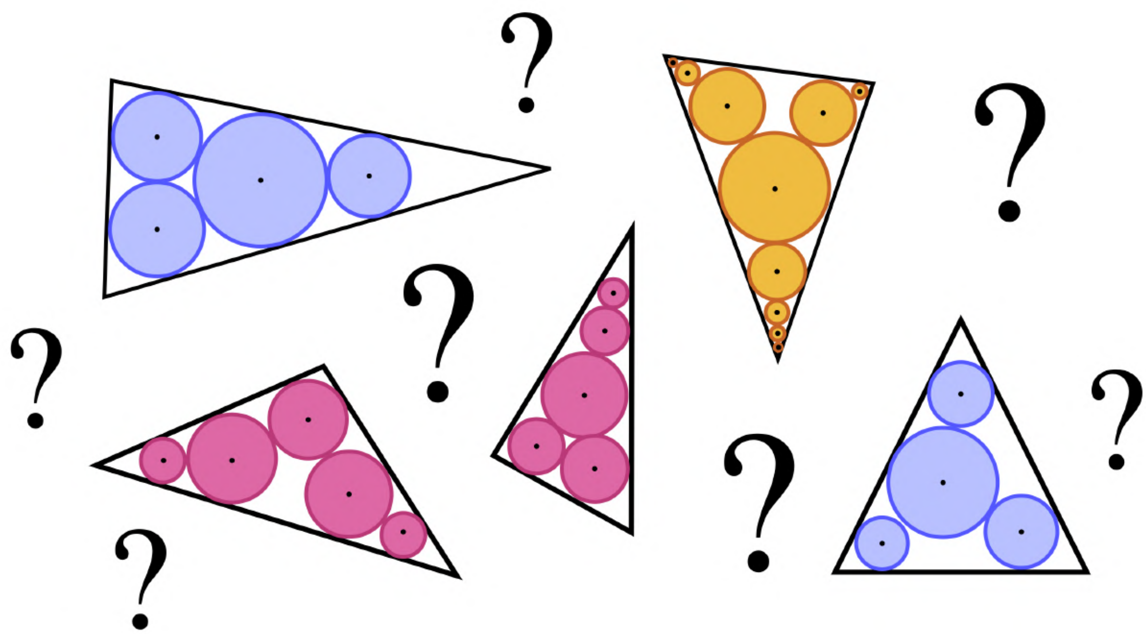
The Simple Problem Mathematicians Got Wrong

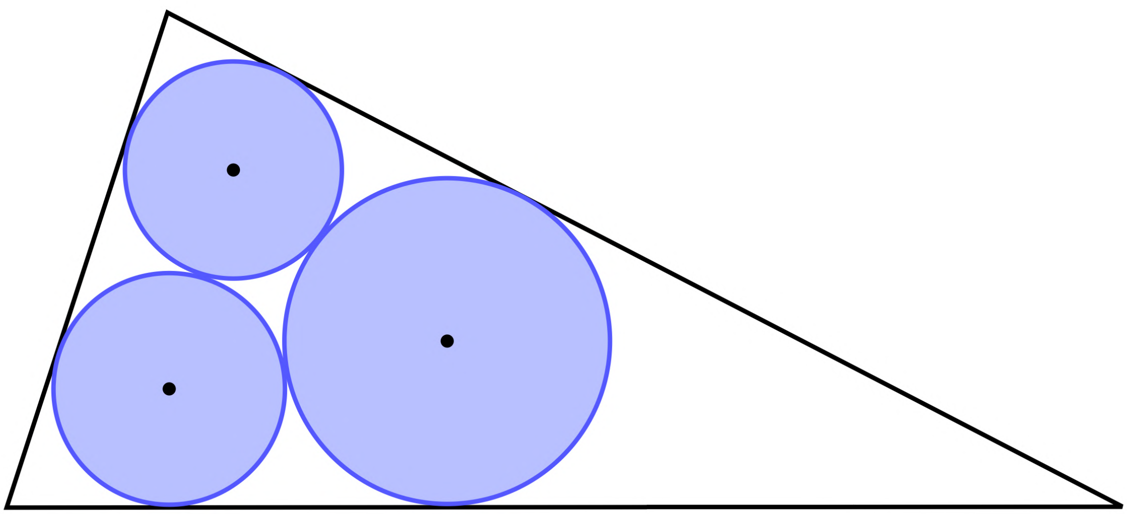
by DiBeos



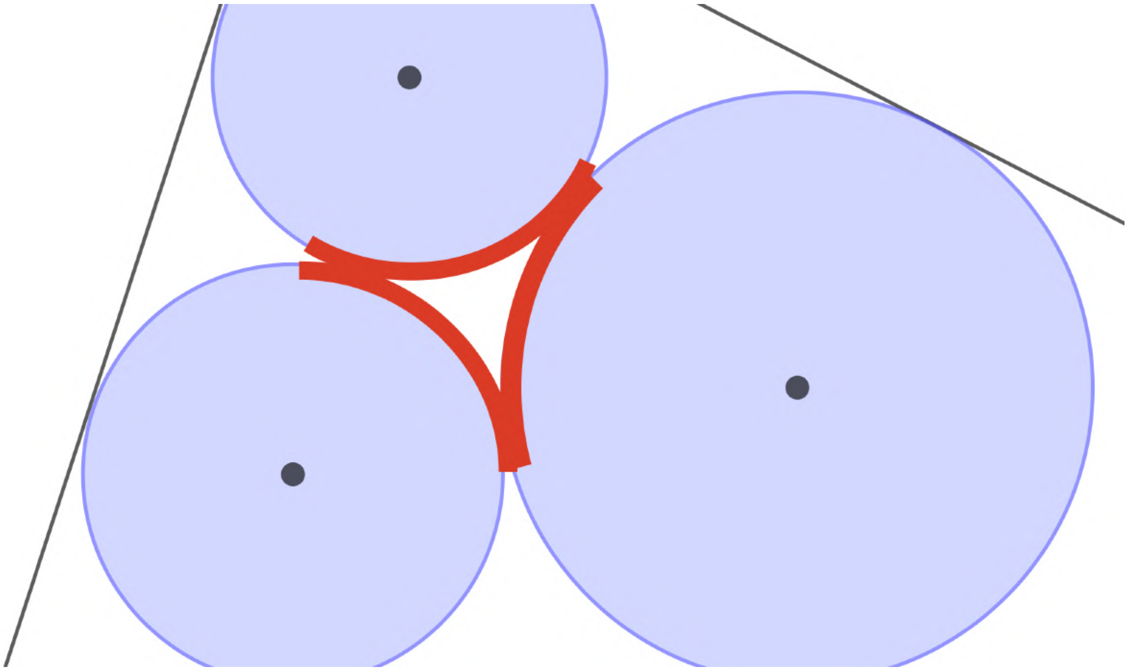
"Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness where the explorers often get lost."
– W.S. Anglin

Note: *You can access the full version of this PDF at FULL-PDF. There you will see the solution of finding the center of one of Malfatti's circles only using synthetic geometry. We also added a deeper explanation about the greedy algorithm, as well as examples and an exercise (with the detailed solution). Thanks for supporting our work.*

