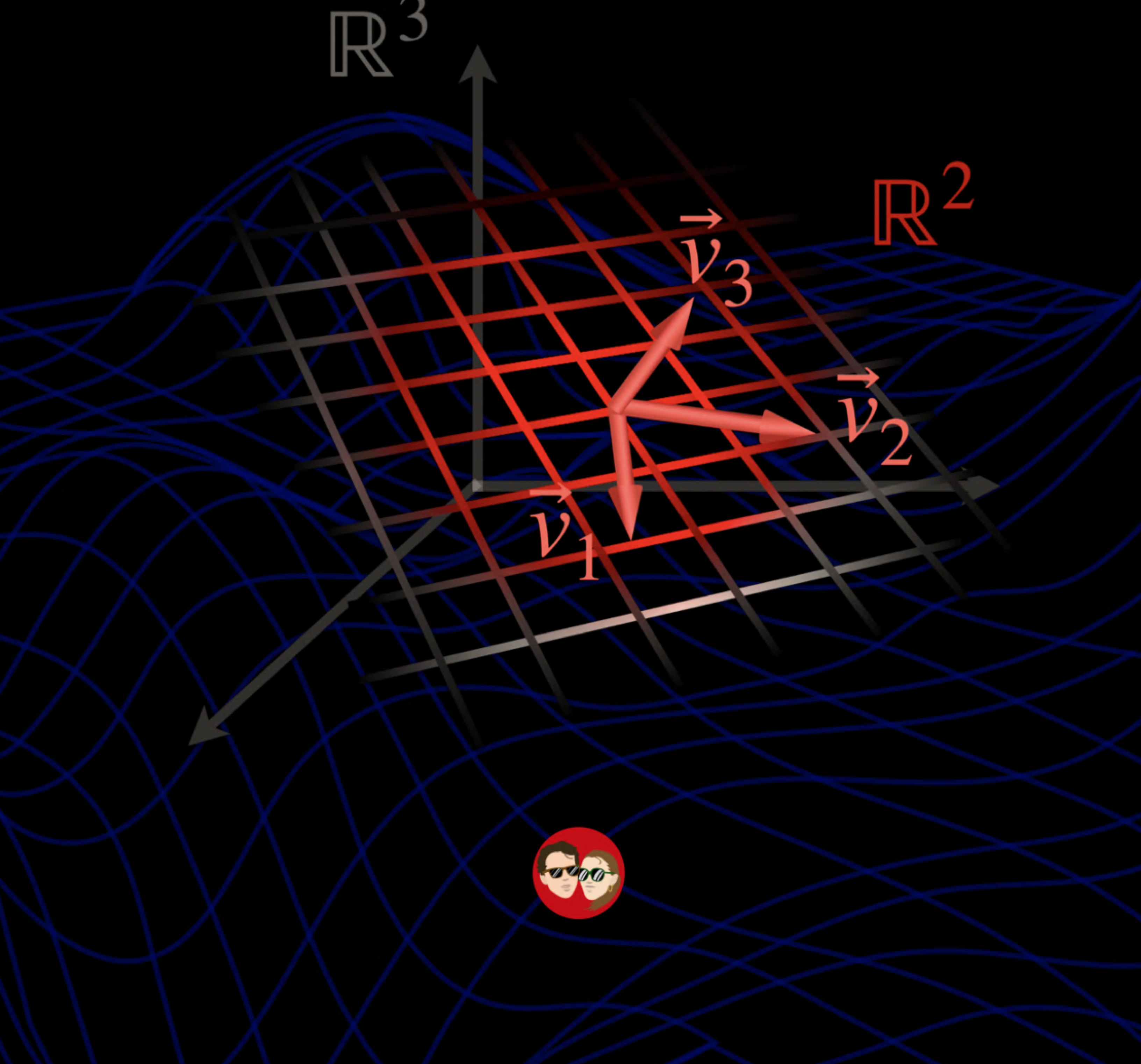
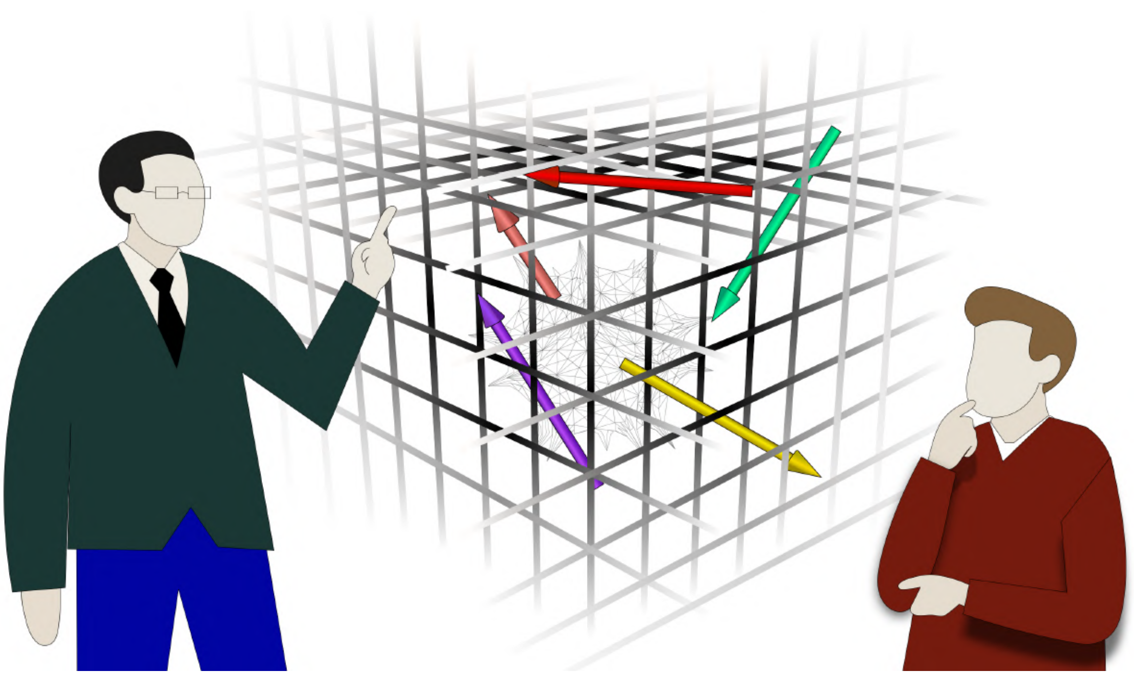
The Roadmap to

