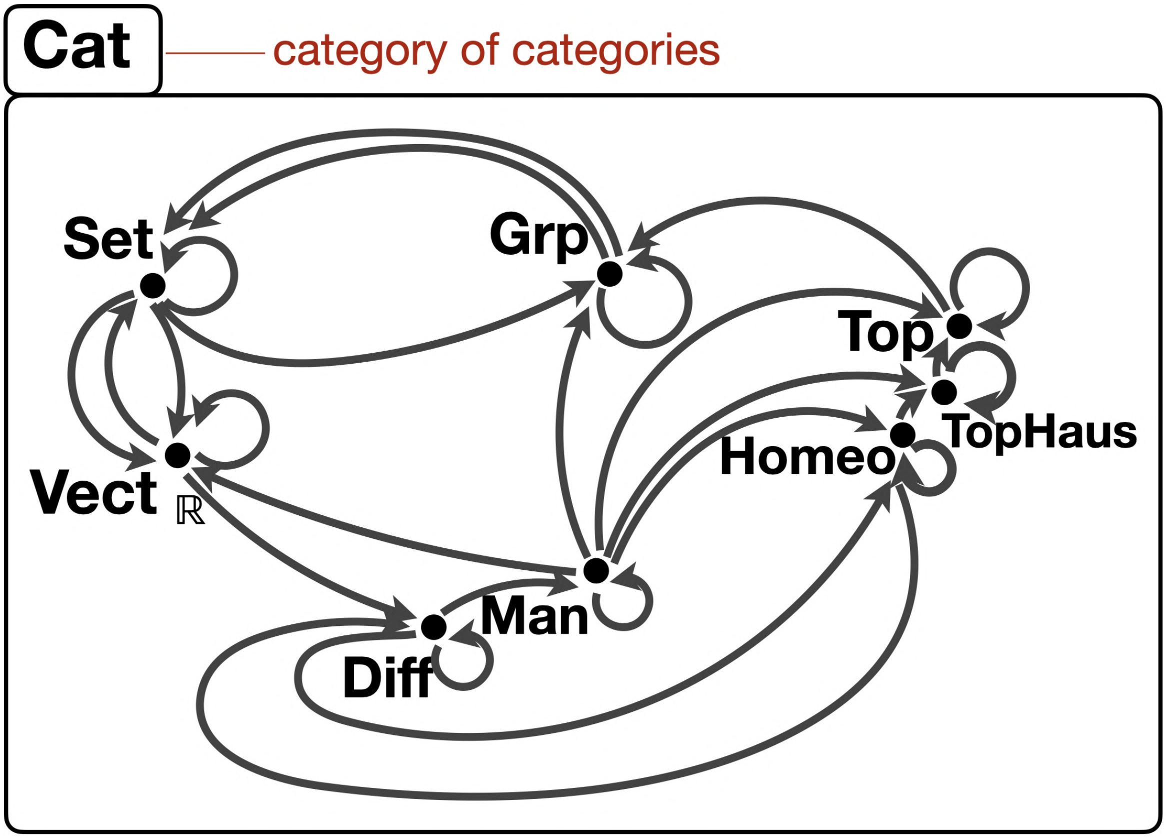
We could produce many other arrows that logically connect these categories.



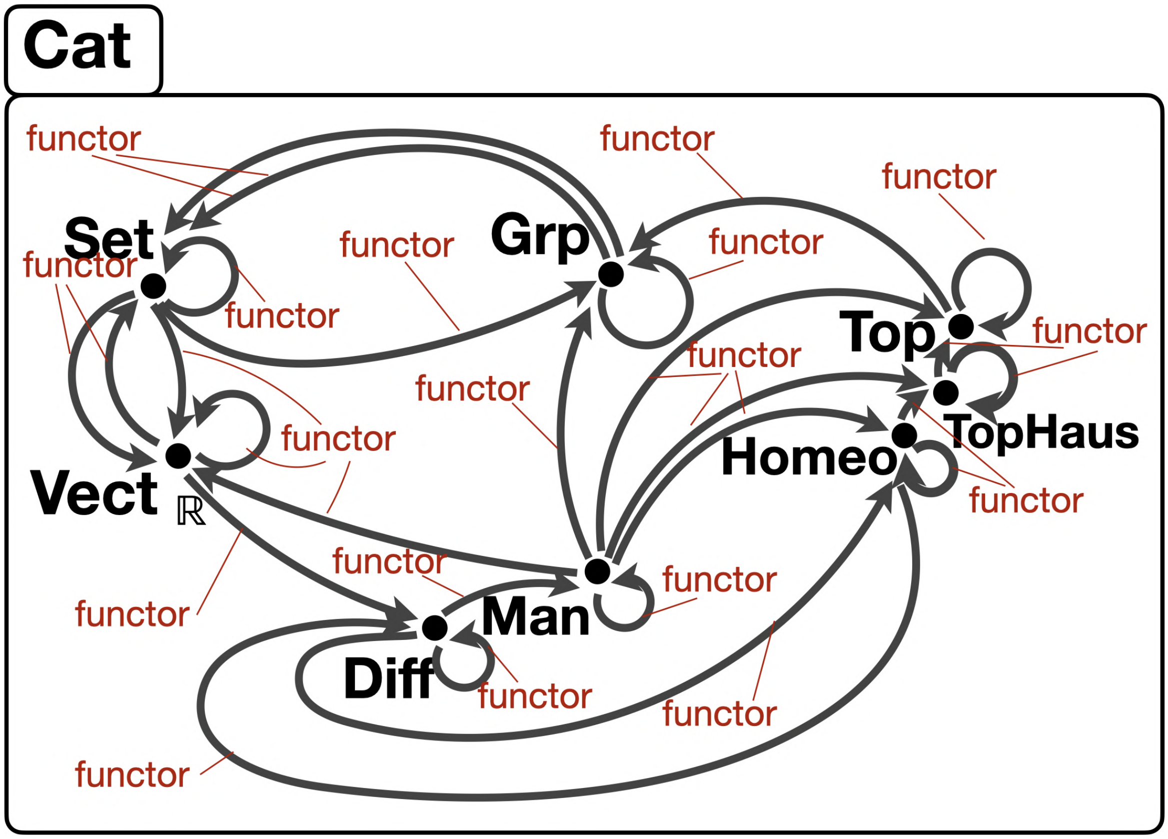
You know, this is getting way too clustered and messy. In order to better visualize what's going on here, why don't we collapse each category to a point, and ignore their internal structure just for a moment…?



I hope by now you get where I'm heading to. Each of these categories can themselves be viewed as objects inside a larger structure called **Cat**, which is the *category of categories*.



The morphisms between these objects are called **functors**, and they are not just arrows we make up as we feel like. These are mappings between categories that preserve their internal structure. They map objects to objects, and morphisms to morphisms, in a way that represents composition and identities.



This perspective opens an important door to a whole new level of abstraction. Instead of just studying structures within a specific category, we can study the relationships between entire mathematical fields themselves.

Welcome to the world of **Category Theory**.

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