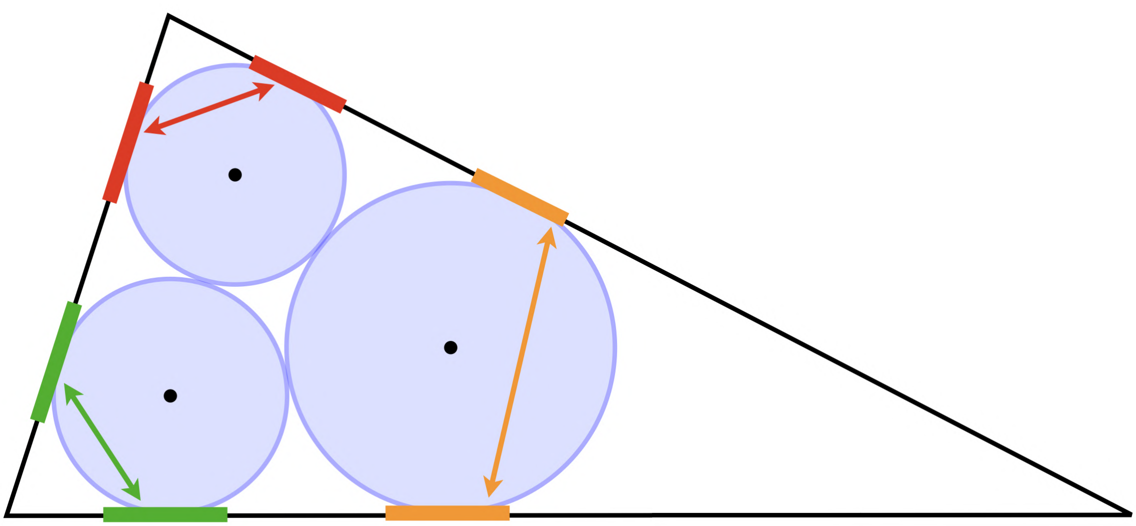
Introduction



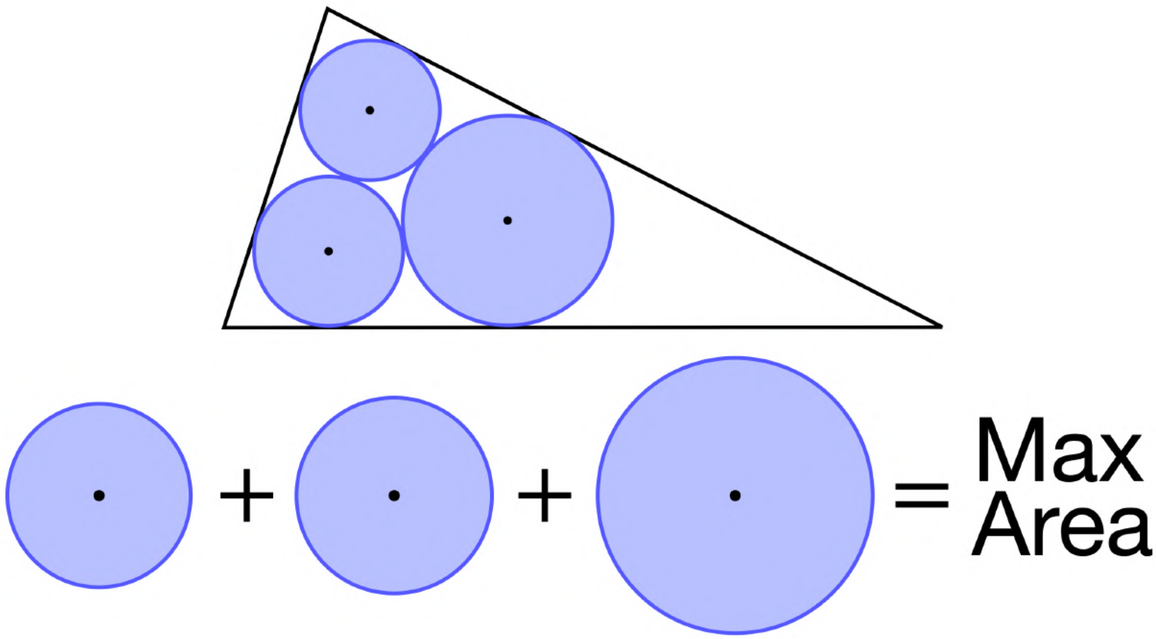
There's a very simple problem that for 100 years, mathematicians just accepted as solved, and yet, no one seemed to question if that solution was really right. Turns out… it wasn't. And there's a version of it that still hasn't been solved to this day.



It is the problem of **Malfatti circles**. Imagine 3 non-overlapping circles inside a triangle, each of which has to touch its neighbors, and each of which has to touch two sides of the triangle.

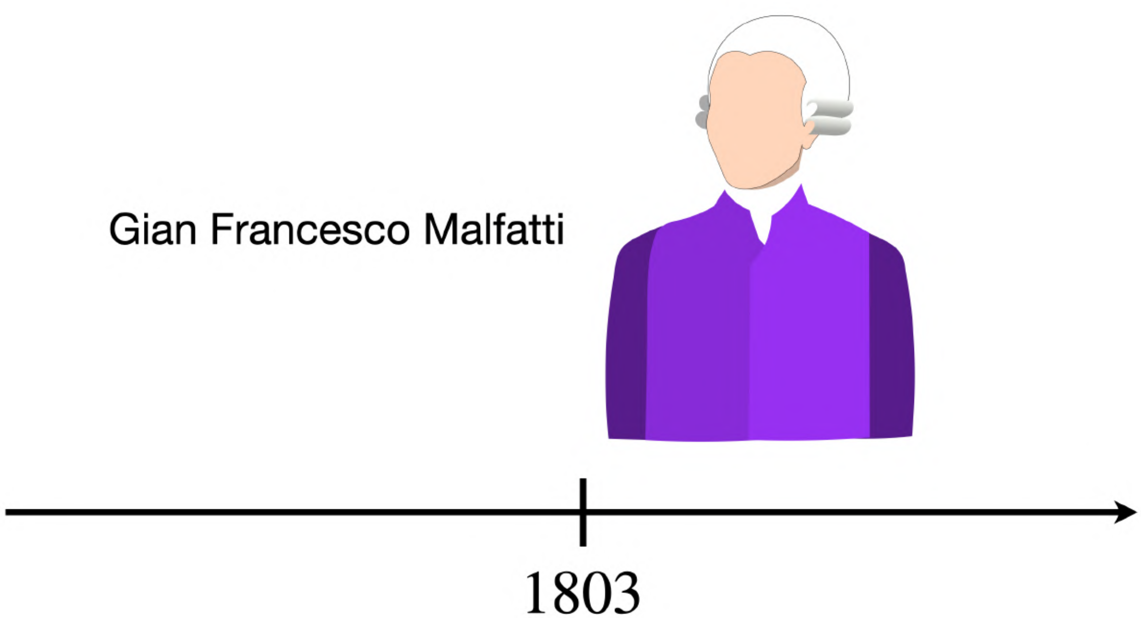


It was believed that any 3 such circles, inscribed within a triangle, would together *maximize the total area* of the 3 circles.

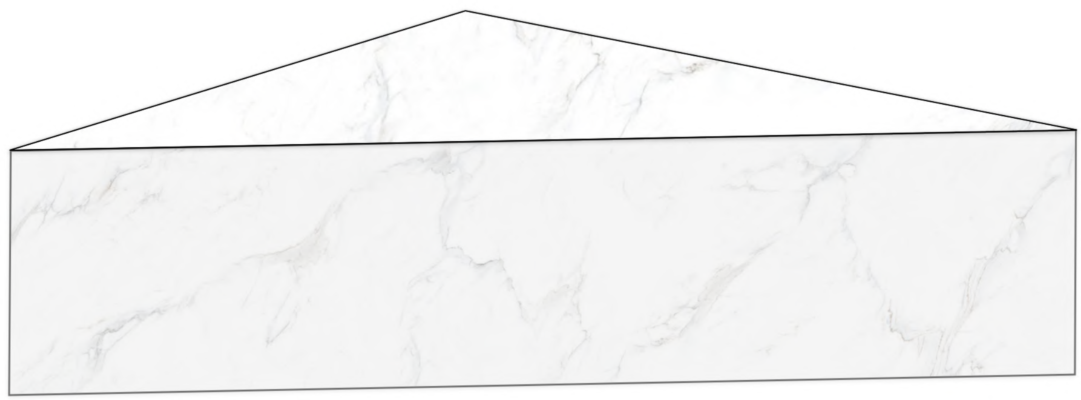


And intuitively, they do look optimal. But… they're not. And the worst part? The proof that they aren't is not even that complicated.

