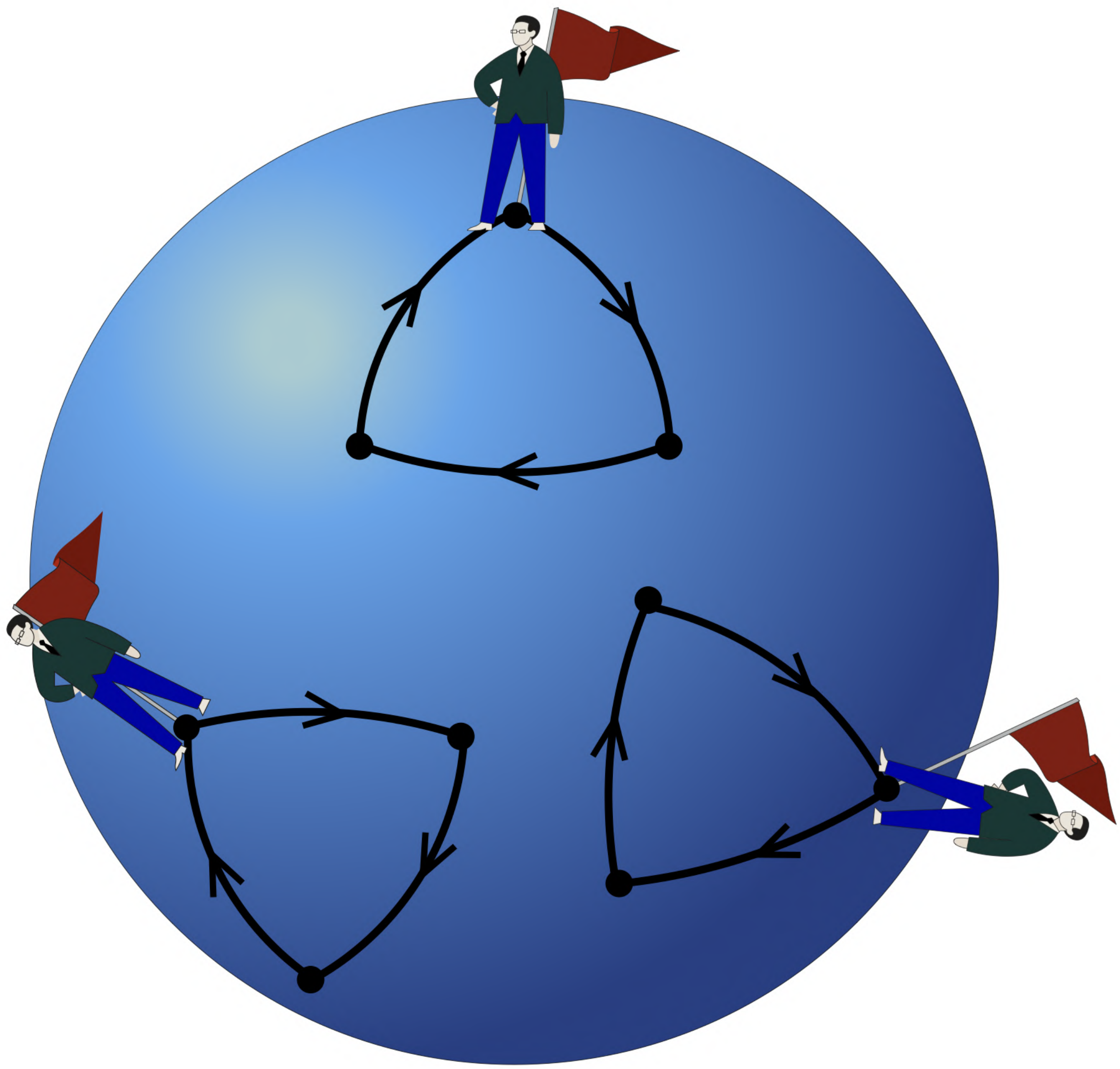
SpaceX Job Interview Problem

by DIBEOS



If you want to support our work, please, consider becoming a **member** *of our YouTube channel! We'd like to keep our videos free of interruptions and sponsors, so that the sole focus is the subject at hand. But in order to do that we need your help. Thanks for supporting us. Also, check out our catalogue of* **PDFs**.

Contents

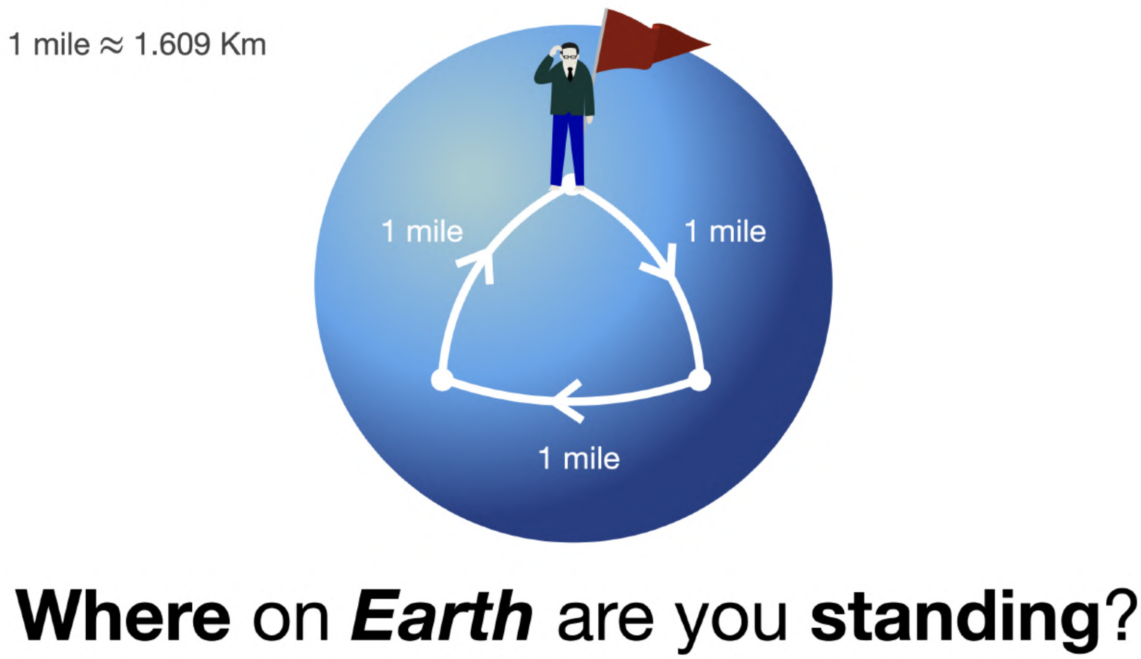
Introduction

There is a very simple geometry problem that most people (including mathematicians) think they understand instantly. But the truth is that, the first answer you will think of probably won't close the question.

1. You walk one mile south.
2. Then one mile west.
3. Then one mile north.

And somehow... you end up exactly where you started.

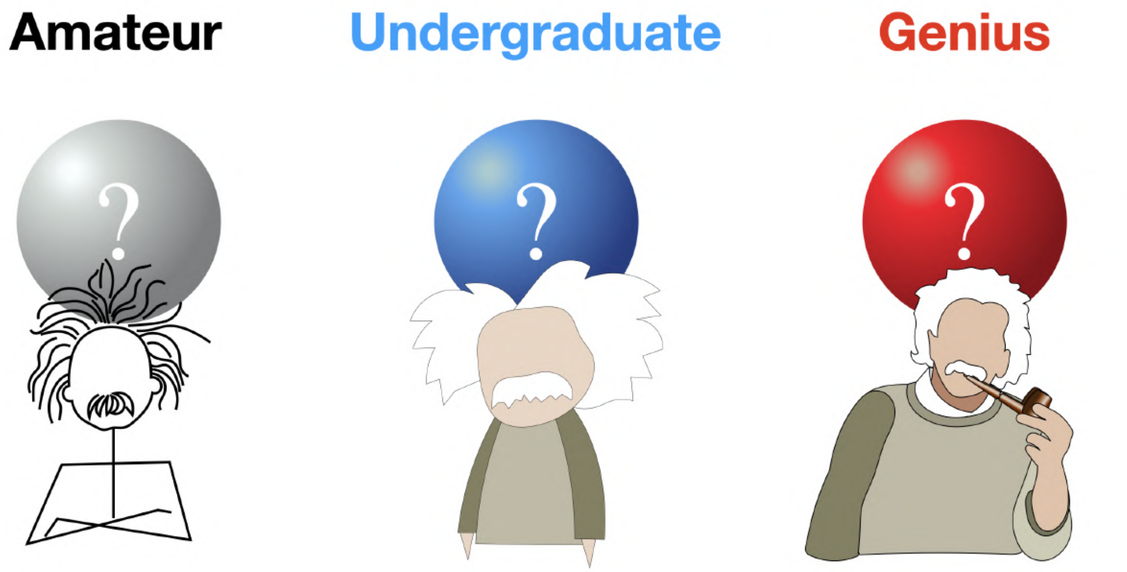
The question is: *"Where on Earth are you standing?"*