Linear Algebra



The Roadmap to Linear Algebra

by DIBEOS



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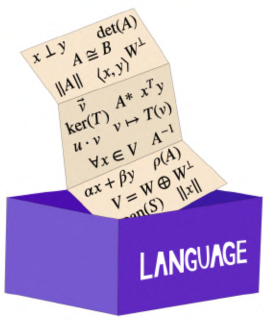

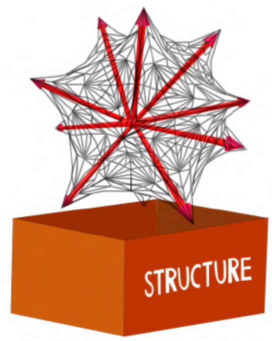

(hidden behaviors)

- **Vectors & Vector Spaces**
- **Orthogonality & Inner Product**
- **Subspaces, Span & Basis**

- **Matrices & Determinants**
- **Linear Transformations**
- **System of Linear Equations**

- **Eigenvalues & Eigenvectors**
- **Spectral Theorem**

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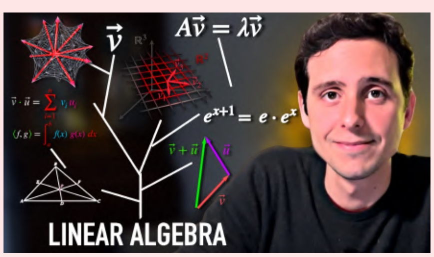
Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.

Gilbert Strang

"

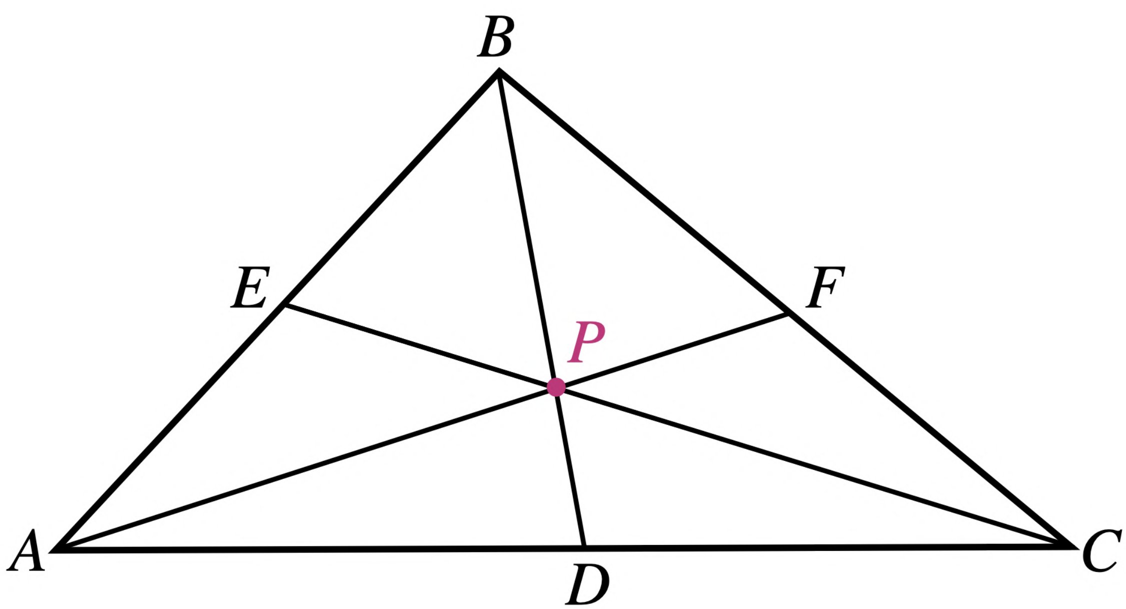
Suggestion



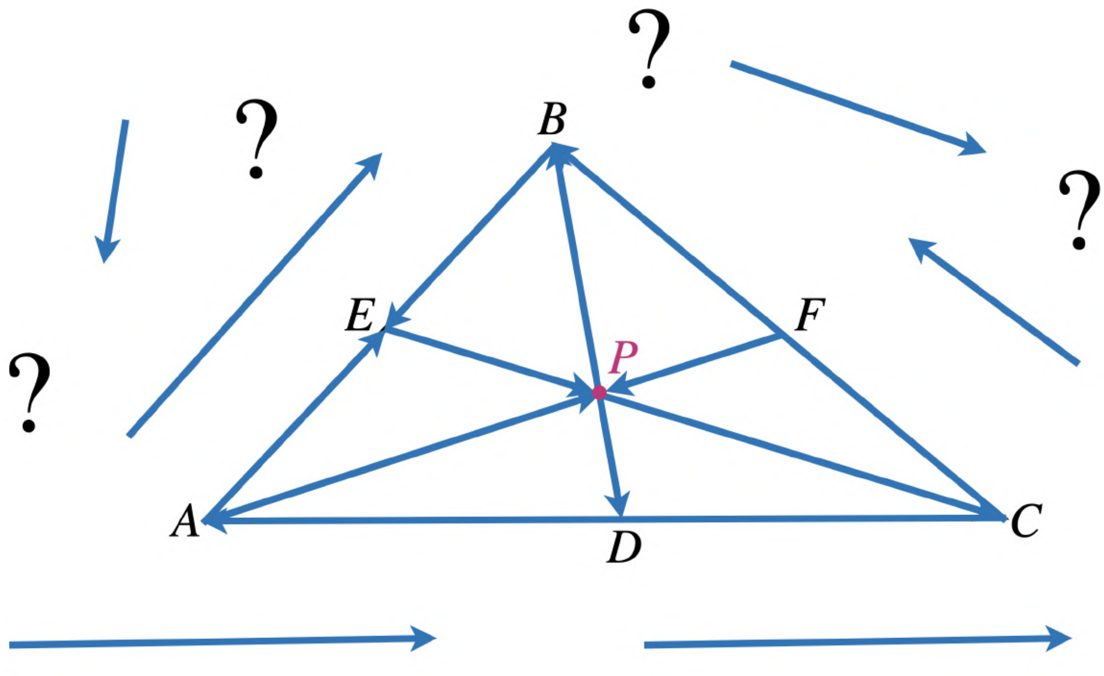
This material is a deeper look at the topics discussed in this YouTube video. We highly recommend watching the video first to get a basic understanding, and then reading this material. Click on the image.