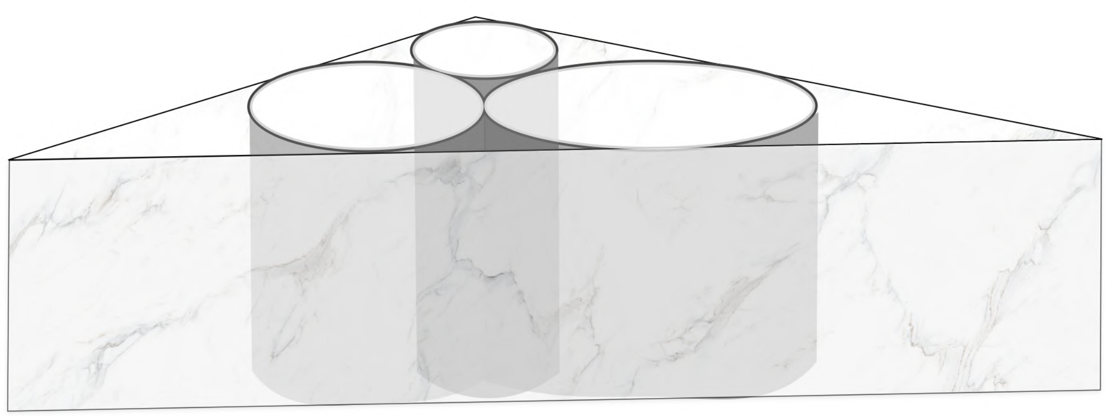
Context



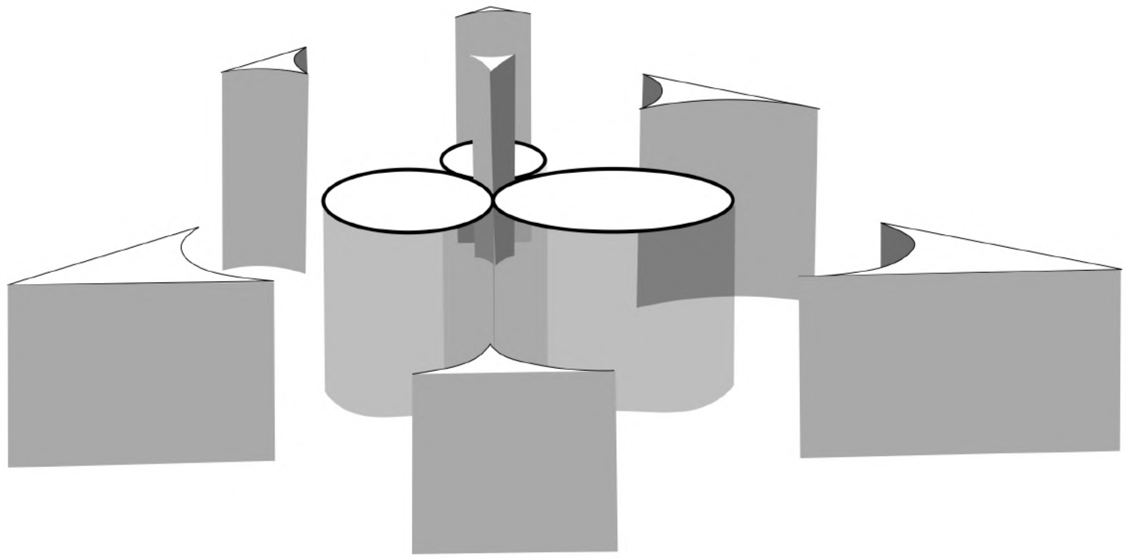
In the year 1803, *Gian Francesco Malfatti* asked what seemed like an innocent question.



He had a prism made of marble, and his goal was to carve out 3 cylindrical columns, in order to optimize the space and not waste any marble.

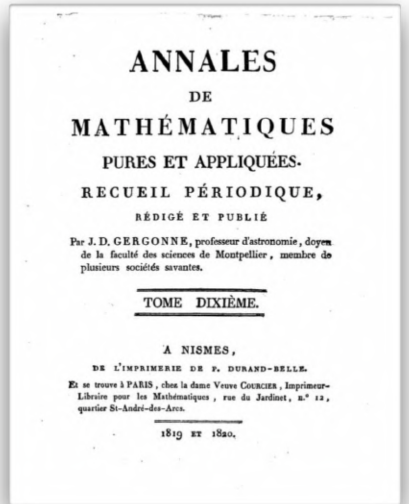


So, what shape do those columns have to take to maximize the total volume removed?



Malfatti assumed that the solution to this problem was given by 3 tangent circles within the triangular cross-section.

Malfatti's assumption seemed to be so elegant that it was quickly accepted. By 1811, French mathematician *Joseph Diaz Gergonne* had published it widely.



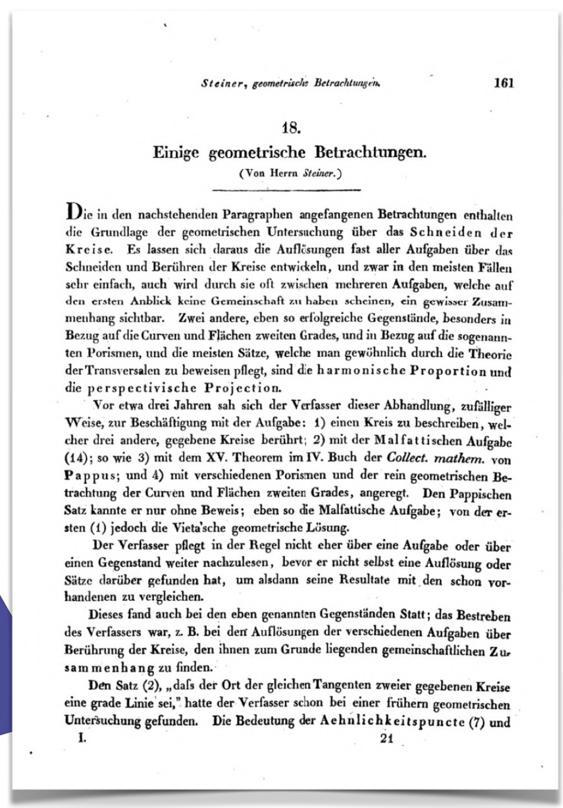
ANNALES
DE
MATHÉMATIQUES
PURES ET APPLIQUÉES.
RECUEIL PÉRIODIQUE,
RÉDIGÉ ET PUBLIÉ
Par J. D. GERGONNE, professeur d'astronomie, doyen de la faculté des sciences de Montpellier, membre de plusieurs sociétés savantes.

TOME DIXIÈME.

A NISMES,
DE L'IMPRIMERIE DE P. DURAND-BELLE.
Et se trouve à PARIS, chez la dame Veuve COURCIER, Imprimeur-Libraire pour les Mathématiques, rue du Jardinet, n.° 12, quartier St-André-des-Arcs.

1819 ET 1820.

And in 1826, *Jakob Steiner*, a giant in geometry, published a construction method that popularized it further, by using synthetic geometry.



Steiner, geometrische Betrachtungen. 161

18.
Einige geometrische Betrachtungen.
(Von Herrn *Steiner.*)

Die in den nachstehenden Paragraphen angefangenen Betrachtungen enthalten die Grundlage der geometrischen Untersuchung über das Schneiden der Kreise. Es lassen sich daraus die Auflösungen fast aller Aufgaben über das Schneiden und Berühren der Kreise entwickeln, und zwar in den meisten Fällen sehr einfach, auch wird durch sie oft zwischen mehreren Aufgaben, welche auf den ersten Anblick keine Gemeinschaft zu haben scheinen, ein gewisser Zusammenhang sichtbar. Zwei andere, eben so erfolgreiche Gegenstände, besonders in Bezug auf die Curven und Flächen zweiten Grades, und in Bezug auf die sogenannten Porismen, und die meisten Sätze, welche man gewöhnlich durch die Theorie der Transversalen zu beweisen pflegt, sind die harmonische Proportion und die perspectivische Projection.

Vor etwa drei Jahren sah sich der Verfasser dieser Abhandlung, zufälliger Weise, zur Beschäftigung mit der Aufgabe: 1) einen Kreis zu beschreiben, welcher drei andere, gegebene Kreise berührt; 2) mit der Malfattischen Aufgabe (14); so wie 3) mit dem XV. Theorem im IV. Buch der *Collect. mathem.* von Pappus; und 4) mit verschiedenen Porismen und der rein geometrischen Betrachtung der Curven und Flächen zweiten Grades, angeregt. Den Pappischen Satz kannte er nur ohne Beweis; eben so die Malfattische Aufgabe; von der ersten (1) jedoch die Vieta'sche geometrische Lösung.

Der Verfasser pflegt in der Regel nicht eher über eine Aufgabe oder über einen Gegenstand weiter nachzulesen, bevor er nicht selbst eine Auflösung oder Sätze darüber gefunden hat, um alsdann seine Resultate mit den schon vorhandenen zu vergleichen.