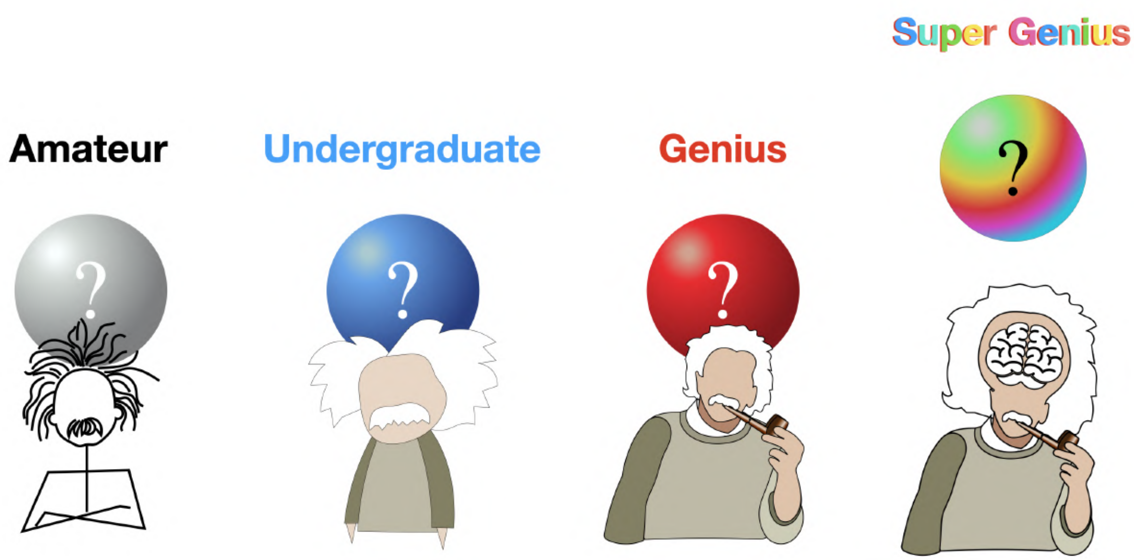
This was used many times as a job interview question at SpaceX.

Yeah, I know, the solution looks simple, but there are actually 3 different levels of "correct answers".

I call the first, the **(1) Amateur Level**, because it provides just one solution that is kind of obvious. The second is the **(2) Undergraduate Level**, and it provides infinite solutions. And the third is the **(3) Genius Level**, which gives us a countably infinite family of sets, each containing uncountably many solutions!



And at the end we will see a bonus explanation, called the **Super Genius Level**.



When I saw this problem for the very first time, before I saw all 3 solutions, I was able to guess the first and second correctly. But for the third level, I have to admit that I probably would've never thought about such an insightful solution…

Now, let's see if you would pass the test.

(1) Amateur Level

To start, we can use a process of elimination. The steps given by the problem are:

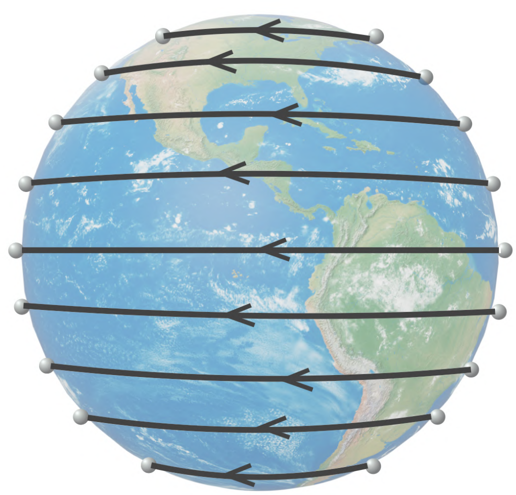
1. 1 mile **south**
2. 1 mile **west**
3. 1 mile **north**

If we start exactly at the South Pole, we can't even complete the first step of the problem, because how can we move farther south if we are already right at the most southern point on Earth? So, this point has to be eliminated as an option.

1. 1 mile **south**
2. 1 mile **west**
3. 1 mile **north**

What about the second step (moving west)? I mean, are there any restrictions to moving west?

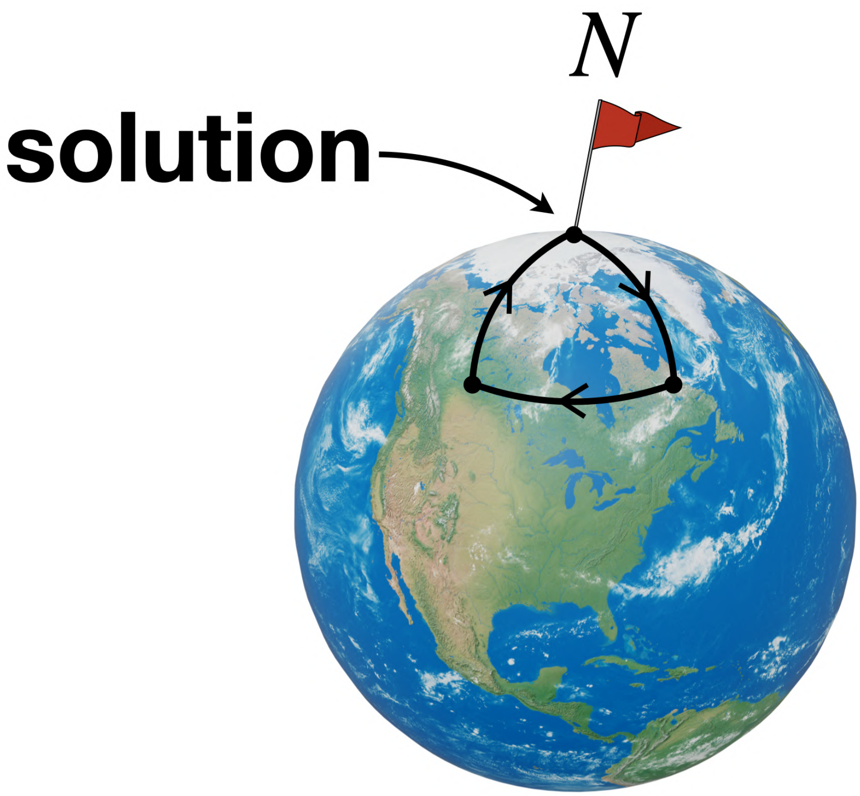
1. 1 mile **south**
2. 1 mile **west**
3. 1 mile **north**



No, because all the parallel latitudes are circles that go around the Earth, except for the poles.