Introduction

Take any triangle. Draw the 3 medians, which are the segments from each vertex to the midpoint of the opposite side.

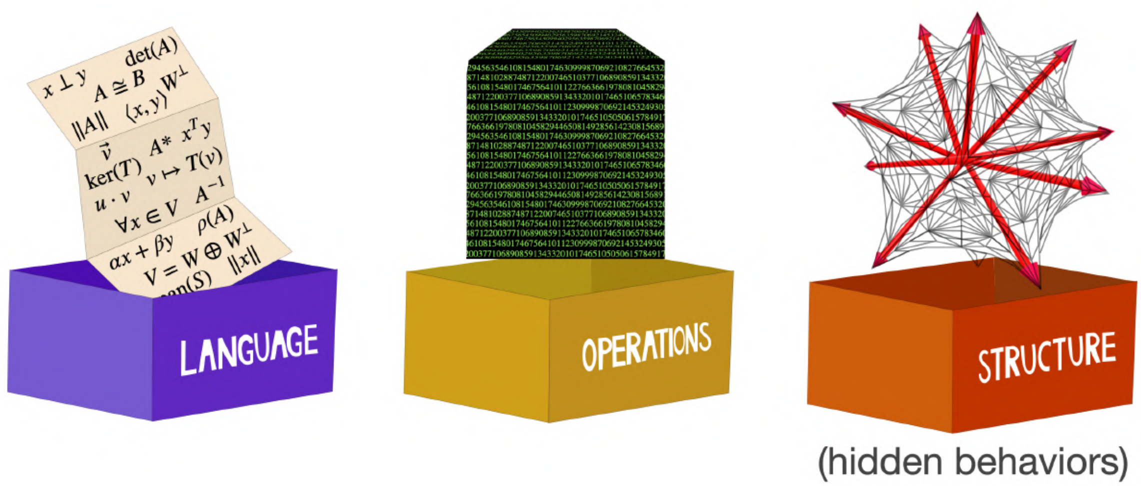


They always intersect at a single point called the **centroid**, or **barycenter** (P). The question is: *"Can we use vectors from Linear Algebra to find its exact location?"* It sounds simple, but there is a beautiful idea hidden in this problem!

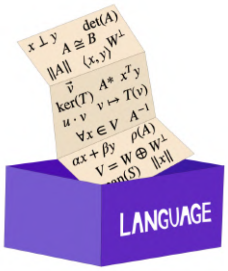


If you want to see the detailed solution of this problem go to exercise 1.

As you will see in this material, **Linear Algebra** is very powerful to solve all sorts of problems. And your roadmap to master it can be boiled down to 3 concepts:



In order to find the solution to this challenge, we have to start at the beginning.



Language

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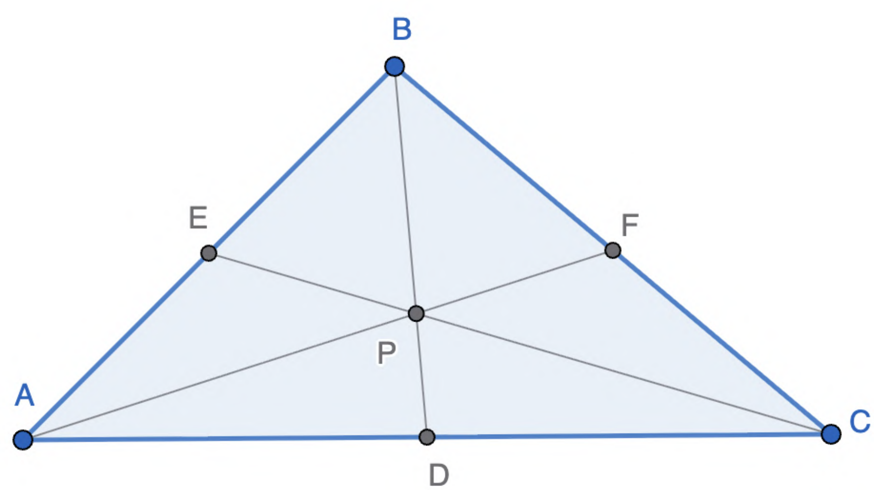
Exercise 1

← Back to main text

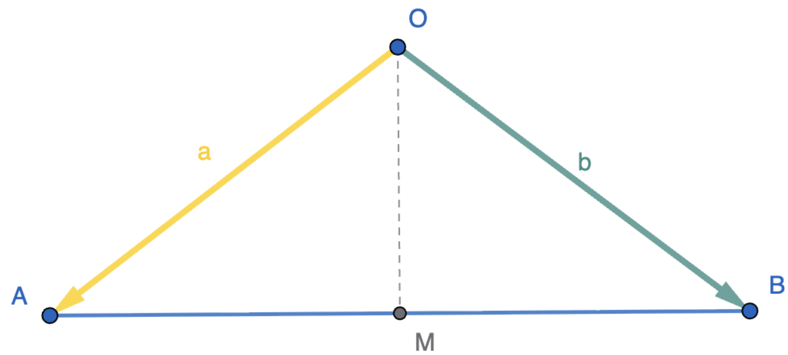
Take any triangle. Draw the 3 medians, which are the segments from each vertex to the midpoint of the opposite side. They always intersect at a single point called the centroid, or barycenter. The question is: *"Can we use vectors from Linear Algebra to find its exact location?"*



Click on the image above to see it in *https://www.geogebra.org/calculator*.