Dieses fand auch bei den eben genannten Gegenständen Statt; das Bestreben des Verfassers war, z. B. bei den Auflösungen der verschiedenen Aufgaben über Berührung der Kreise, den ihnen zum Grunde liegenden gemeinschaftlichen Zusammenhang zu finden.

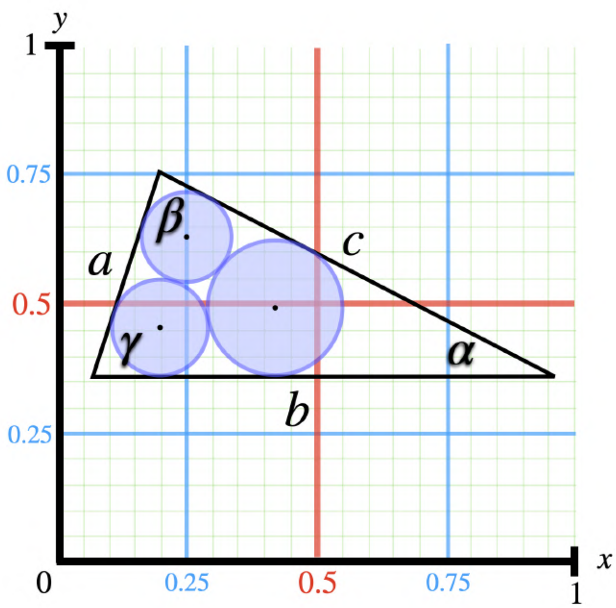
Den Satz (2), „dafs der Ort der gleichen Tangenten zweier gegebenen Kreise eine grade Linie sei," hatte der Verfasser schon bei einer frühern geometrischen Untersuchung gefunden. Die Bedeutung der Aehnlichkeitspuncte (7) und

I. 21

Around this time, many geometers in Europe tended to favor **synthetic geometry** over **analytical geometry** because of their philosophical preference. They basically believed that synthetic geometry preserved purity, visual intuition, and the elegance of geometry.

Jakob Steiner was one of the biggest spreaders of this view. His work on the Malfatti circles was something beyond just another mathematical advancement. It was also a philosophical statement, it was his own way of emphasizing the power of reasoning purely geometrically, without relying on algebra or coordinates.

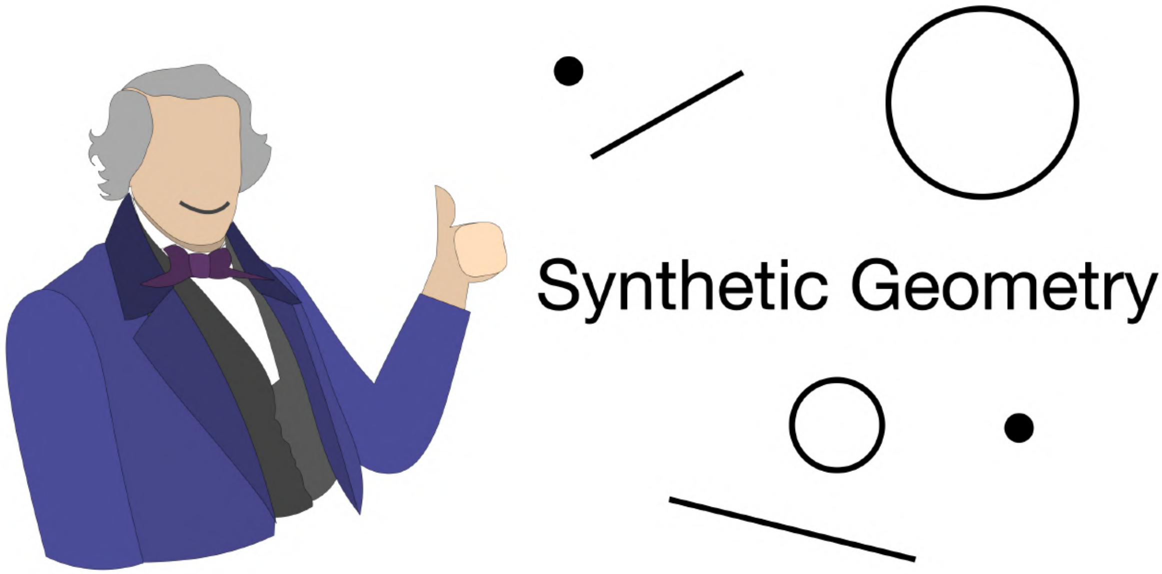
To make it clearer, *analytical geometry* (which is also called *coordinate geometry*) maps geometric figures onto algebraic expressions by placing them in a Cartesian coordinate system. It lets geometric problems be translated into equations, and then to be solved using calculus or some kind of algebraic method.



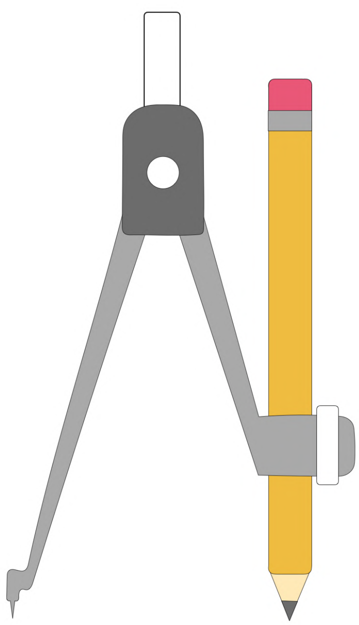
$$c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$$

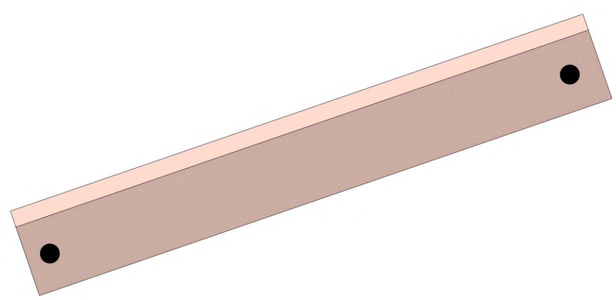
Analytical geometry is a great tool, but, to some classical or pure geometers like Steiner, it felt too much like algebra in disguise.

These mathematicians tended to favor *synthetic geometry*, which works with geometric primitives like points, lines, and circles by using logical deductions and constructions.



Synthetic geometry uses methods which involve compasses and straight-edges, and don't have references to numerical coordinates.