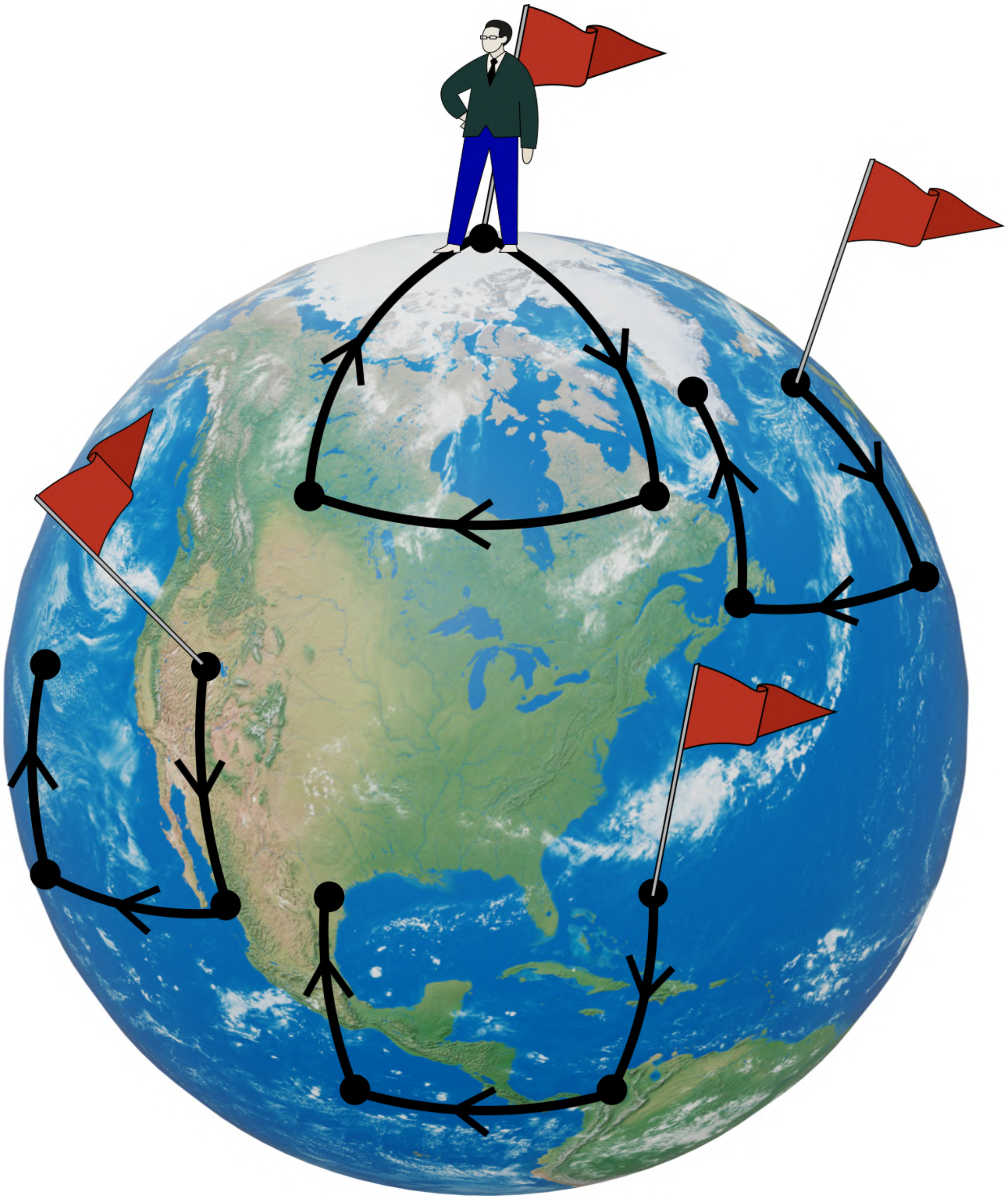
This doesn't mean that we have to eliminate the North Pole as a solution, because all it tells us is that we cannot find ourselves at the poles after completing step 1, i.e. for step 2 specifically, we can't find ourselves on the poles.

Ok, that's a helpful observation. But what is the solution after all? If you follow these 3 steps and end up at the same point you started at, where were you standing at the beginning (and at the end)? Have you solved it?

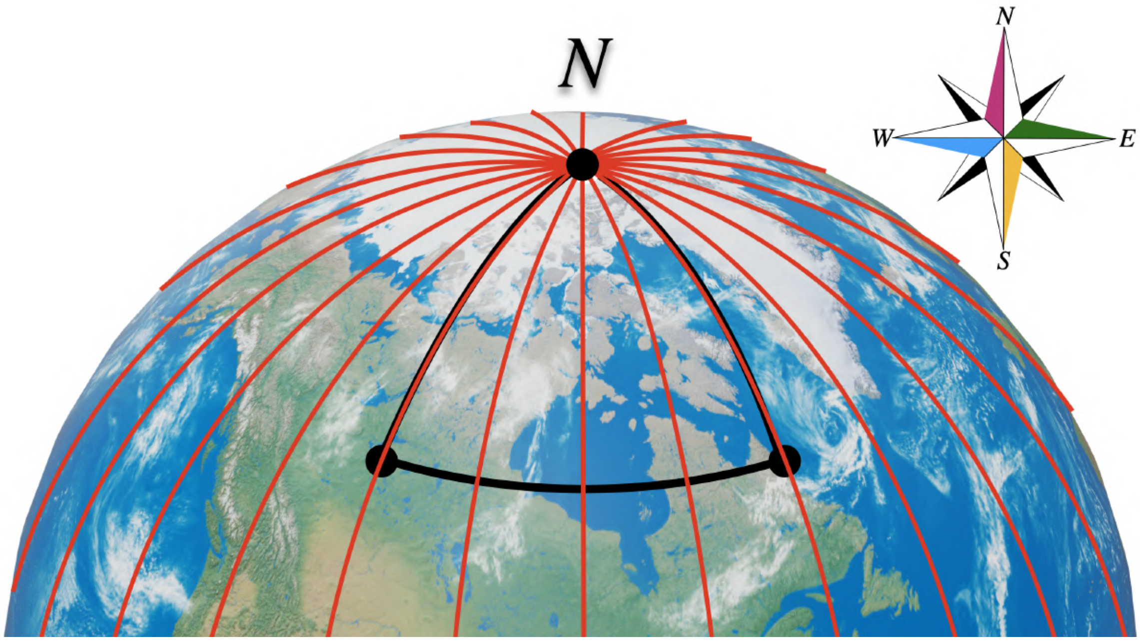


The **Amateur Level** of solution is the **North Pole**. You can see how moving 1 mile south, then 1 mile west, and finally 1 mile north will take you back to the original point.

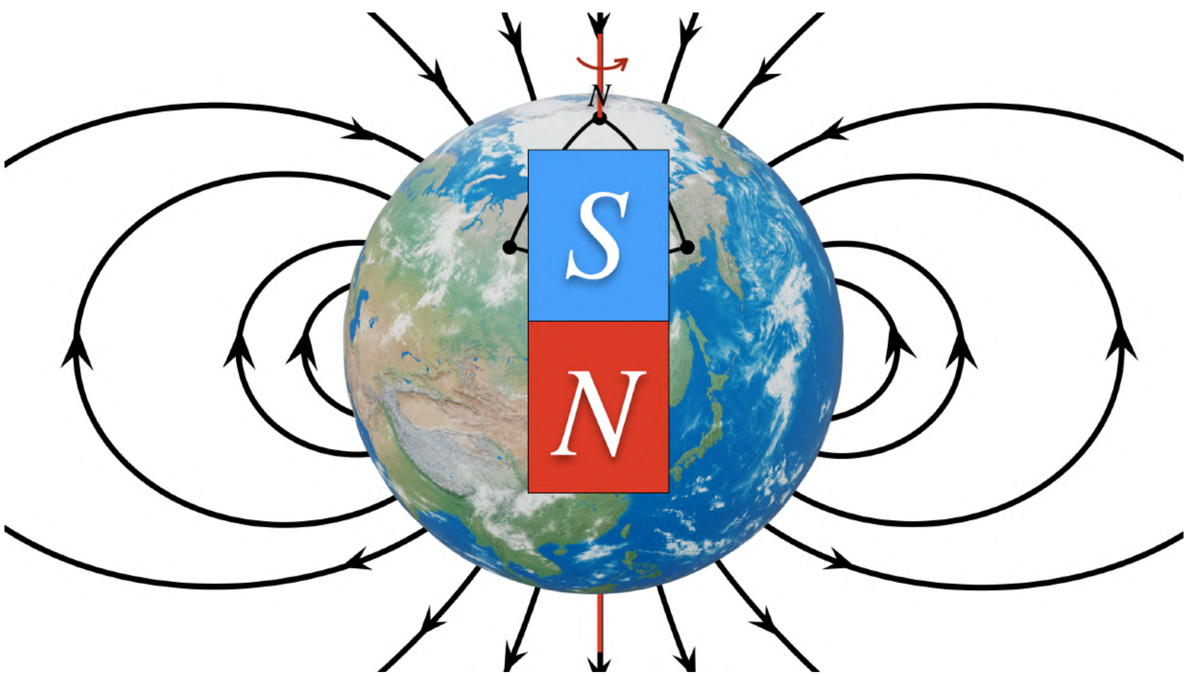
Notice how this would not work for most points on Earth.



There is nothing physically magical about the North Pole on a perfect sphere. What makes it special is our global definition of directions: all longitudes converge there.