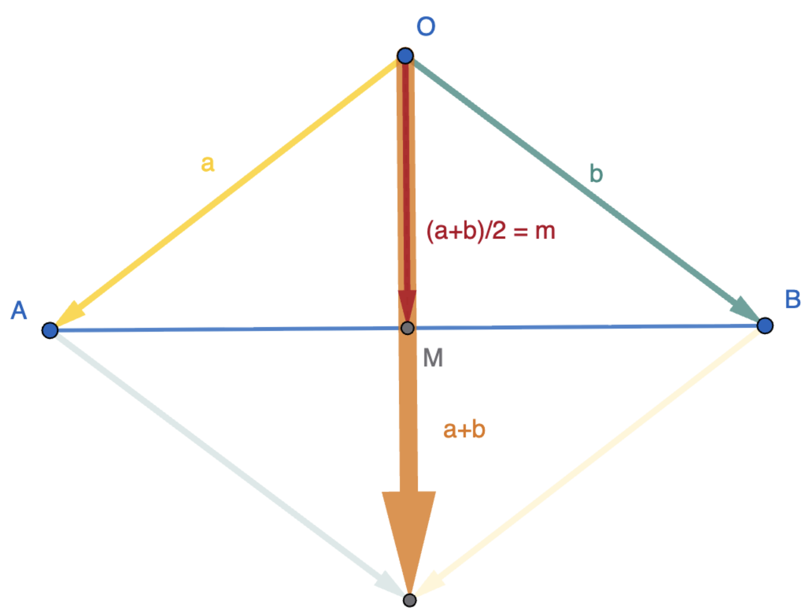
<u>Solution</u>:

Let us start with a much simpler problem, and build up to it. Imagine two points A and B connected by a line segment. We can choose any point right above the midpoint M of $\overline{AB}$, and denote it by O, which stands for *origin*. That is the point with respect to which our vectors will be defined.



Click on the image above to see it in *https://www.geogebra.org/calculator.*

Now, we consider the vectors $\vec{a}$ and $\vec{b}$ (from the origin O) pointing to A and B, respectively. Using the *parallelogram law of vector addition*, we get the vector $\vec{a} + \vec{b}$ and (of course) a parallelogram. The diagonal lines connecting opposite vertices of this parallelogram meet at the centroid, which turns out to be M.

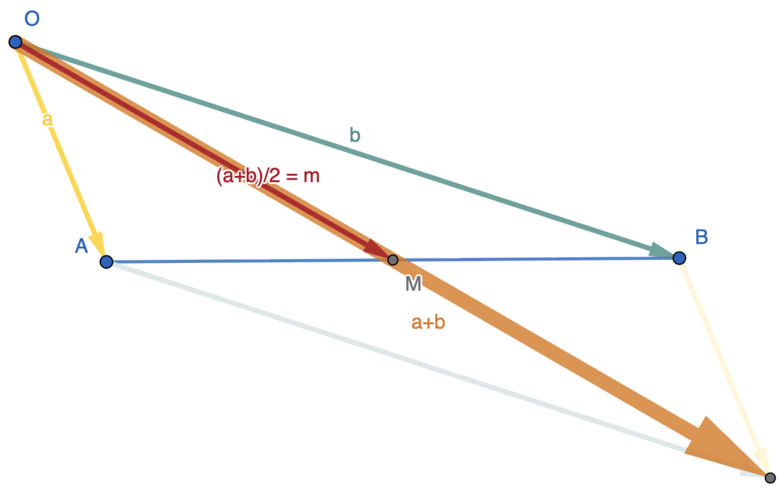


Click on the image above to see it in *https://www.geogebra.org/calculator.*

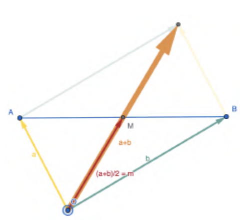
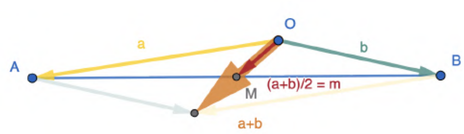
Since the length of the vector $\vec{a} + \vec{b}$ corresponds to the length of the vertical diagonal, then $\frac{\vec{a}+\vec{b}}{2}$ is the vector connecting the origin O to the centroid M. We will denote it by:

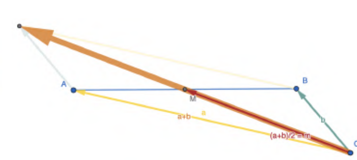
$$\vec{m} := \frac{\vec{a} + \vec{b}}{2}$$

It turns out that there is nothing special about our choice of where to position the origin O. I.e. for any random position you pick for O, you would still get a parallelogram (with some other shape) that has its diagonal lines meeting at the centroid, corresponding to the midpoint M.