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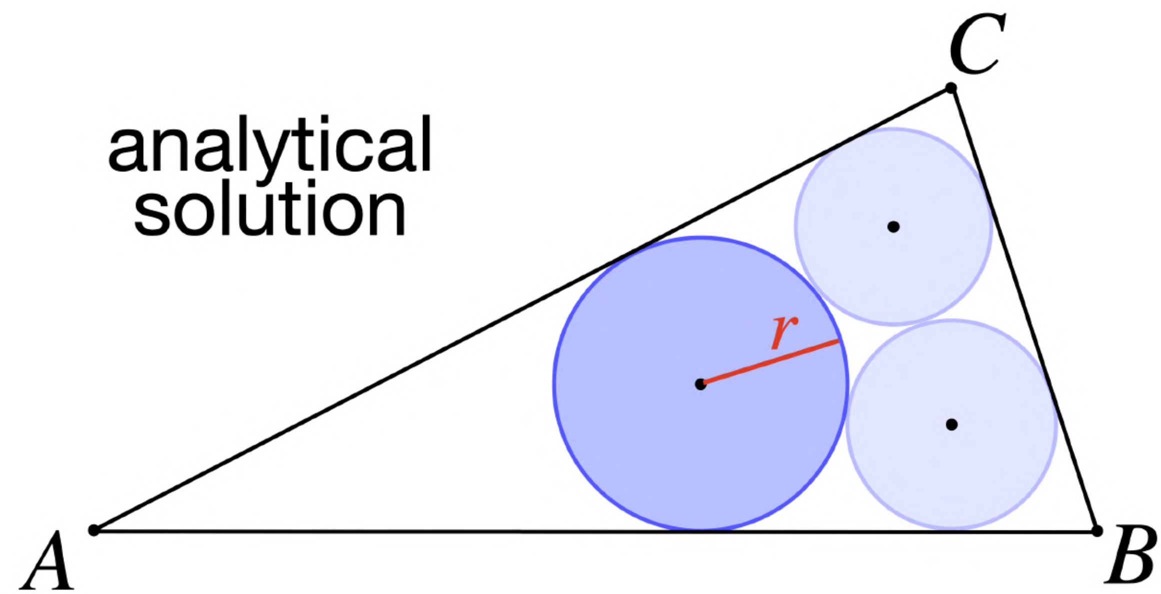
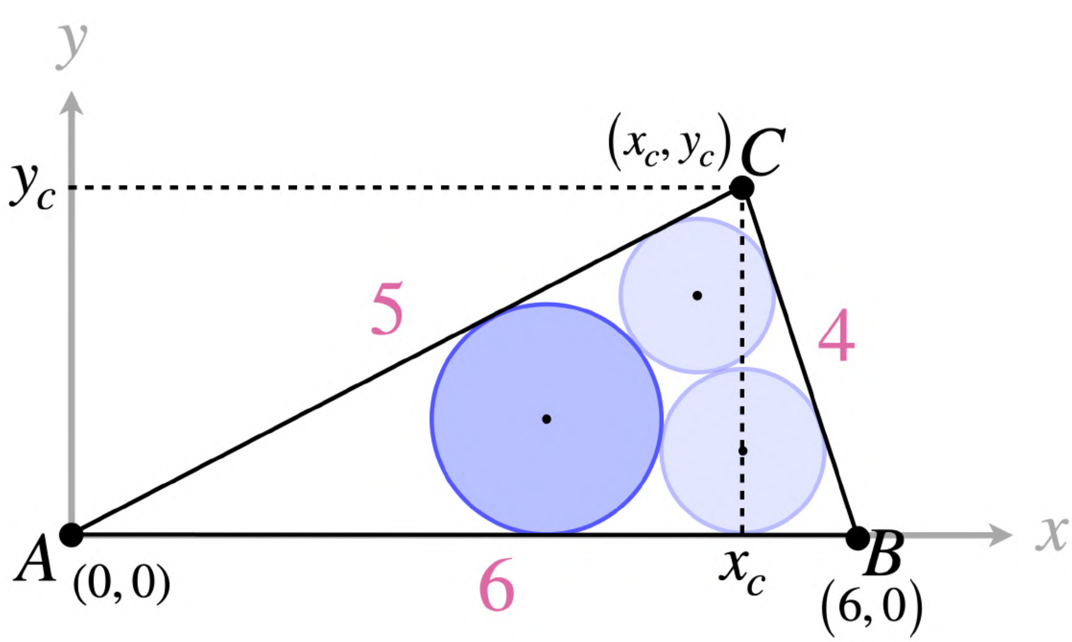
Example

Say we want to find the center of one of the Malfatti circles (with radius r) in an acute triangle ΔABC. We could first solve it **analytically**:

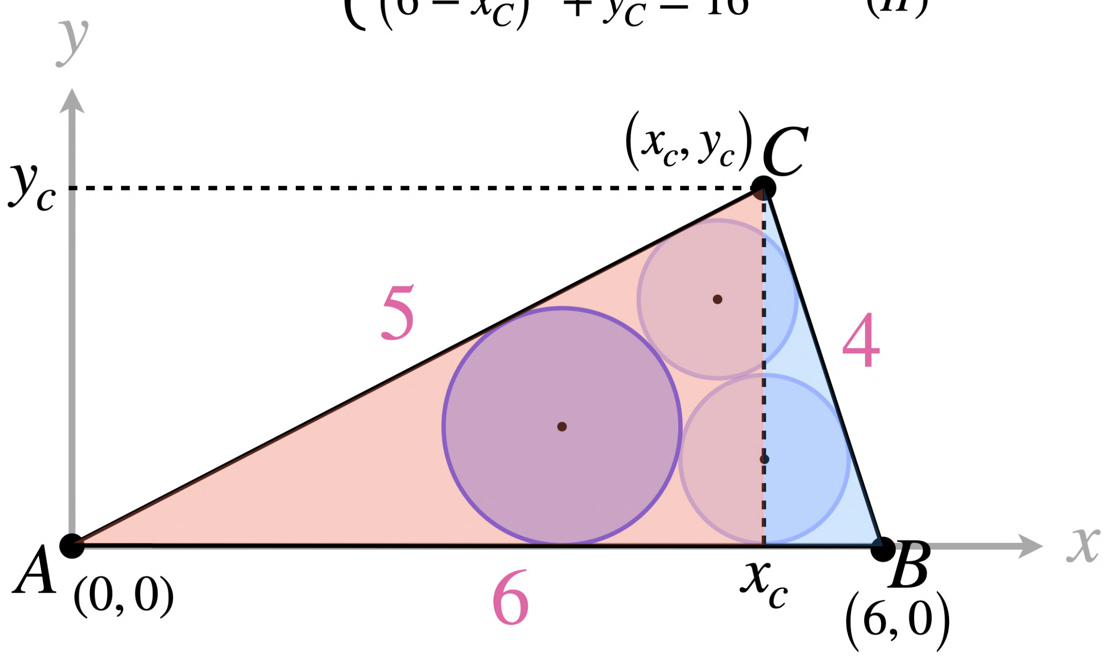


Let's define a Cartesian plane with origin at point A, and the triangle has sides $4, 5, 6$. As a consequence, we can write all vertices in terms of their coordinate pairs. A and B are easy to find, but we need to do some math in order to find the coordinates of C.



Using the Pythagorean theorem for these two right triangles (show below), we can write these equations:

$$\begin{cases} x_C^2 + y_C^2 = 25 & (I) \\ (6 - x_C)^2 + y_C^2 = 16 & (II) \end{cases}$$



That's a system with 2 *equations* and 2 *unknowns*.