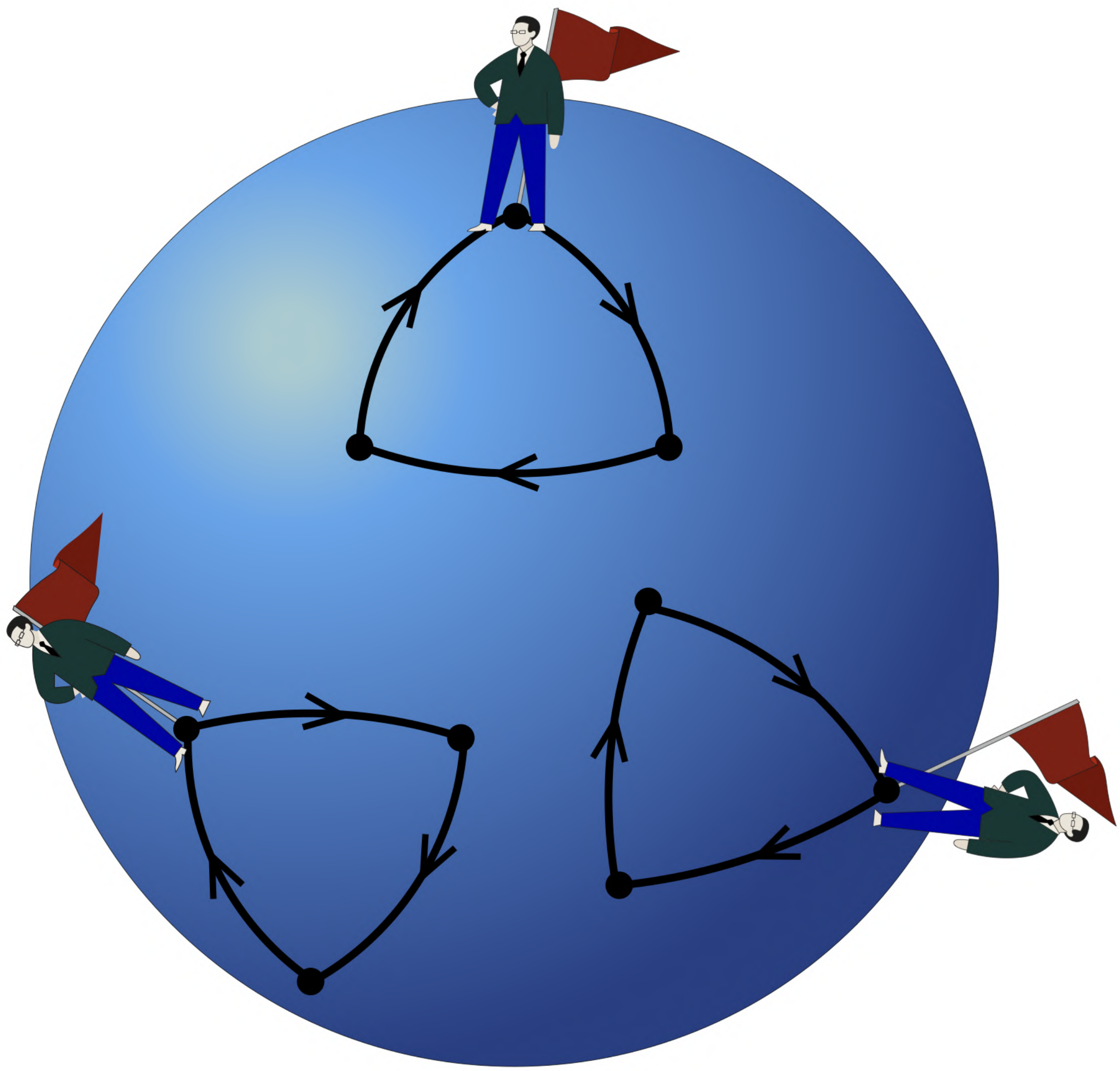
Of course, there are some physical meanings behind this convention, like the Earth's rotation, and its magnetic field, but ultimately the cardinal directions are a human choice, out of convenience.



A more local definition of directions would provide different results.

For example, say we define north as moving forward with respect to a person's orientation, south is backwards, west means moving to the left and east is moving to the right.

Then, any point on Earth would produce the same result. You could follow the 3 step guide and you would always end up at the same point you started at.



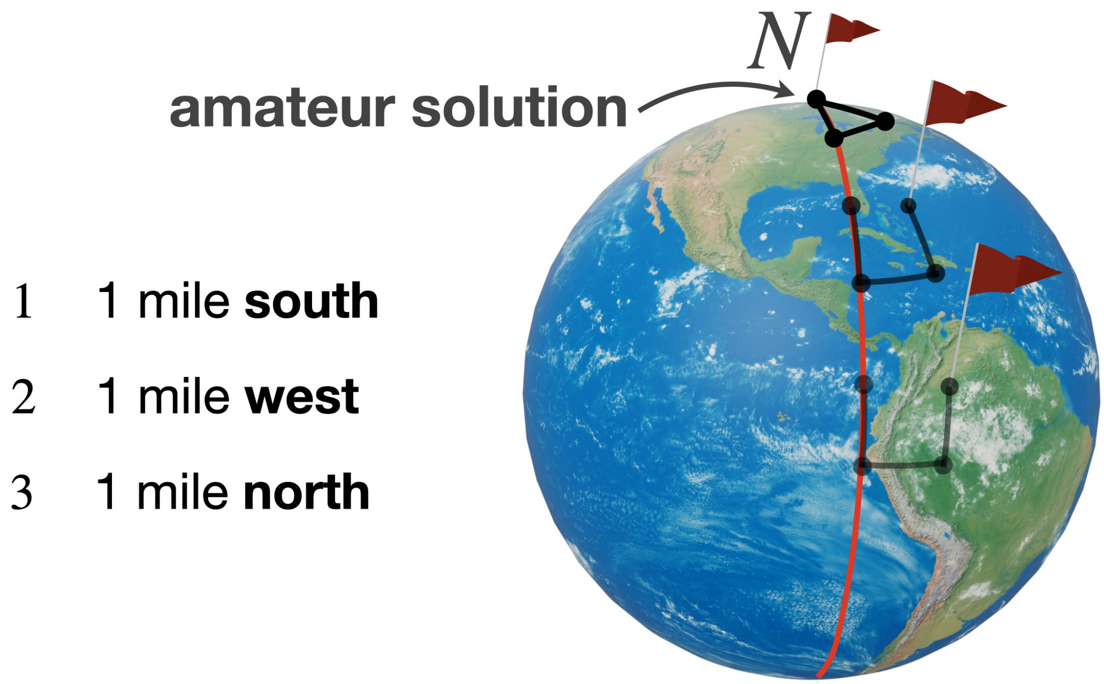
But let us go back to our global convention of cardinal directions in order to see the infinite solutions produced by the **Undergraduate Level**.

(2) Undergraduate Level

Undergraduate

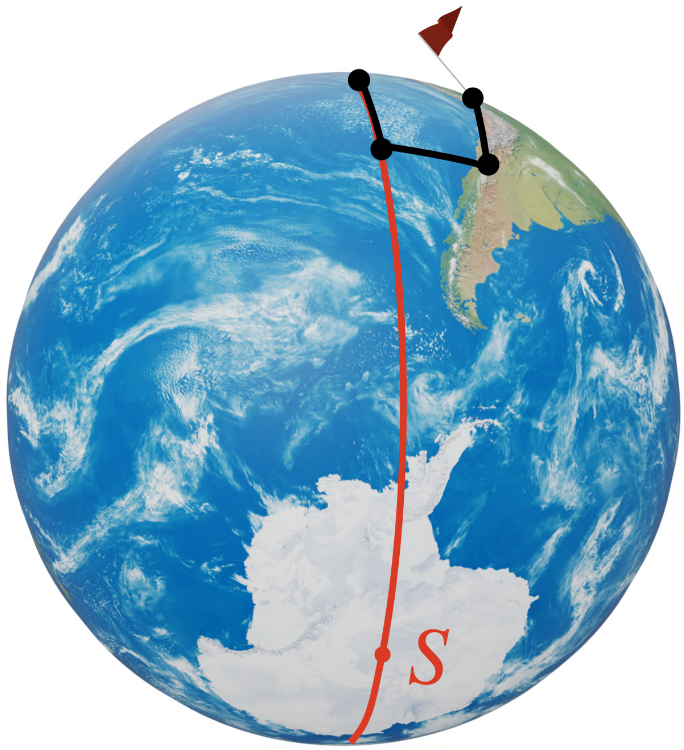
We can start at a point on the equator and follow the 3 step guide. This would take us to a different point, i.e. points on the equator are definitely not the solutions we are looking for.

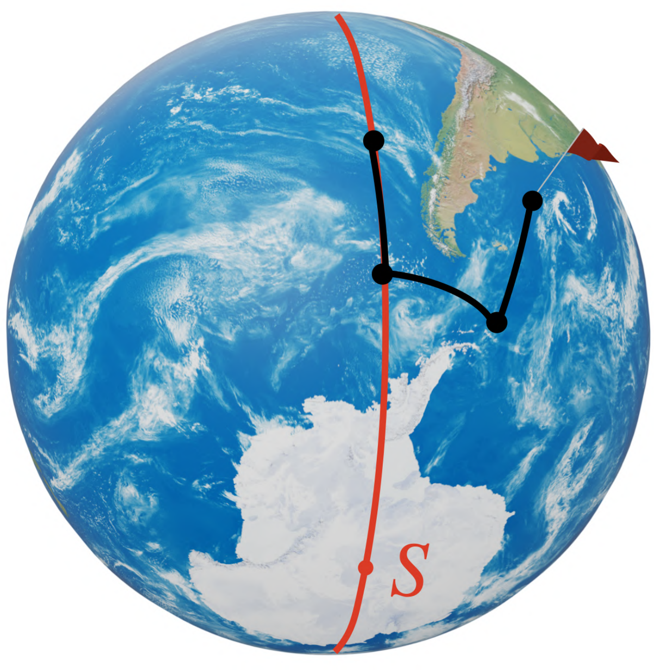
But notice what happens to this sort of incomplete square when we move along a specific longitude.