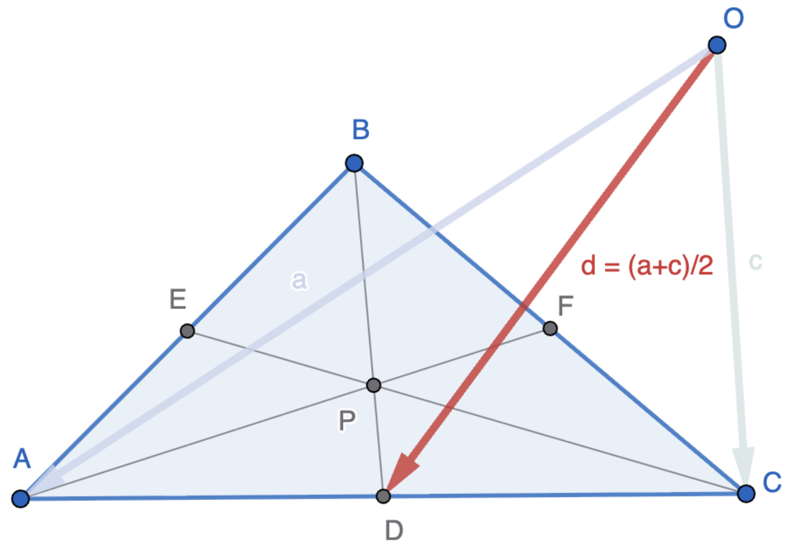


And most importantly, $\vec{m}$ would still be the vector connecting O to the centroid M. Click on any of the images below and drag the point O around. You will see how the vector $\vec{m}$ is always land on the midpoint M, no matter where you position the origin.



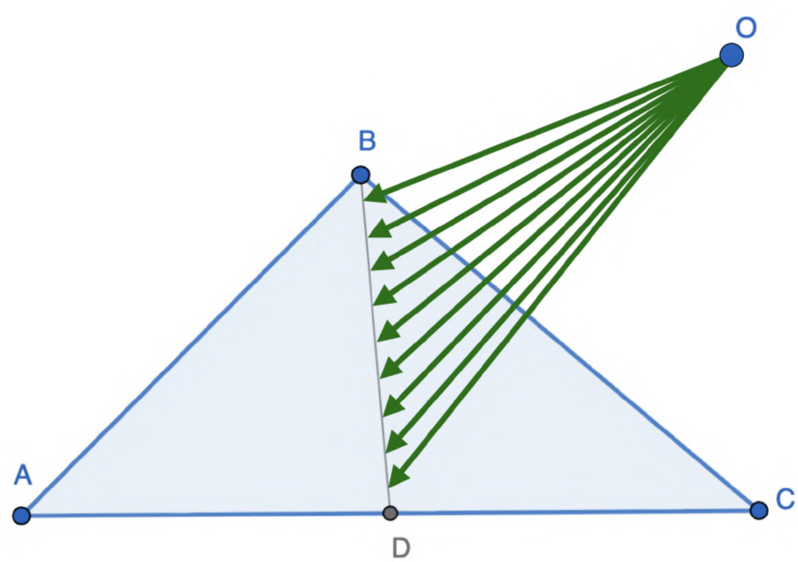
Once you are convinced of this fact, we can move on to generalize this problem to 3 points (A, B, C):



Let us define an origin O somewhere and take the midpoint D of the line segment $\overline{AC}$, i.e.

$$\vec{d} := \frac{\vec{a} + \vec{c}}{2}$$

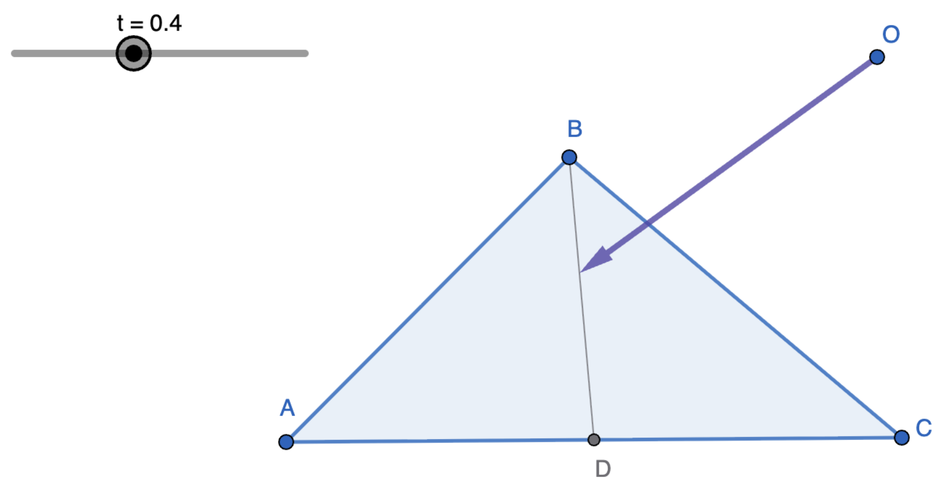
Now we will do something very interesting. We want to create a family of vectors, all of which come out of the origin O and land on points in the segment $\overline{BD}$.



We will do that through something called a ”parametrization”, which is a way of describing all points of a line segment (like $\overline{BD}$, in our case) by using a single variable $t \in \mathbb{R}$.

You can think of t as being like time in physics, because as t changes, the point moves and traces out the line segment. This makes the idea very intuitive. But in mathematics, t does not have to mean actual time at all. It is just a parameter, i.e. a real number that we let vary in some interval $[t_0\,,\ t_1] \in \mathbb{R}$. Each value of t gives one point of $\overline{BD}$, and as t runs through the interval, we get all points we want.