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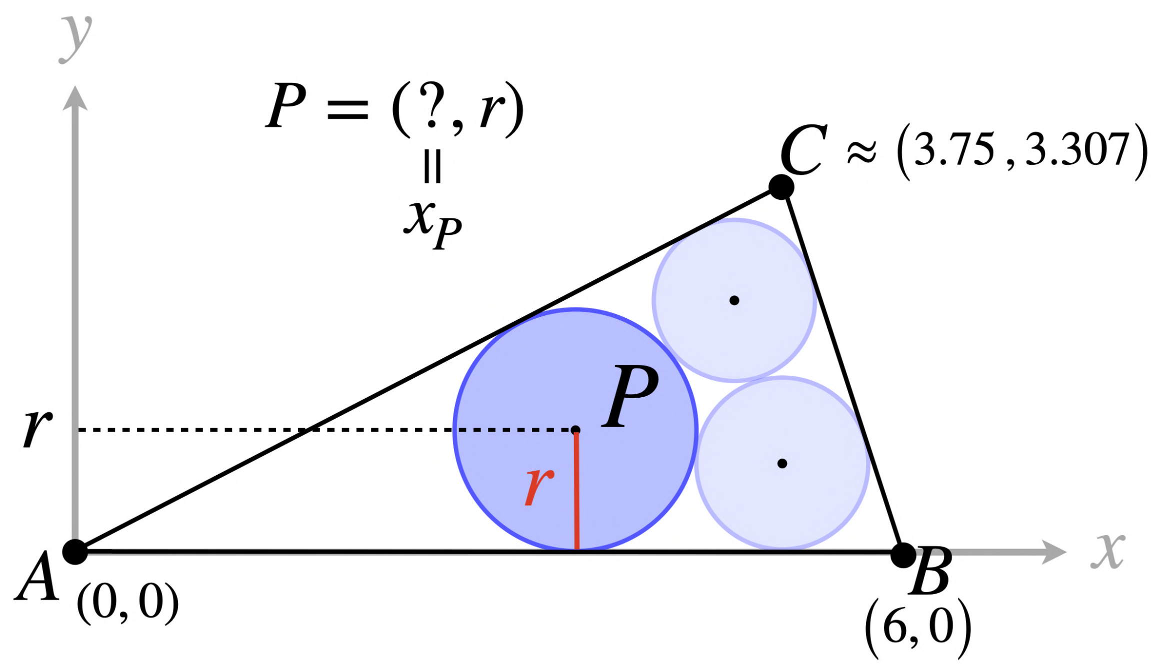
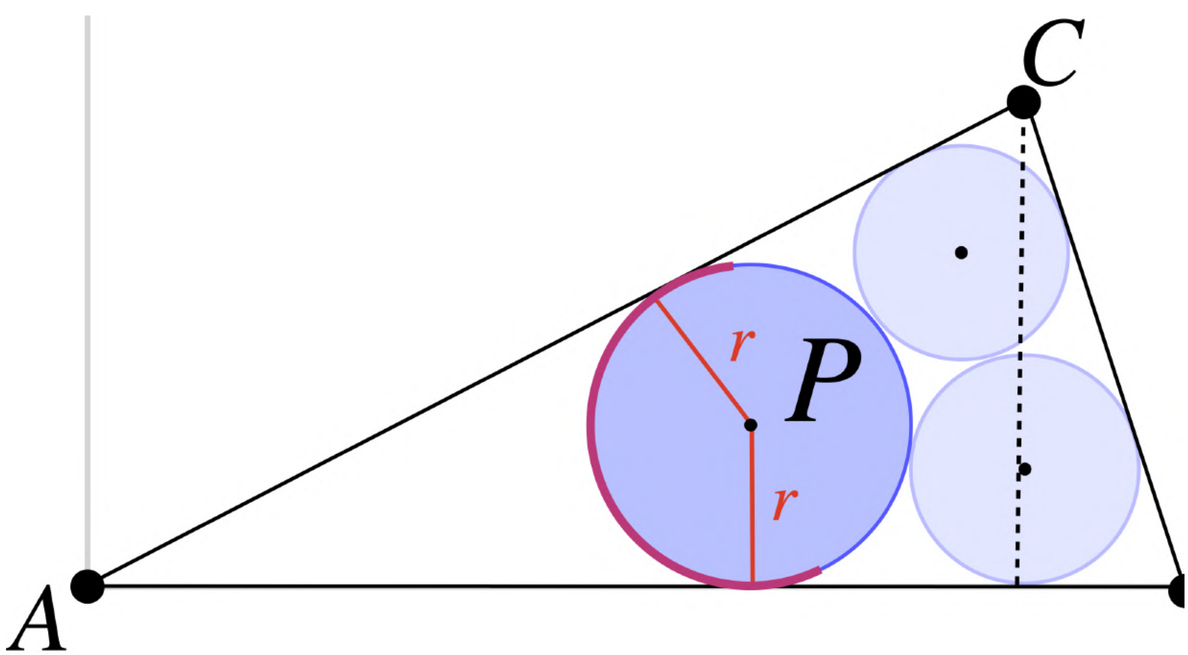
After solving it in detail in the FULL-PDF, we found the following:

$$(x_C, y_C) \approx (3.75, 3.307)$$

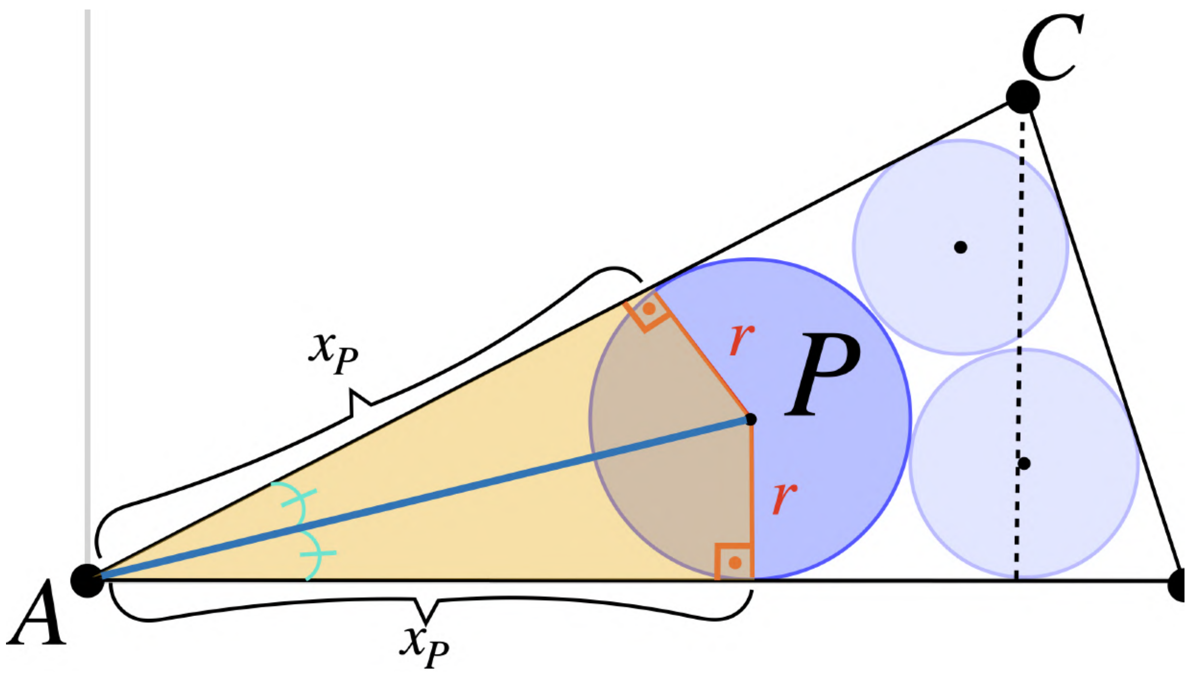
Now, I remind you guys that we are looking for the center P of the circle touching sides AB and AC. Well, since its radius is r, it follows that $P = (x_P, r) = (?, r)$. We just need to find the x component of point P, and then we're done.



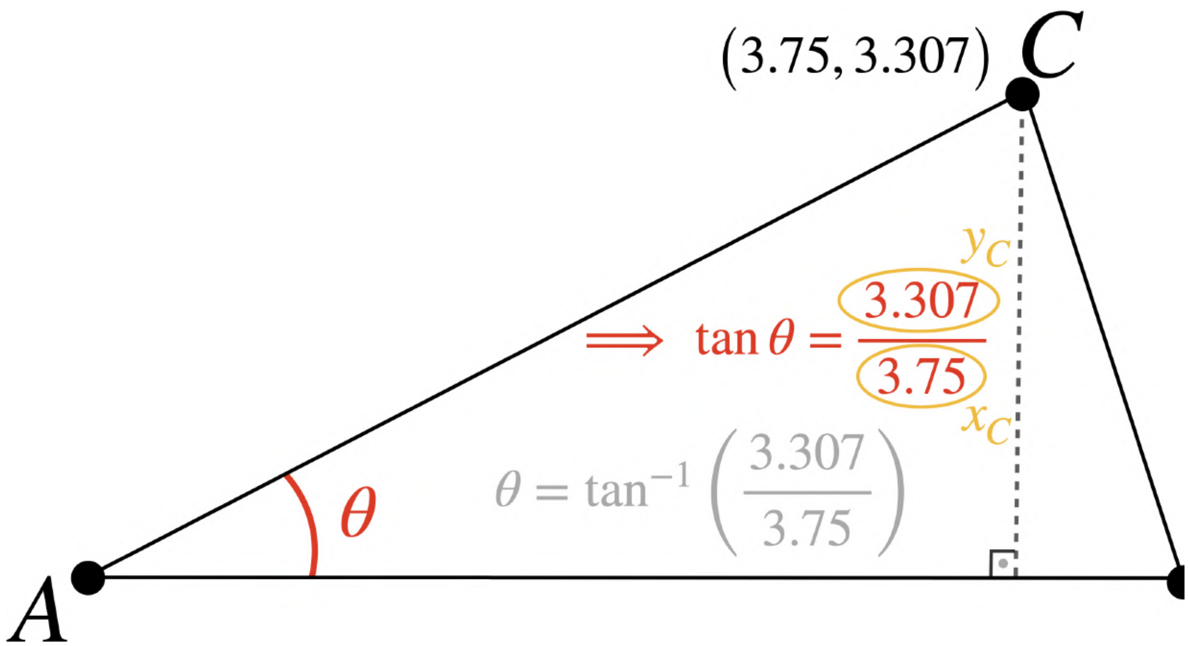
Notice how these two lengths are r (below).