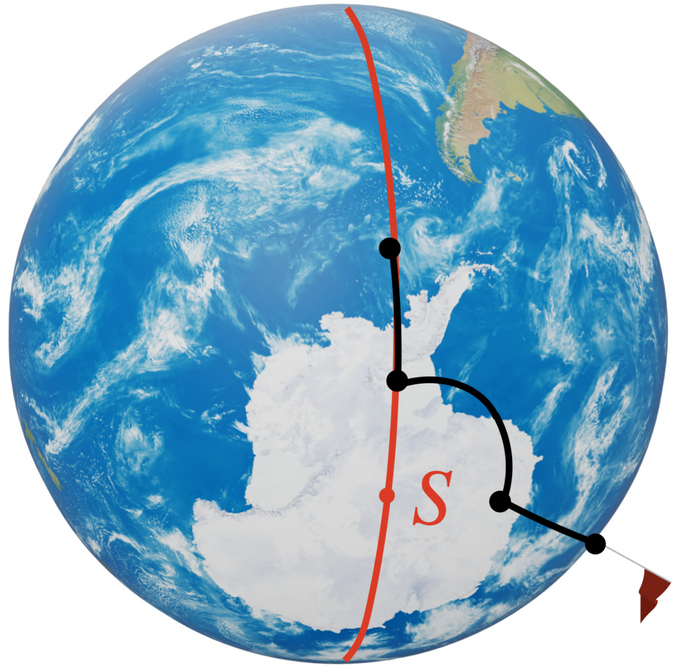
The extreme points converge such that at the North Pole we get a closed figure. A sort of curved triangle. This brings us back to the **Amateur Level** solution.

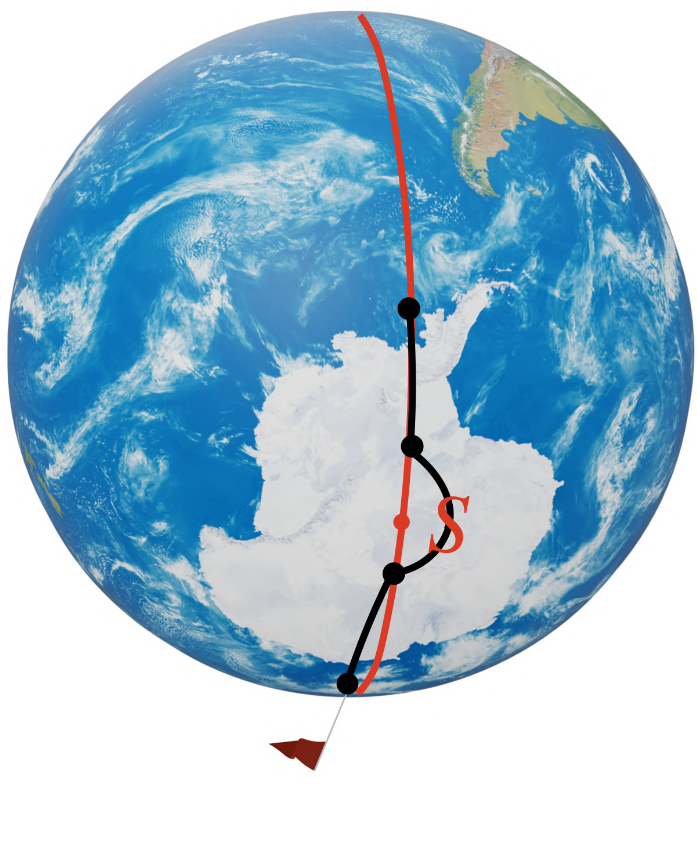
Interesting! What if, instead of moving north along the same longitude from the equator, we moved south? What would happen?



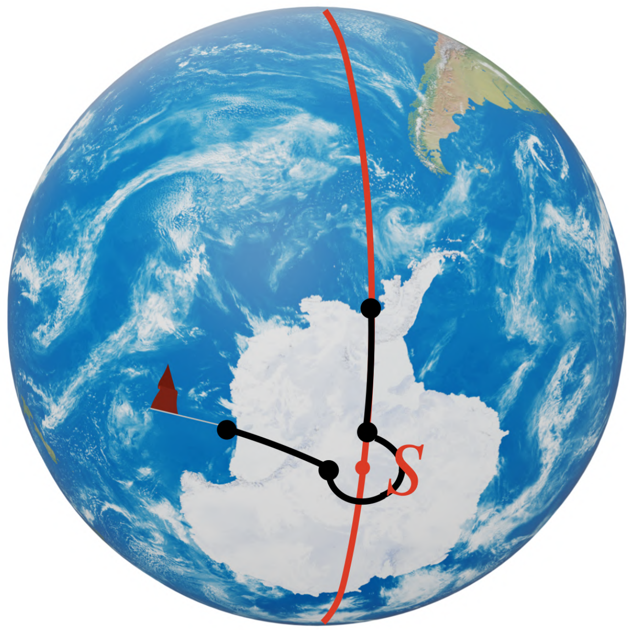


Instead of converging now, the start and endpoints diverge at first, with the endpoint moving west.





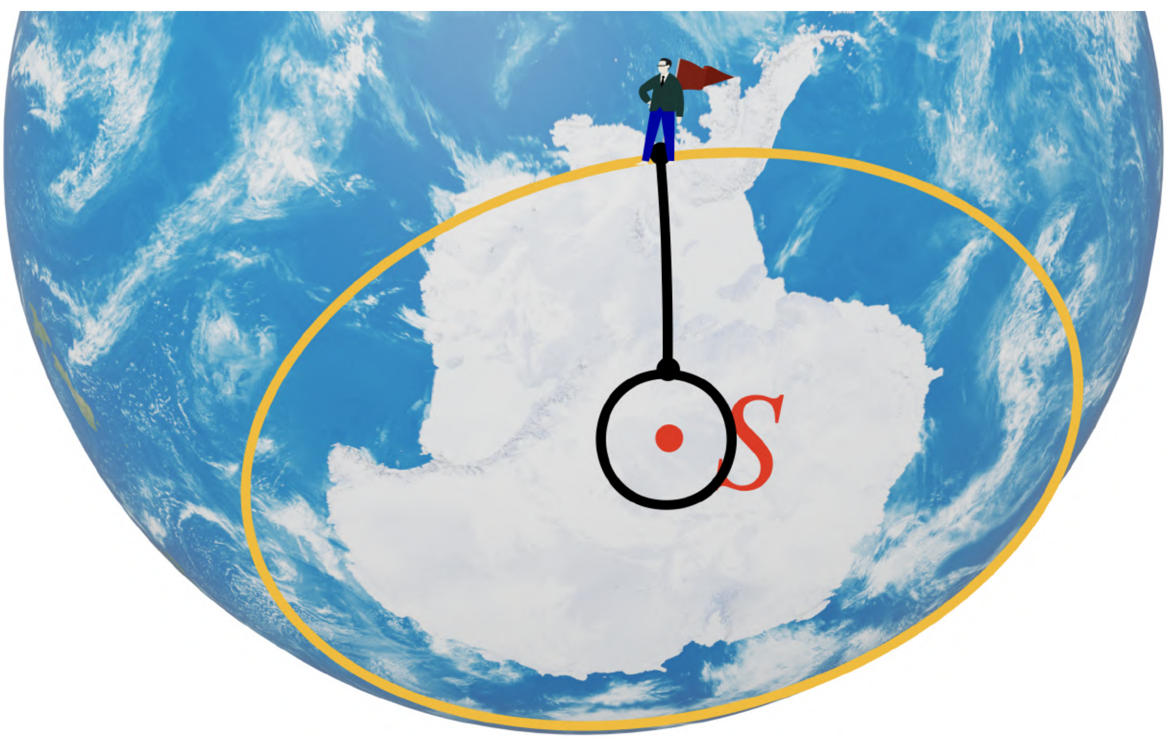
But this can't continue forever, because paths in the west direction eventually wrap around the sphere in a circle, which means that at a specific latitude the start and endpoints coincide.



And, by definition of our 3 step problem, this is a valid solution.

As you can see, the magic just happens when this little circle around the South Pole has circumference of 1 mile. And the distance from the initial point to the little circle, corresponding to the first and third steps, must also measure 1 mile.