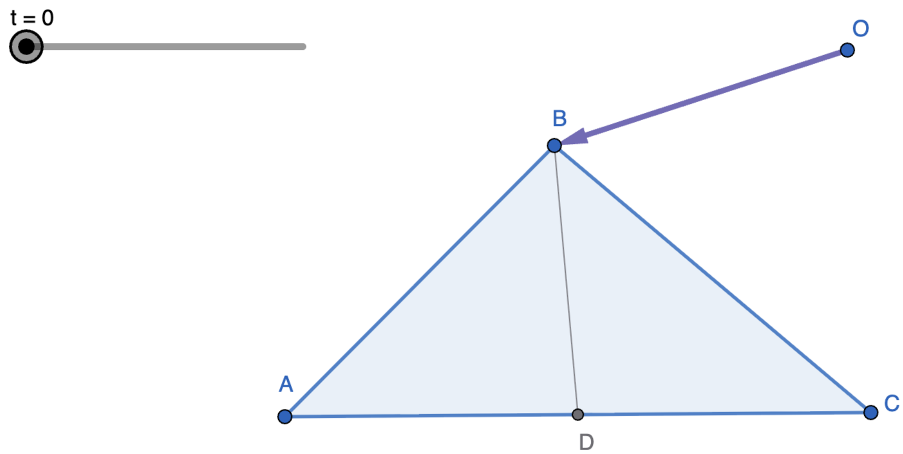
So the parameter plays the role of a label that tells us where we are on the segment. Click on the image below and press the play button to see the parametrization in action:

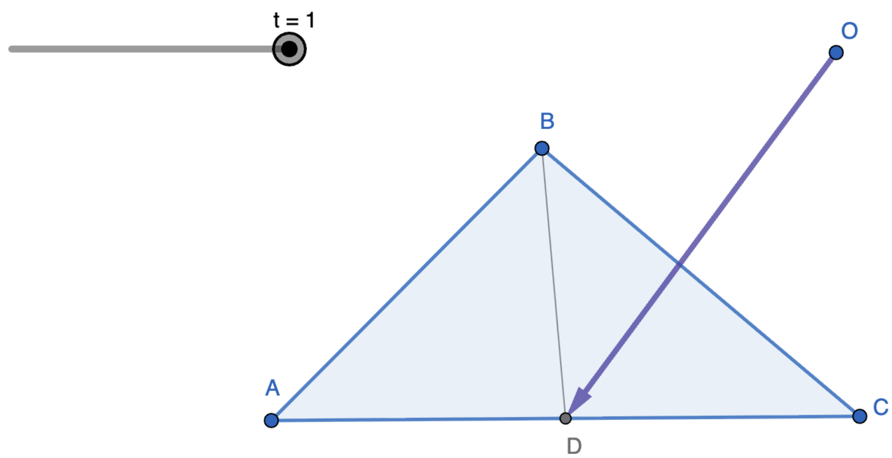


This is the parametrization that gives us all vectors from the origin O to points of $\overline{BD}$:

$$\boxed{(1-t)\,\vec{b} + t\,\vec{d}} \qquad , \qquad t \in [0, 1]$$

Notice how $t = 0$ gives us $\vec{b}$ (i.e. the vector from O to B), and $t = 1$ gives us $\vec{d}$ (i.e. the vector from O to D):

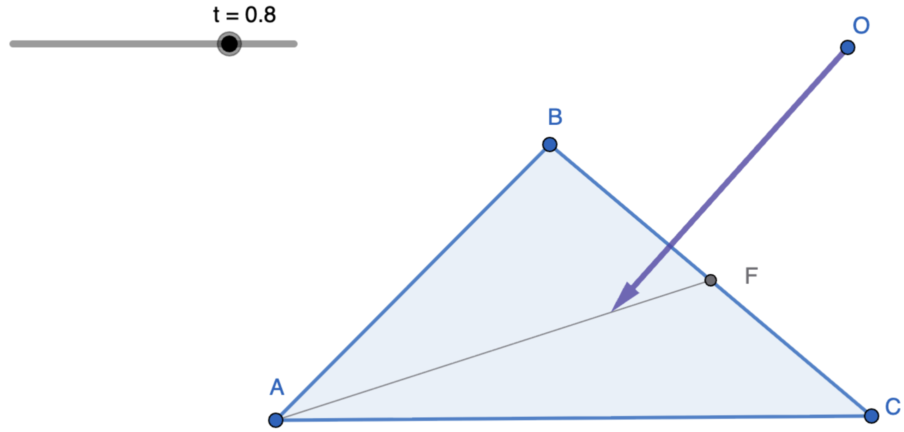




But since we saw earlier that $\vec{d} = \frac{\vec{a}+\vec{c}}{2}$, we can substitute it in the parametrization formula and obtain a new expression of the same family of vectors:

$$(1-t)\,\vec{b} + t\,\vec{d} \qquad \Longrightarrow \qquad \boxed{(1-t)\,\vec{b} + t\cdot\frac{\vec{a}+\vec{c}}{2}} \qquad (I)$$

Now, we do the same for the segment $\overline{AF}$, with F being the midpoint of $\overline{BC}$:



The parametrization gives us:

$$(1-t)\,\vec{a} + t\,\vec{f} \quad \Longrightarrow \quad \boxed{(1-t)\,\vec{a} + t \cdot \frac{\vec{b}+\vec{c}}{2}} \quad (II)$$

Since we know that the centroid P (which we are trying to locate) is at the intersection of $\overline{BD}$ with $\overline{AF}$, all we have to do is impose that $(I) = (II)$ in order to find the vector they have in common. This common vector points to the exact location of P:

$$(I) = (II) \quad \Longrightarrow \quad \boxed{(1-t)\,\vec{b} + t \cdot \frac{\vec{a}+\vec{c}}{2} = (1-t)\,\vec{a} + t \cdot \frac{\vec{b}+\vec{c}}{2}}$$

However, this equation will not be satisfied for all values of t. Only one of them will give us the correct "time" t where the vectors of each parametrization align perfectly to pinpoint the centroid P. Let us make a few attempts:

<u>For $t = 0$</u>:

$$\vec{b} = \vec{a} \quad \Longrightarrow \quad A = B \qquad \text{(contradiction)}$$

<u>For $t = 1$</u>:

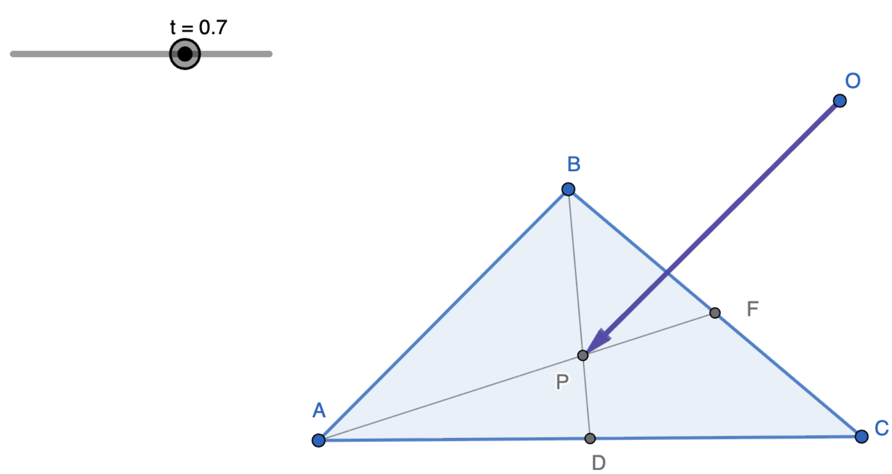
$$\frac{\vec{a}+\vec{c}}{2}=\frac{\vec{b}+\vec{c}}{2} \implies \frac{\vec{a}}{\cancel{2}}+\cancel{\frac{\vec{c}}{2}}=\frac{\vec{b}}{\cancel{2}}+\cancel{\frac{\vec{c}}{2}} \implies \vec{a}=\vec{b} \qquad \text{(contradiction)}$$

<u>For $t=\frac{1}{2}$:</u>

$$\frac{1}{2}\vec{b}+\frac{1}{2}\cdot\frac{\vec{a}+\vec{c}}{2}=\frac{1}{2}\vec{a}+\frac{1}{2}\cdot\frac{\vec{b}+\vec{c}}{2} \implies \frac{\vec{b}}{2}+\frac{\vec{a}}{4}+\cancel{\frac{\vec{c}}{4}}=\frac{\vec{a}}{2}+\frac{\vec{b}}{4}+\cancel{\frac{\vec{c}}{4}} \implies$$
$$\implies \frac{\vec{b}}{2}-\frac{\vec{b}}{4}=\frac{\vec{a}}{2}-\frac{\vec{a}}{4} \implies \frac{\vec{b}}{\cancel{4}}=\frac{\vec{a}}{\cancel{4}} \implies \vec{b}=\vec{a} \qquad \text{(contradiction)}$$

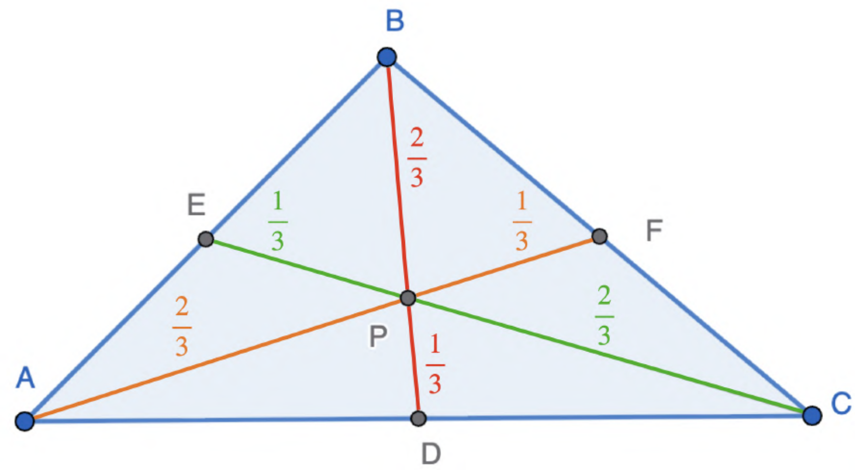
<u>For $t=\frac{2}{3}$:</u>

$$\frac{1}{3}\vec{b}+\frac{2}{3}\cdot\frac{\vec{a}+\vec{c}}{2}=\frac{1}{3}\vec{a}+\frac{2}{3}\cdot\frac{\vec{b}+\vec{c}}{2} \implies \frac{\vec{b}}{3}+\frac{2\vec{a}}{6}+\cancel{\frac{2\vec{c}}{6}}=\frac{\vec{a}}{3}+\frac{2\vec{b}}{6}+\cancel{\frac{2\vec{c}}{6}} \implies$$
$$\implies \frac{\vec{b}}{3}+\frac{\vec{a}}{3}=\frac{\vec{a}}{3}+\frac{\vec{b}}{3} \implies \vec{0}=\vec{0} \qquad \text{(satisfied)} \checkmark$$



Click on the image above and notice how the two parametrized vectors overlap only at the centroid.

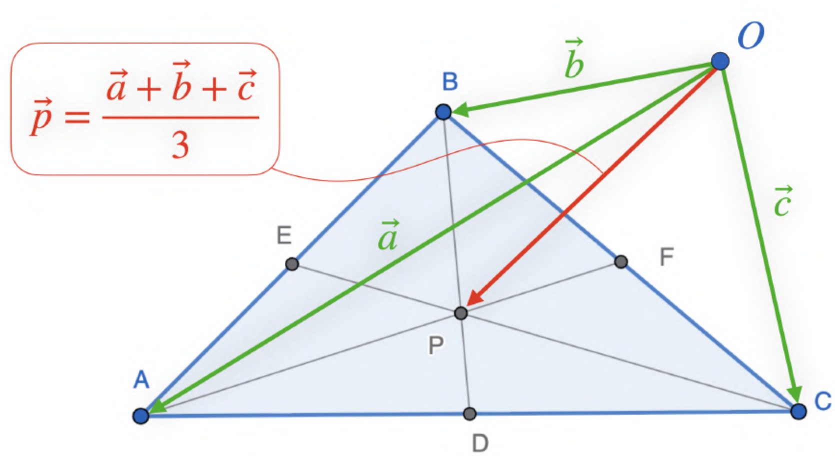
In fact, it is a well known fact that the distance between any of the vertices of a triangle to the centroid is $\frac{2}{3}$ of the distance of the whole segment.