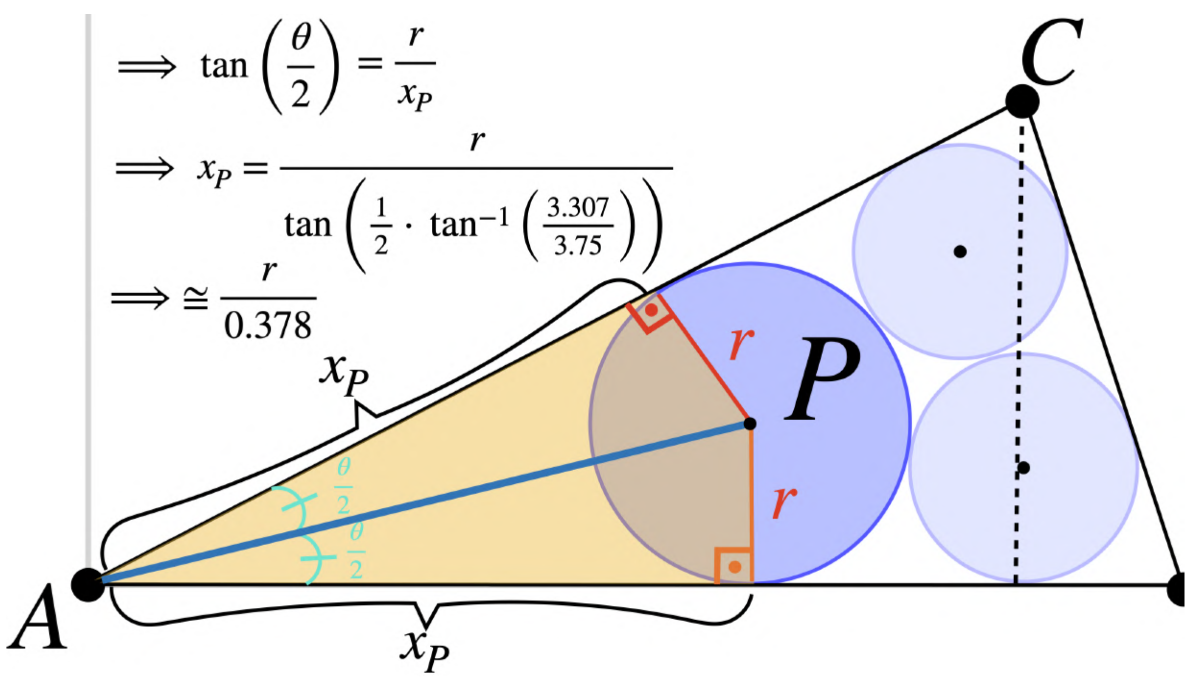
And since the circle is tangent to the sides AB and AC, we also know that these are 90° angles (below). Look at what we just did! We found two congruent triangles. Both of these sides measure x_P, which corresponds to the x component of point P, i.e. our unknown!



Using the formula of the tangent of the angle $\angle A =: \theta$, we can equate it to its opposite side (which is the y component of point C) divided by the adjacent side (which is the x component of point C). This way we can find the angle θ.



Another thing to notice here, is that we just split the angle θ in half. And using the formula of the tangent once again (but this time for a half of θ), we can equate it to its opposite side (which measures r) divided by the adjacent side (which is our unknown x_P).



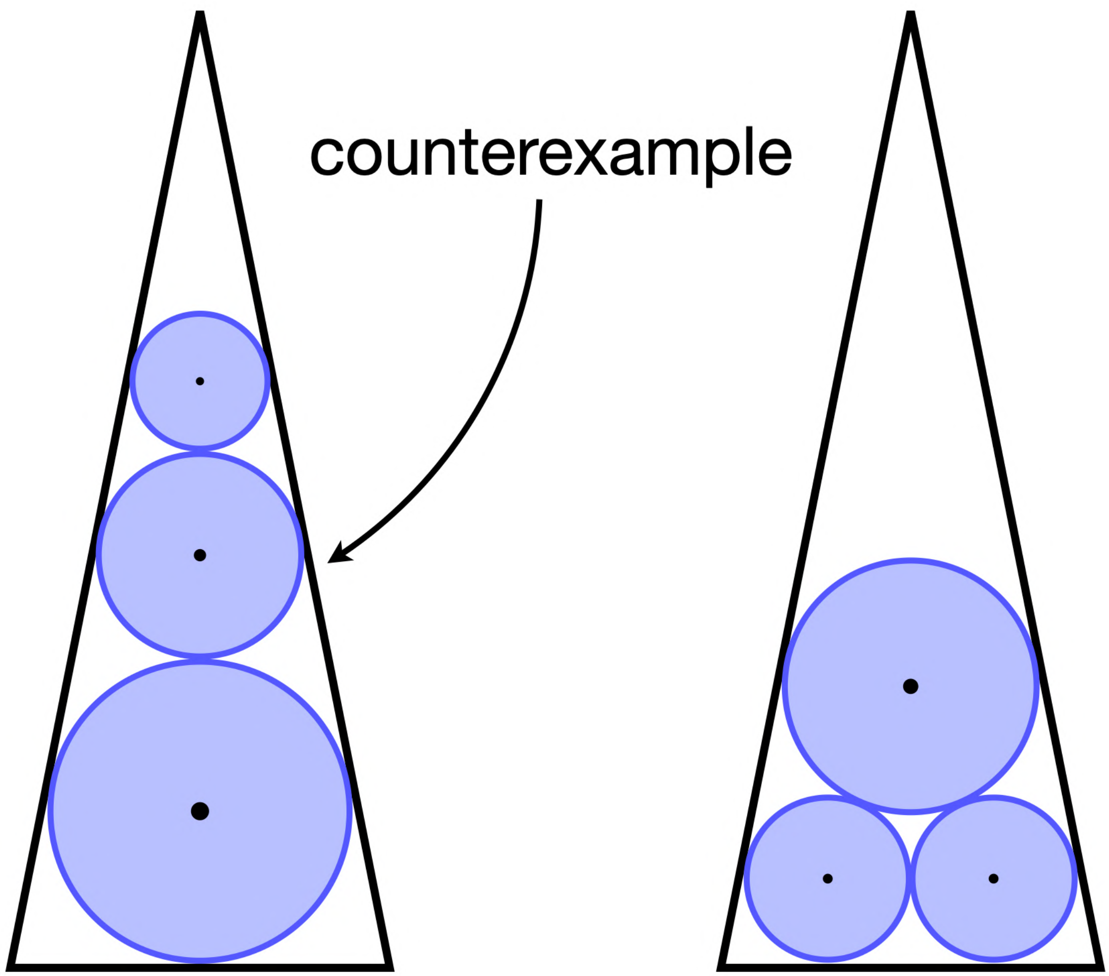
Finally, applying the appropriate inverse transformation, we find the value of the x component of point P in terms of r.

We also solved it using only synthetic geometry in FULL-PDF.

The Greedy Algorithm

Anyway, the problem is beautiful, but there was one little issue. Malfatti's assumption was never actually... *proven*. He just believed (following his own intuition) that mutual tangency = maximal area.

In 1930, a few mathematicians revisited Malfatti's original work, and what they found was shocking. For certain triangles, especially tall, narrow isosceles ones, the Malfatti circles didn't maximize area at all. And they showed a counterexample.