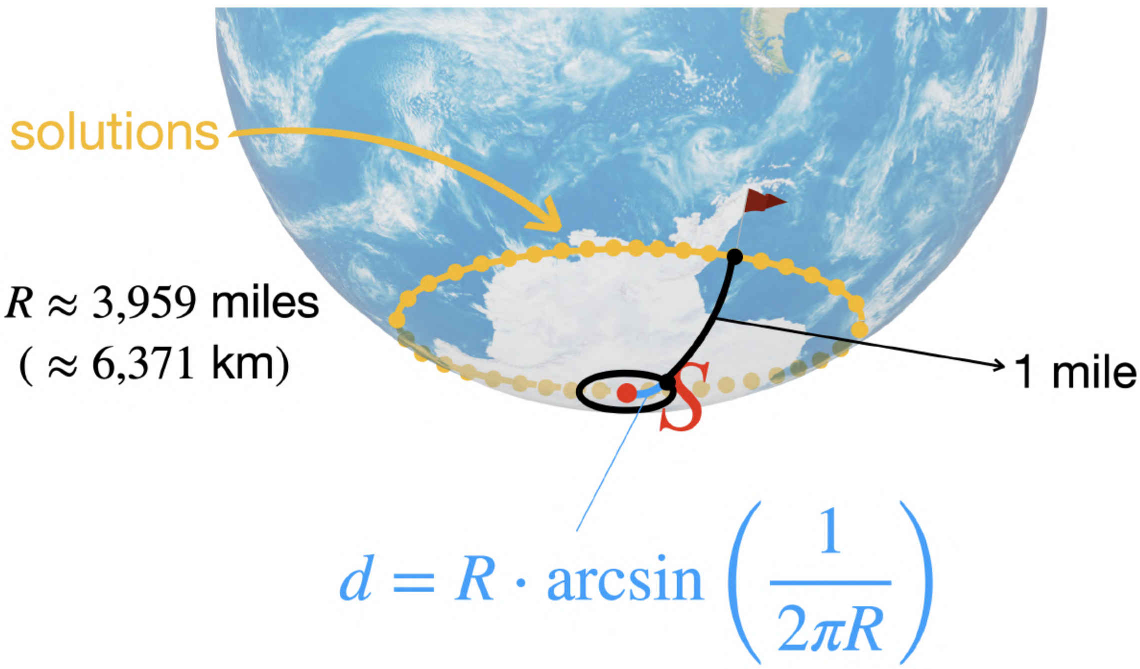
It is not very hard to see that this solution would work for all possible longitudes along this fixed latitude, i.e. all infinite points along this larger circle (image below) are solutions to our problem.



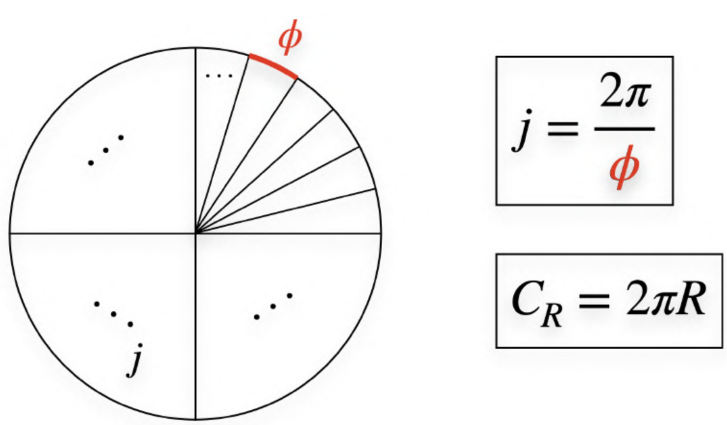
The first and third step distance is 1 mile, and the distance d, from the South Pole to points on the circle with circumference 1 mile, is:

$$d = R \cdot \arcsin\left(\frac{1}{2\pi R}\right)$$

, where $R \approx 3,959$ miles (or $\approx 6,371$ km) is the radius of the Earth.



This can be calculated with the *arc length formula*. Let us see how to derive it:



For an angle ϕ in a circle of circumference

$$\boxed{C_R := 2\pi R} \quad ,$$

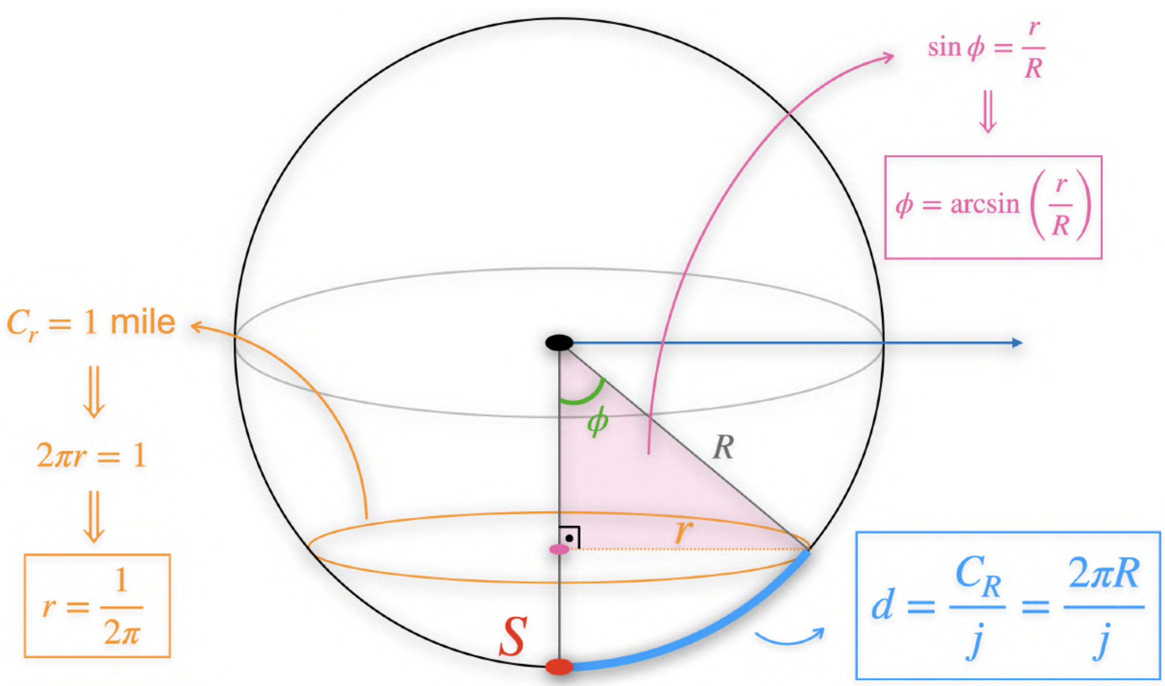
we can "fit in" $j \in \mathbb{N}$ slices inside of it such that it is completely filled in.

Here the number of slices can be calculated in the following way:

$$\boxed{j := \frac{2\pi}{\phi}}$$

And the arc length d of the arc formed by the angle ϕ, can be calculated by dividing the entire circumference by the number of slices:

$$\boxed{d := \frac{C_R}{j}}$$



Using all these equations that we found, we can calculate the distance d from the South Pole to all points of our circle of circumference 1 mile:

$$\boxed{r = \frac{1}{2\pi}} \qquad \boxed{\phi = \arcsin\left(\frac{r}{R}\right)} \qquad \boxed{C_R = 2\pi R}$$

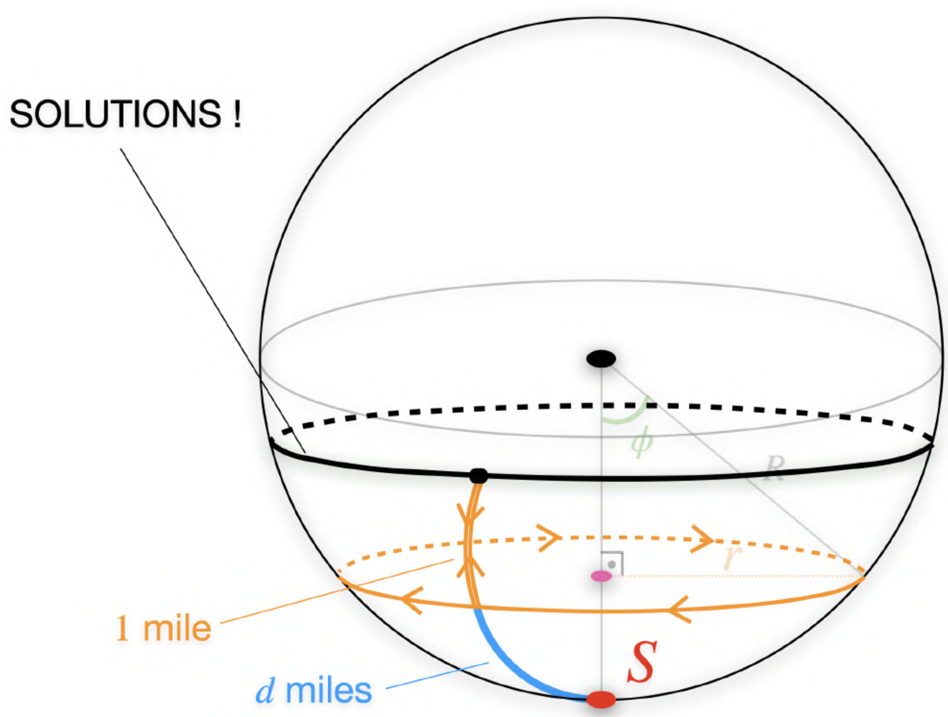
$$\boxed{j = \frac{2\pi}{\phi}} \qquad \boxed{d = \frac{C_R}{j}}$$

$$\Longrightarrow \qquad d = \frac{\cancel{2\pi} R}{\cancel{2\pi}} \cdot \phi = R \cdot \arcsin\left(\frac{r}{R}\right) \qquad \Longrightarrow$$

$$\Longrightarrow \qquad \boxed{\boxed{d = R \cdot \arcsin\left(\frac{1}{2\pi R}\right)}}$$