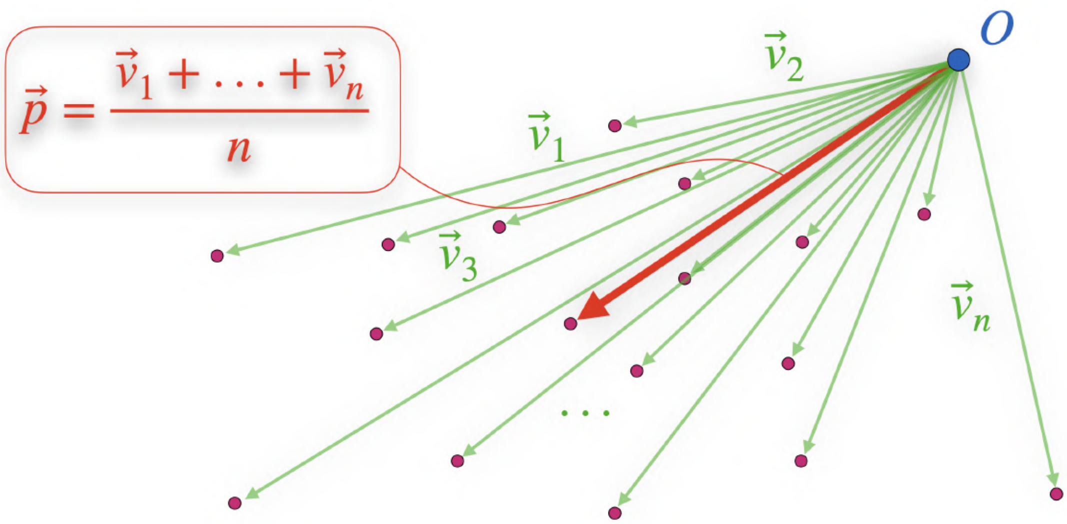
So to conclude the exercise of locating the centroid P, we can substitute $t = \frac{2}{3}$ into any of the parametrization formulas we have seen, and doing so will allow us to find the vector $\vec{p}$ from the origin to the centroid P:

$$(1-t)\,\vec{a} + t \cdot \frac{\vec{b}+\vec{c}}{2} \quad \xRightarrow{t=\frac{2}{3}} \quad \boxed{\vec{p} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}}$$

In other words, no matter where you defined your origin O to be, the centroid of the triangle is always located at the *arithmetic mean* of the 3 vertex vectors.

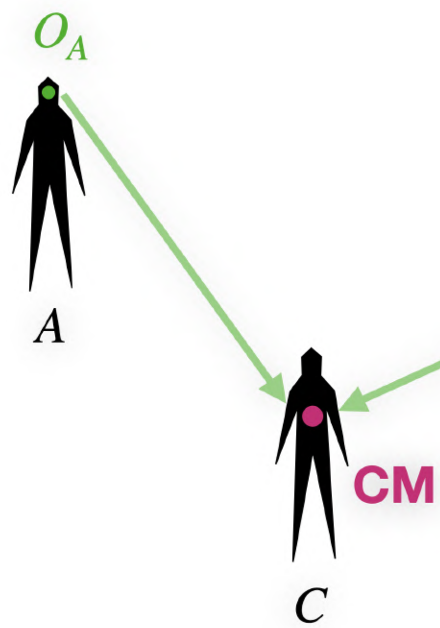


By the way, we could extend this idea to any collection of points. If we have 4, 5, 6, ..., n points, the centroid is always obtained by taking the arithmetic mean of the position vectors. It is the average location of all points.



And this is strongly related to the notion of **center of mass** (CM) in physics. If all the masses are equal, then the CM is found in exactly the same way. You just have to average the position vectors.

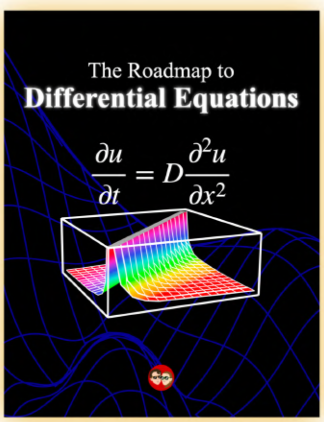
An important fact here must be highlighted: ***the choice of origin is arbitrary***.

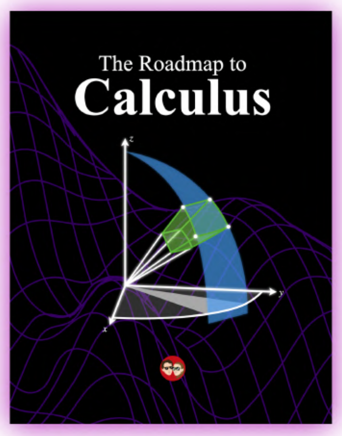


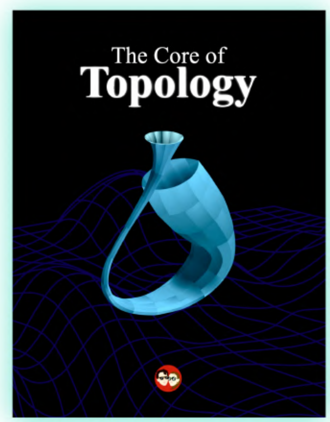
If a person A and a person B observe the same person C from different points in space (from different origins), they will use different coordinates (different bases), but they will still be referring to the same CM. The CM does not depend on a particular choice of basis. It is a true physical property of the body itself.

CLICK HERE TO CONTINUE READING

Do not forget to check out our catalogue of learning material. You might find something interesting. Just click on one of these covers:







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