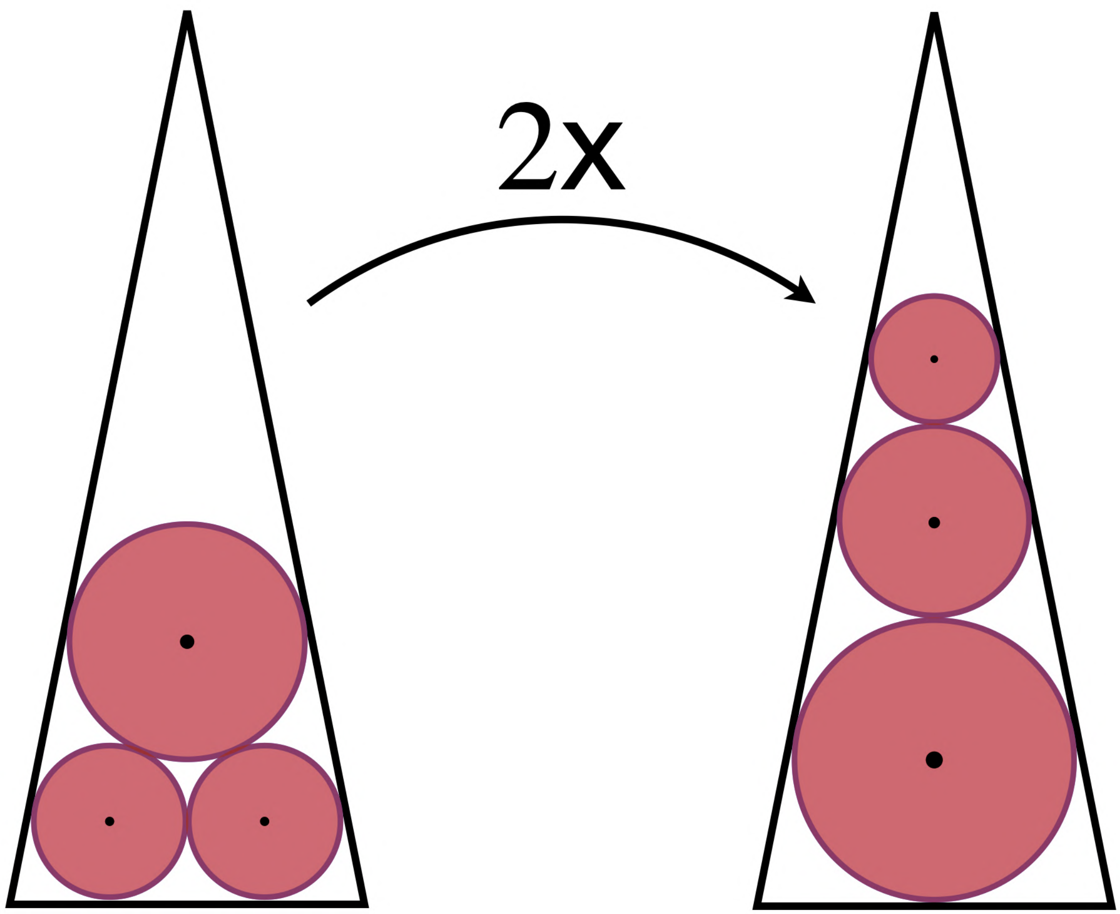
And eventually, a method was found that did work. It is called the **greedy algorithm**.

Here's the idea:

1. Place the largest circle you can inside the triangle.
2. Then place the next largest in the remaining space.
3. Then a third in what's left.

Pretty simple, huh?!

In tall triangles, the greedy method gave almost *twice* the area of the traditional Malfatti circles!



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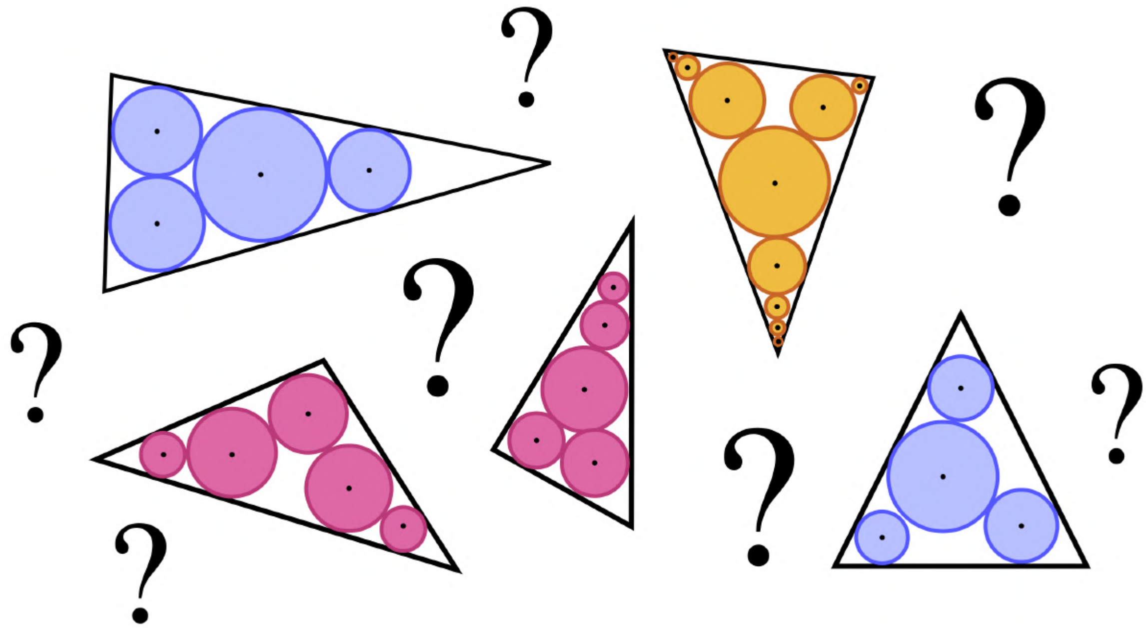
Conclusion

So… why wasn't this discovered sooner?

Part of it was visual deception. The Malfatti circles look so perfect. In an equilateral triangle, the difference in area is barely 1%. Almost indistinguishable. But in skewed or tall triangles? The Malfatti circles are clearly wasteful. And what about a formal proof? Can we prove that the greedy algorithm is always better?

In the 1960s, numerical methods finally confirmed it. The greedy algorithm consistently outperformed Malfatti's construction. And the proof came in 1968, when it was finally, rigorously established that Malfatti's method is never optimal. Not sometimes. Not even in equilateral cases. In 1994, the greedy algorithm was proven to always produce the maximum area for 3 circles in any triangle.

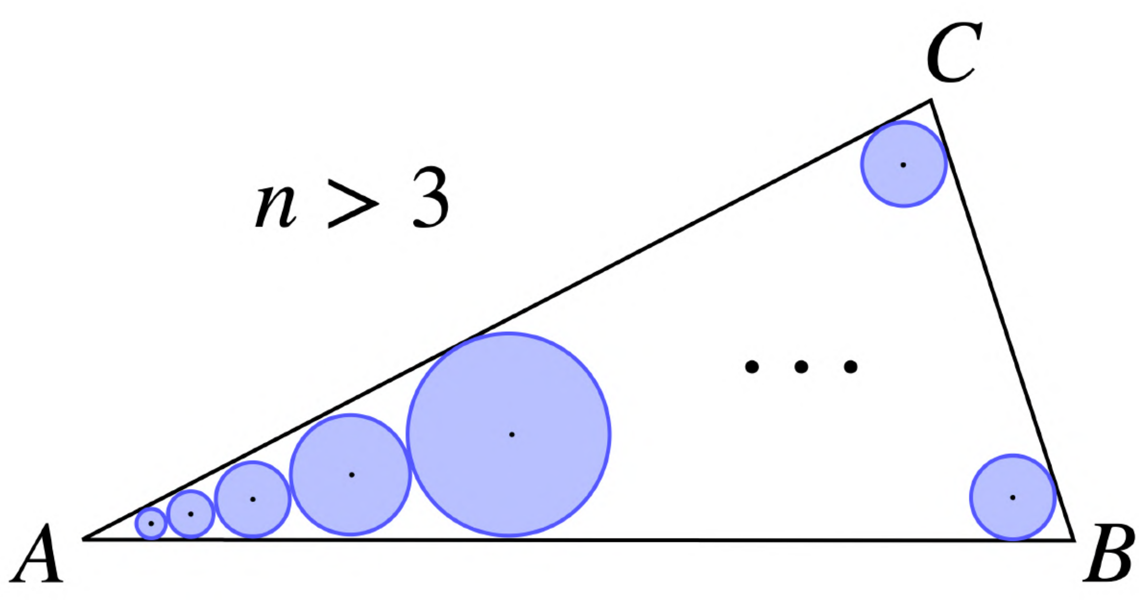
But wait.



We've answered the question for 3 circles. But what about 4? 5? 10?

Here's the twist: we still don't know.

The greedy algorithm is very successful for 3, but it hasn't been proven for higher numbers. We think it works. It might work. But what if our intuition is deceiving us once again?



The problem of placing n area-maximizing circles within a triangle is still unsolved for $n > 3$.

We're still guessing. Just like Malfatti did over 200 years ago.

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