Therefore, the solutions of the **Undergraduate Level**, in precise mathematical language, would be:

All points along a circle whose points lie at a geodesic distance $1 + d$ *miles from the South Pole.*



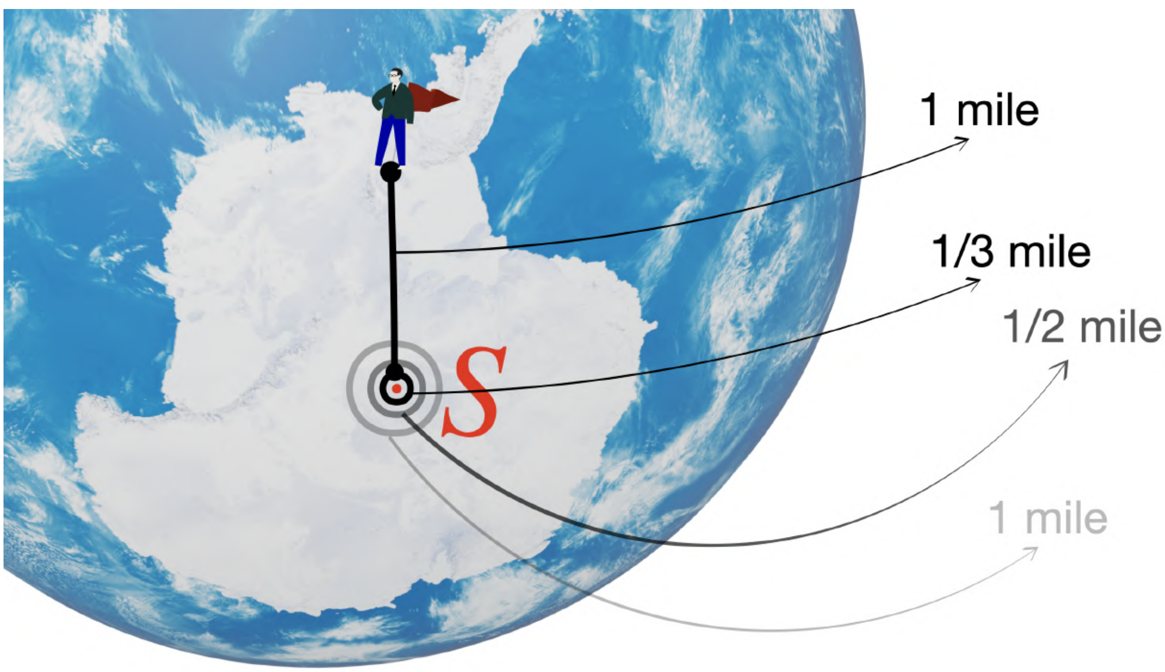
The images above are obviously not in scale.

After calculating it, we get an estimate of:

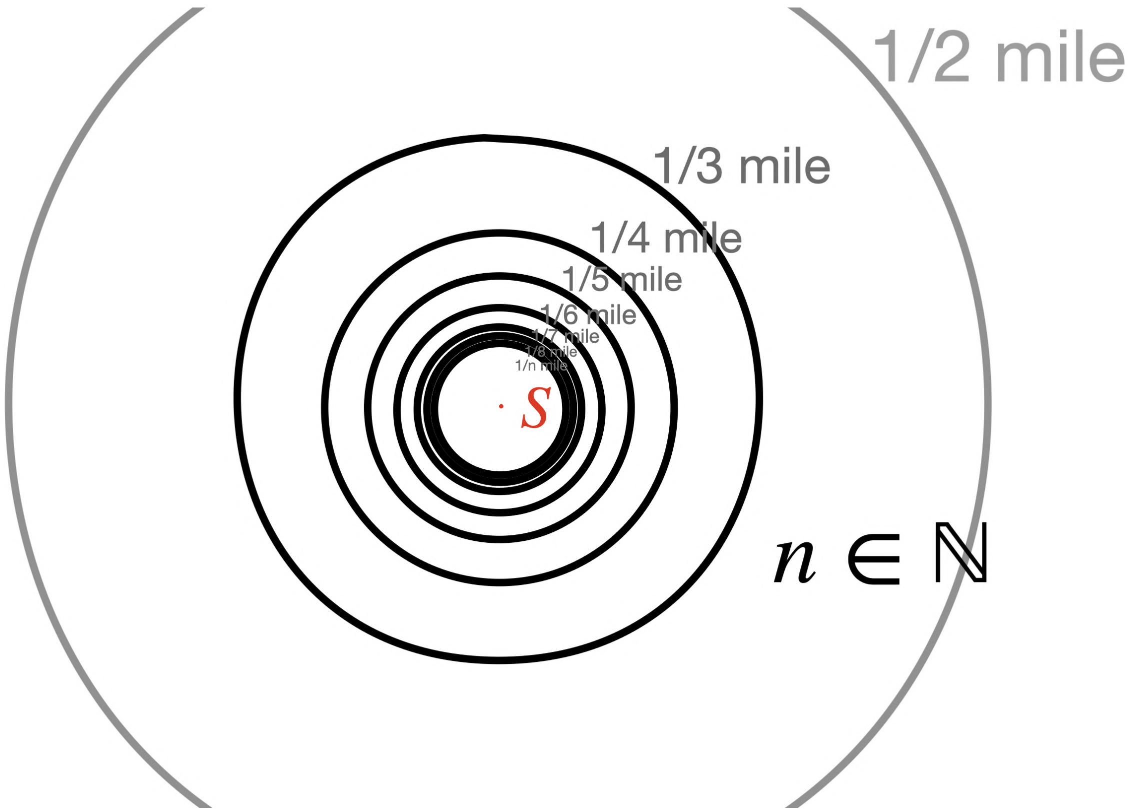
$$1 + d \approx 1.15915 \text{ miles (or)} \approx 1.865 \text{ km}$$

Awesome! Time to move on to the climax:

(3) Genius Level



What if we find a circle closer to the South Pole, that has a circumference of 1/2 mile (0.5 mile)?



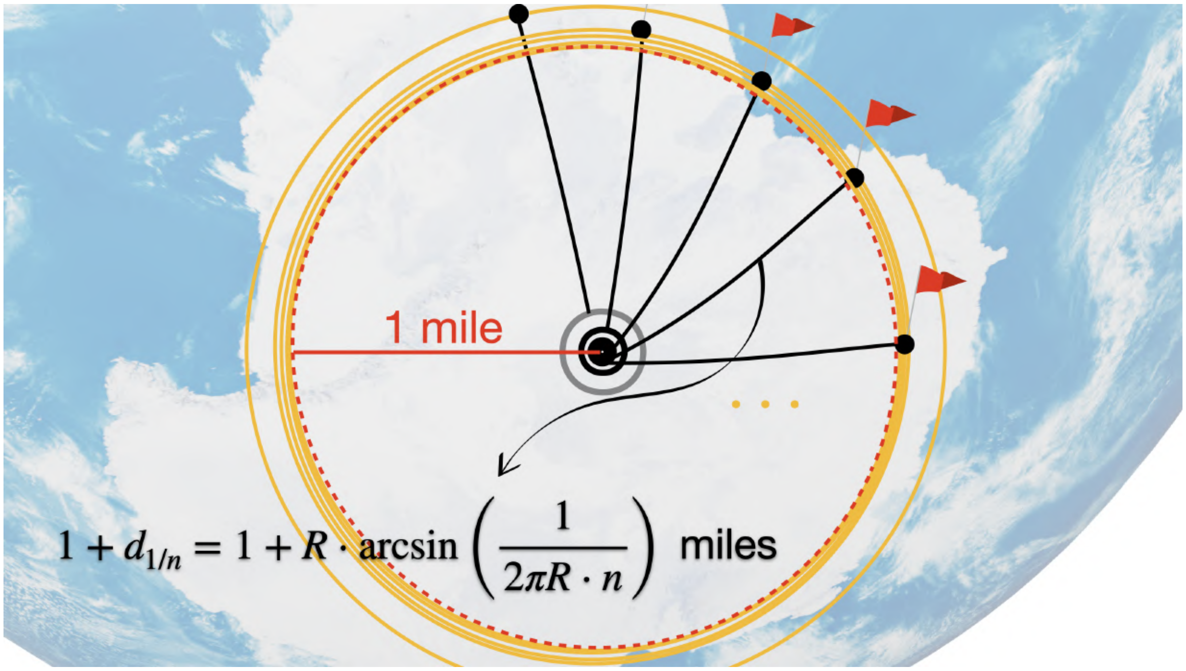
Then, we just found another solution! Let us use our 3 step guide:

1. Move one mile south.
2. One mile west, which means going around the circle twice.
3. And one mile north.

As a consequence, all points at the geodesic distance

$$1 + d_{1/2} = 1 + R \cdot \arcsin\left(\frac{1}{2\pi R \cdot 2}\right) \text{ (in miles)}$$

are solutions to our problem.

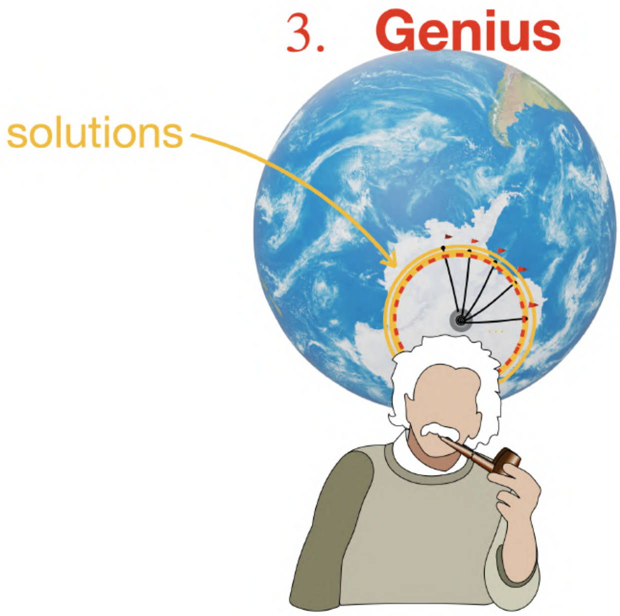


It is straightforward to see that there is nothing special about the number 1/2, other than the fact that the denominator is a natural number. We can find another circle of circumference 1/3 and repeat the same process.

And so on for all natural numbers $n \in \mathbb{N}$.

Rigorously stating it:

There are uncountably many solutions to this problem, corresponding to points along a countable infinite family of circles.



These solution points can be found at the following geodesic distances from the South Pole:

$$1 + d_{1/n} = 1 + R \cdot \arcsin\left(\frac{1}{2\pi R \cdot n}\right) \quad \text{(in miles)}$$

Let us derive this expression, $\forall n \in \mathbb{N}$:

First, we need to modify one of the equation:

$$C_r = \frac{1}{n}\ \text{mile} = 2\pi\, r \qquad \Longrightarrow \qquad \boxed{r = \frac{1}{2\pi\, n}}$$

Now, we can calculate the geodesic distance d from the South Pole to all solutions:

$$d_{1/n} = R \cdot \arcsin\left(\frac{r}{R}\right) = R \cdot \arcsin\left(\frac{1}{2\pi\, n\, R}\right)$$

Therefore,

$$\boxed{1 + d_{1/n} = 1 + R \cdot \arcsin\left(\frac{1}{2\pi R \cdot n}\right)}$$

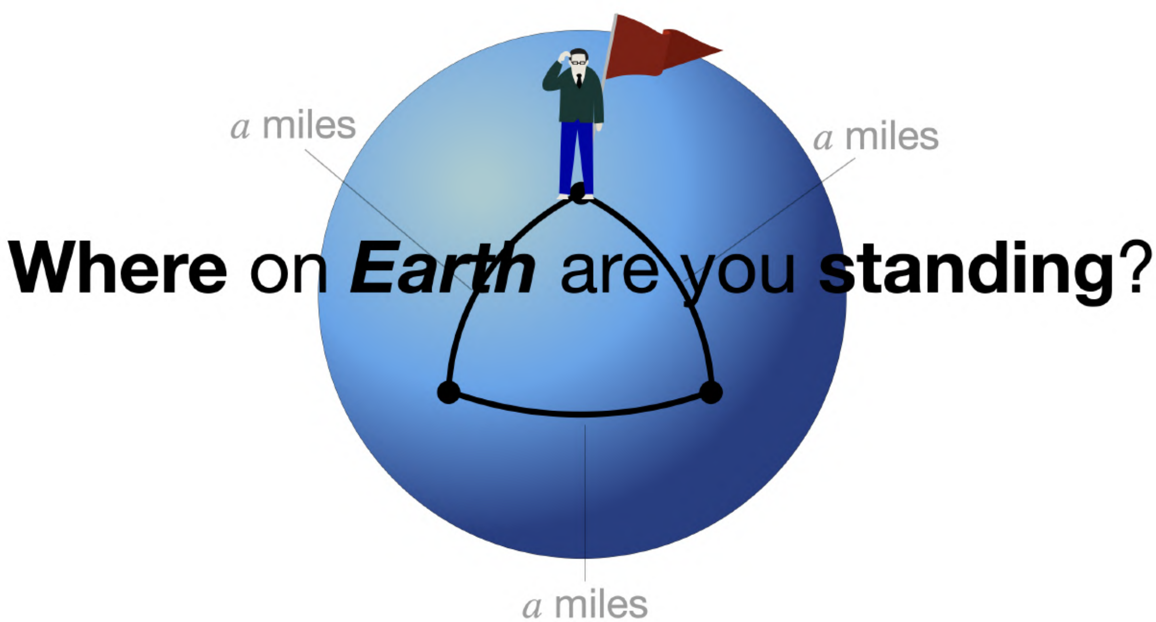
Cool! Now, we can level up the discussion with a slight modification of this problem. Try to solve it on your own first.

Bonus problem: Super Genius Level

1. You walk a miles south.
2. Then a miles west.
3. Then a miles north.

And you end up exactly where you started.

The question is: *"Where on Earth are you standing?"*



The only difference now is that we didn't fix the distance moved at each step to be 1 mile. Now, we can do it for any real number a of miles ($a \in \mathbb{R}$).

We need to generalize our previous solutions:

$$C_r = \frac{a}{n} \text{ miles} = 2\pi\, r \quad \Longrightarrow \quad \boxed{r = \frac{a}{2\pi\, n}}$$

$$d_{a/n} = R \cdot \arcsin\left(\frac{r}{R}\right) = R \cdot \arcsin\left(\frac{a}{2\pi\, n\, R}\right)$$

Therefore, the geodesic distance from the South Pole to solutions is:

$$\boxed{a + d_{a/n} = a + R \cdot \arcsin\left(\frac{a}{2\pi R \cdot n}\right)}$$

This is very interesting because looking at this expression we can see that there are some restrictions to the length a. It actually can't be just any real number we desire.

We know that the arcsin is a function that takes as an argument any real number between -1 and 1.

$$-1 \leq \frac{a}{2\pi R n} \leq 1$$

But also, since we are using spherical coordinates, the argument of the arcsin can be only between 0 and 1:

$$0 \leq \frac{a}{2\pi R n} \leq 1$$

Not only that. Since we want a to represent physical distances, then it must be positive:

$$0 < a \leq 2\pi R n$$

But also, since a cannot be greater than half of the circumference of the Earth, then this is the true valid interval of possibilities:

$$0 < a < \pi R$$

$$a < \frac{\cancel{2}\pi R}{\cancel{2}} = \pi R$$

$$\Downarrow$$

$$0 < a < \pi R$$

Finally, that's how some of the solutions of this generalized version of the SpaceX job interview problem look like on Earth, for different values of a ranging from 0 (excluded) to πR, and different natural numbers n:

$$a + d_{a/n} = a + R \cdot \arcsin\left(\frac{a}{2\pi R \cdot n}\right)$$

(Notice that not all combinations of $a \in \mathbb{R}$ and $n \in \mathbb{N}$ produce solutions, though.)

If you want to support our work, please, consider becoming a **member** *of our YouTube channel! We'd like to keep our videos free of interruptions and sponsors, so that the sole focus is the subject at hand. But in order to do that we need your help. Thanks for supporting us. Also, check out our catalogue of* **PDFs**.

`If you found this document useful let us know. If you found typos or things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com`